$$|g|^2 \leq \frac{\text{const}}{\ln\{[1-\alpha_\rho(0)]\ln(1/\epsilon)\}^{-1}} \qquad (4.12)$$

in very close analogy to the result of Finkelstein and Kajantie.[1]

ACKNOWLEDGMENTS

It is a pleasure to thank Dr. R. C. Brower and Dr. J. Weis for many interesting and useful conversations during the course of this work.

*This work is supported in part through funds provided by the Atomic Energy Commission under Contract No. AT(11-1)-3069.

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PHYSICAL REVIEW D VOLUME 6, NUMBER 4 15 AUGUST 1972

Calculations of the Pomeranchuk Residue at $t=0$ Using Intermediate-Energy Bootstrap Dynamics Based on Average Nonlinear Duality*

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By imposing nonlinear forms of average duality at intermediate energies it is possible to obtain simple inhomogeneous relations between Regge residues. This reopens the possibility of using duality to make genuine bootstrap calculations. We consider two different forms of nonlinear duality, both of which are applied to $\pi\pi$ scattering, in combination with the usual linear duality. The first (Type A) asserts that, if σ^c is the cross section for a particular reaction c, $\sigma^c_{\text{resonance}} = \sigma^c_{\text{Regge}}$, when averaged appropriately over one or more resonances. Applying it at the g resonance, we obtain a total g width of 123 MeV, in agreement with experiment. The second (Type B) uses the optical theorem and states that $A_{\text{Regge}} = \sum_c K\sigma^c_{\text{Regge}}$ on the average, where A is the forward $\pi\pi$ absorptive part and K is a kinematic factor. The sum is over all reactions c making up the total cross section, which we take to be $\pi\pi \to \pi\pi$, $\pi\pi \to \rho\rho$, $\pi\pi \to \rho\epsilon$, and $\pi\pi \to \epsilon\epsilon$ below the 3ρ threshold. The last three are treated in a model-independent way, assuming only semilocal linear duality and the dominance of $I=1$ exchanges, such as the π and A_2. Calculations are then made in which Pomeranchuk exchange is included in A_{Regge} and different energy intervals are selected for the averaging procedure. For example, a semilocal calculation around the g resonance gives a ρ Regge residue corresponding to a ρ-meson width of 133 MeV, and a Pomeranchuk residue corresponding to an asymptotic $\sigma_{\text{tot}} = 13.5$ mb; the only input parameters are the resonance masses, which can be fixed by using the partial conservation of axial-vector current.

I. INTRODUCTION

There has recently been a certain revival of interest in the use of duality for making bootstrap calculations.[1-4] At first sight this may not appear to be very promising. For example, in the familiar linear Dolen-Horn-Schmid average absorptive-part duality condition[5]

$$\int A_{\text{resonance}} = \int A_{\text{Regge}}\,, \qquad (1)$$

both sides of the equation are proportional to a Regge residue function (excluding the Pomeranchukon). The over-all scale of such functions cannot therefore be determined from such conditions alone, and so we can have at best only a partial bootstrap. This is explicitly evident in the dual Veneziano model,[6] for which the normalization is completely arbitrary. Fortunately, this objection does not apply to nonlinear forms of duality, which lead to inhomogeneous conditions on residues and reopen the

possibility of using duality to make more complete bootstrap calculations.

Nonlinear duality conditions, which bring in unitarity and involve squares of amplitudes as well as linear terms, are not in fact completely new. Various forms have been suggested over the years[7,8] and have even been discussed within bootstrap frameworks. It is only recently, however, that attempts have been made to use them directly to make bootstrap calculations. At least one form[8] also seems to be capable of yielding the Pomeranchuk residue at the same time[2,3]; we shall be primarily concerned with this latter form here.

One of the main advantages of a bootstrap dynamics based on duality is its simplicity, which in turn permits us to look at more realistic models. In contrast to both low-energy N/D (or potential) dynamics or to high-energy multiperipheral dynamics, it does not necessitate solving any differential or integral equation. It also has the advantage that all calculations are carried out in the physical region and are done, moreover, at intermediate energies, where there already exists a considerable amount of experimental data. These data are usually much easier to analyze in detail than experiments at very high energies, especially when we are dealing with production amplitudes. They can therefore be more reliably incorporated into a calculation or used as a guide in setting up specific models.

We will use the same basic approach as in Ref. 3, but will formulate it somewhat differently and avoid as much as possible any direct phenomenological input or reliance on any specific model. In particular, instead of using a simple peripheral one-pion-exchange model for our production processes, we will only assume that they are dominated by $I=1$ exchanges. Most of our results would therefore follow from a broad class of models. As in Refs. 2 and 3 we will use two different forms of average nonlinear duality. The first (Type A) states that

$$\int K\sigma^c_{\text{resonance}} = \int K\sigma^c_{\text{Regge}}, \tag{2a}$$

where σ^c is the cross section for a particular reaction or channel c, $K\sigma^c$ is the corresponding contribution to the forward elastic absorptive part in the optical theorem, and the integral is the usual duality integral over one or more resonances. This form was probably first introduced by Chew and Pignotti, who used it to justify the multi-Regge model.[7] Recently a more direct application of it was recommended by Veneziano.[1] It has also been shown[3] that a generalized version of it is capable of approximately reproducing some of the results originally obtained by Schmid on the more dubious basis of local duality.[9]

Equation (2a) has to be handled with a certain amount of caution. Chew, Rogers, and Snider[10] have shown that it can lead to difficulties when used at very low energies. Our hope is that it becomes meaningful at intermediate energies, where inelastic effects are already important. In Sec. II we use it in $\pi\pi$ scattering to calculate the total width of the g meson. In this calculation, as in all others in this paper, we assume that the Lovelace-Veneziano model[11] is a reasonable description of $\pi\pi$ scattering, at least in some semilocal sense.

The second type of nonlinear duality which we shall use (Type B) is a way of implementing the optical theorem and has the form

$$\int A_{\text{Regge}} = \int \sum_c K\sigma^c_{\text{Regge}}, \tag{2b}$$

where A is the forward elastic absorptive part, and the sum is over all reactions c which contribute to the total cross section. Equation (2b) can also be generalized to $t\neq 0$, as well as to inelastic amplitudes. At intermediate energies, the dominant reactions c would be quasi-two-body processes with single Regge exchange, but if we want to use Eq. (2b) at higher energies, we would presumably have to include the contribution of more complicated production graphs, such as diffraction-dissociation or multiperipheral diagrams. Indeed this type of duality was originally proposed as an approximate way of solving multiperipheral-type integral equations.[8,2,3]

Although Eq. (2b) can be derived from a combination of linear duality and a generalized version of Type-A duality, it is really a weaker form than the latter. It does not even use the resonance description, so that both sides of Eq. (2b) are relatively smooth functions. It should therefore be more reliable than Eq. (2a), particularly as we go to higher energies. Since it uses unitarity (via the optical theorem) we might also expect that the Pomeranchuk (P) trajectory would have to be included in A_{Regge}. In fact, it was seen in Refs. 2 and 3 that it is possible to use Eq. (2b) to calculate some of the parameters of this trajectory.

In Sec. III, we discuss our Type-B duality in the specific context of $\pi\pi$ scattering. We then apply it to the s-channel isospin $I_s=2$ state. Since this is an exotic state it is presumably dominated by P exchange,[12] and duality can be used even at low energies. In Sec. IV we find that we have a simple relation between the t-channel isospin $I_t=1$ and $I_t=0$ absorptive parts below the 3ρ threshold, assuming only semilocal linear duality and $I=1$ exchange dominance in the inelastic processes which contribute to the right-hand side of Eq. (2b). Such a relation would be valid in a model with π and A_2 Regge

exchange, for example. In Sec. V we combine it with Eq. (2b) at the g resonance mass. If we require the resulting relation to be consistent with Sec. III we obtain both the forward P residue and the normalization of the Lovelace-Veneziano model.

In Sec. VI we again calculate the forward P residue, but instead of assuming semilocal duality, use a finite-energy sum rule (FESR) in which linear and nonlinear duality are combined. This permits us to do our duality averaging over a wider interval. We also relax or modify the demands of exchange degeneracy and the Freund-Harari hypothesis. In Sec. VII we repeat the calculation of Sec. VI but replace one of our positive-moment FESR's with a negative-moment sum rule. This requires a knowledge of the amplitude at zero energy, for which we use the partial conservation of axial-vector current (PCAC). Finally, in Sec. VIII we discuss various possible improvements and generalizations of some of our techniques.

II. TYPE-A DUALITY AND THE TOTAL WIDTH OF THE g MESON

It is possible to generalize Eq. (2a) to $t \neq 0$ by rewriting it in the form

$$\int_{s_0}^{s_1} ds[a_{\rm res}^{\pi\pi}(s,t) - a_{\rm Reg}^{\pi\pi}(s,t)]\nu^n = 0\,, \tag{3}$$

where s, t, u are the usual Mandelstam variables, $\nu = \frac{1}{2}(s-u)$, s_0 and s_1 are points either at or midway between resonances, and $a_{\rm res}^{\pi\pi}$ and $a_{\rm Reg}^{\pi\pi}$ are the contributions to the absorptive part of Figs. 1(a) and 1(b), respectively, keeping only the $\pi\pi$ intermediate state. In what follows, we will only consider $n=0$ and $t=0$, however. In that case

$$a_i^{\pi\pi}(s,0) = \frac{1}{2q\sqrt{s}} \int_{-4q^2}^{0} dt |T_i(s,t)|^2\,, \tag{4}$$

where q is the magnitude of the c.m. three-momentum, while $T_{\rm res}$ is the resonance and $T_{\rm Reg}$ is the Regge form of the $\pi\pi$ scattering amplitude. In Refs. 2 and 3, Eqs. (3) and (4) were only used in combination with a model-dependent version of Type-B duality to calculate the total width of the g. In this section we will use them by themselves alone.

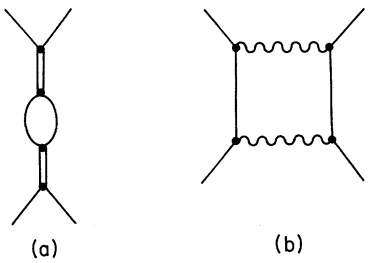

FIG. 1. (a) Unitarity diagram giving $a_{\rm res}^{\pi\pi}$. Single lines denote pions and double lines denote resonances (R). (b) Unitarity diagram giving $a_{\rm Reg}^{\pi\pi}$. Wavy lines denote Regge exchanges.

To calculate the two forms of T, we use the Lovelace-Veneziano model.[11] In the $I_s = 1$ state this gives

$$T^{I_s=1}(s,t) = F(s,t) - F(s,u)\,, \tag{5}$$

with

$$F(x,y) = -\bar{\beta}\,\frac{\Gamma(1-\alpha(x))\Gamma(1-\alpha(y))}{\Gamma(1-\alpha(x)-\alpha(y))} \tag{6}$$

and $\alpha(x) = \alpha_0 + \alpha' x$. By requiring the trajectory α to give the correct ρ mass m_ρ, we obtain $\alpha' = (1-\alpha_0)/m_\rho{}^2$. Actually, since the pion has a negligibly small mass, m_ρ (or α') is not a true parameter but merely serves to fix the energy scale in our problem. The value of α_0 can be fixed either by requiring the g meson[13] to have the correct mass $m_g \simeq \sqrt{5}\, m_\rho$ or requiring the $I_t = 2$ amplitude $F(s,u)$ to satisfy PCAC[11] (the situation for $I_t = 0$ is complicated by the presence of the Pomeranchukon). In either case we get $\alpha_0 \simeq 0.5$. This result can also be obtained by demanding a semilocal version of Eq. (1) to hold exactly at $t=0$ in the $I_t = 2$ state. Since there are no trajectories dual to any resonances in this state, this means that all s- and u-channel resonances must cancel each other out, at least when we average over all those having the same mass. At $t=0$ this is possible only with $\alpha_0 \simeq 0.5$.

Equations (5) and (6) give an asymptotic behavior of the form

$$T_{\rm Reg}^{I_s=1}(s,t) = \beta(t)e^{-i\pi\alpha(t)}\nu^{\alpha(t)} - \beta(u)e^{-i\pi\alpha(u)}\nu^{\alpha(u)}\,. \tag{7}$$

Instead of using Eq. (7) directly, however, we found it more convenient to use the fact that $F(t,u)$ is a smooth function of s for $s>0$ and can be written as

$$F(t,u) \simeq \beta(t)\nu^{\alpha(t)} + \beta(u)\nu^{\alpha(u)}\,. \tag{8}$$

Now, even at the sort of intermediate energies we shall be concerned with, the first and second terms in Eqs. (7) and (8) dominate only near the forward and backward directions, respectively, and can thus be expected to have little interference between them. Thus, as far as the integral in Eq. (4) is concerned, we can write

$$|T_{\rm Reg}^{I_s=1}(s,t)|^2 \approx |F(t,u)|^2\,. \tag{9}$$

The integral itself was evaluated numerically at $s=s_0$. For $s>s_0$, however, we used the fact that, except for logarithms, Eqs. (4) and (8) lead to an approximate energy dependence of $\nu^{2\alpha(0)-1}$. We thus have

$$a_{\rm Reg}^{\pi\pi}(s,0) \simeq a_{\rm Reg}^{\pi\pi}(s_0,0)(\nu/\nu_0)^{2\alpha(0)-1}\,, \tag{10}$$

where ν_i is the value of ν at $s=s_i$.

To evaluate $a_{\rm res}^{\pi\pi}$ we shall again use Eqs. (5) and (6). Now Eq. (6) cannot be used directly in Eq. (4) unless an imaginary part is added to α, since we would then have double poles in the integrand. We shall, instead, simply use the fact that, if T is dominated by a single Breit-Wigner resonance with elasticity $r(s)$ at a given energy, then Eq. (4) gives

$$a_{\rm res}^{\pi\pi}(s,0)=r(s)\,{\rm Im}\,T_{\rm res}(s,0). \tag{11}$$

At $s=m_g{}^2$, it is easy to check explicitly that the g resonance does in fact dominate by simply going back to Eq. (5) and (6), although Eq. (11) continues to be valid even if several resonances are important, provided each of them has approximately the same elasticity. If we use Eqs. (5) and (6) for $T_{\rm res}$, Eq. (11) reduces to a sum of δ functions.

In practice, we took $(s_0, s_1)=(4m_\rho{}^2, 6m_\rho{}^2)$, which are the halfway points between the g meson and the resonances immediately below and above it in the Lovelace-Veneziano model. The value of $r(m_g{}^2)$ was then calculated from Eq. (3), using Eqs. (11), (5), (6), (10), (4), and (9). This gave $r=0.57\bar{\beta}$. Now the residue of the pole in Eqs. (5) and (6) at the g mass corresponds to a partial width of $r\Gamma_{\rm tot}=70\bar{\beta}$ MeV for the decay of the g into $\pi\pi$. Combining these two results, we obtain $\Gamma_{\rm tot}=123$ MeV, a result which is independent of the value of $\bar{\beta}$ and agrees with experiment to within experimental error.[13]

Once we have a value for $\Gamma_{\rm tot}$ we also have a constraint on the partial widths, which must add up to $\Gamma_{\rm tot}$. If there are enough homogeneous relations between these partial widths, either from some kind of symmetry[4] or from linear duality, we could then determine their actual values. A more reliable procedure, however, would probably be to use Type-B duality.

III. TYPE-B DUALITY AND A SIMPLE APPLICATION TO AN EXOTIC STATE

The first problem we are faced with in using Eq. (2b) is deciding what to choose for the channels c. We must certainly include the elastic $\pi\pi\to\pi\pi$ process of Fig. 2(a). At intermediate energies, however, we will also have inelastic processes such as the $\pi\pi\to RR$ quasi-two-body reaction of Fig. 2(b), where R is any accessible resonance in the Lovelace-Veneziano model (including the ρ). Alternatively, we could also have reactions like $\pi\pi\to\pi\omega$ and $\pi\pi\to\pi A_2$. This set is actually dual to the $\pi\pi\to RR$ set, so that we cannot pick both at the same time but must choose one or the other, at least in any given region of phase space. It is not clear which is better on the average; if this duality is a good approximation it should not matter, however. We will pick the $\pi\pi\to RR$ set for simplicity and also because it is the correct choice below the 3ρ threshold if we were using a generalized Reggeized Amati-Bertocchi-Fubini-Stanghellini-Tonin (ABFST) model.[14] Such a model takes into account the experimental fact that if the final particles from a multiple-production reaction at high energies are ordered according to longitudinal momenta, the mean subenergy of a neighboring pion pair is in the resonance region.

In evaluating Fig. 2(a), we will assume that the dominant exchanges are given by the ρ and f^0 trajectories, with Lovelace-Veneziano[11] residue functions. In contrast to what was done in Ref. 3, we will not assume any specific model for Fig. 2(b) but simply assume that it is dominated by $I=1$ exchange. This includes therefore the exchange of the π, A_1, and A_2 trajectories. It does not include ω Regge exchange. However, the comparatively small branching ratio of the g meson for decay into $\pi\omega$ suggests that this neglect may not be unreasonable.[13] In applying Eq. (2b) we will restrict ourselves to energies below the 3ρ threshold. If we did not, we would have to include diagrams like Fig. 3, which would complicate the calculation considerably.

Equation (2b) can be generalized away from the forward direction by rewriting it as

$$\int_{s_0}^{s_1} ds\sum_i b_i(t)\nu^{\alpha_i(t)+n}=\int_{s_0}^{s_1} ds\sum_c a_{\rm Reg}^c(s,t)\nu^n, \tag{12}$$

where $b_i\nu^{\alpha_i}$ is the contribution of the Regge trajectory α_i to $A_{\rm Reg}$, and $a_{\rm Reg}^{\pi\pi}$ and $a_{\rm Reg}^{RR}$ are the contribu-

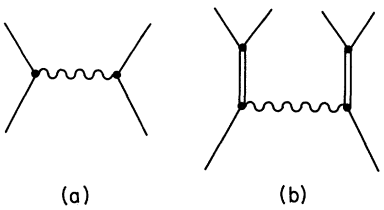

FIG. 2. (a) Diagram giving $T_{\rm Reg}$ for $\pi\pi$ scattering. (b) Quasi-two-body production amplitude for $\pi\pi\to 4\pi$.

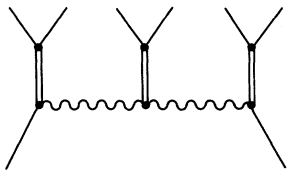

FIG. 3. A Regge generalization of the ABFST production amplitude for $\pi\pi\to 6\pi$.

tions to the absorptive part of Figs. 1(b) and 4, respectively. In what follows, however, we will again restrict ourselves to $n=0$ and $t=0$, in which case $a^{\pi\pi}$ is again given by Eq. (4). The quantities s_0 and s_1 could be chosen in the same way as in Sec. II, but since resonances do not come into Eq. (13) unless we make further approximations, it may be more appropriate to take them to lie at, or halfway between, channel thresholds. By a happy coincidence it turns out that most of the thresholds we will encounter happen to lie at or midway between resonances anyway.

Perhaps the simplest application of Eq. (12) would be to the $I_s=2$ state. Since there are no resonances in this state, our duality should be valid at much lower energies. In fact, if we take $s_1=4m_\rho^2$ we do not have to worry about the RR intermediate state at all. This leaves us with $a^{\pi\pi}_{\rm Reg}$, which can again be evaluated from Eq. (4) using the fact that, even at fairly low energies, we have

$$T^{I_s=2}_{\rm Reg}(s,t)\simeq F(t,u) \tag{13}$$

in the Lovelace-Veneziano model. If we now assume, as usual, that only the P trajectory contributes to the left-hand side of Eq. (12) in this exotic state we obtain the simple sum rule

$$\frac{b_P(0)}{\alpha_P(0)+1}\left[\nu_1{}^{\alpha_P(0)+1}-\nu_0{}^{\alpha_P(0)+1}\right] = \int_{s_0}^{s_1} ds\,\frac{1}{2q\sqrt{s}}\int_{-s+4}^{0} dt|F(t,u)|^2 . \tag{14}$$

With a P intercept of $\alpha_P(0)=1$, this immediately gives us the P residue $b_P(0)$ and hence the total high-energy cross section $\sigma_{\rm tot}$ (via the optical theorem). Thus

$$\sigma_{\rm tot}=36.6\bar\beta^2 \text{ with } (s_0,s_1)=(0,4m_\rho^2) \tag{15a}$$

$$=32.2\bar\beta^2 \text{ with } (s_0,s_1)=(2m_\rho^2,4m_\rho^2). \tag{15b}$$

We see that the result is not too sensitive to the specific choice of the interval chosen.

The value of $\bar\beta$ can be fixed by requiring the Lovelace-Veneziano model to give a ρ resonance with the experimental width of 125 MeV.[13] This gives $\bar\beta=0.6$ and so we have

$$\sigma_{\rm tot}=13.2 \text{ mb with } (s_0,s_1)=(0,4m_\rho^2)$$

$$=11.6 \text{ mb with } (s_0,s_1)=(2m_\rho^2,4m_\rho^2).$$

The corresponding "experimental" value, as obtained from factorization and the πp and pp total cross sections, is 15 mb.

Apart from our use of the Lovelace-Veneziano model, the main assumption which has gone into the above calculation is that an exotic cross section can be well approximated on the average by Pomeranchukon exchange even in the elastic region. To get some idea of the accuracy involved in making such an approximation, we could look at another exotic process, such as K^+p and pp scattering, for which a considerable amount of data already exists. Thus, if we compare the high-energy total cross section with the *average* cross section in the elastic region, we see that the difference is of the order of 30% (see, for example, pages 54 and 59 of the particle data booklet accompanying Ref. 13). Although the situation might be slightly better for $\pi\pi$ scattering, where the region of elastic dominance (in the s variable) is about twice as large as it is for K^+p and pp scattering, it would clearly be necessary to take a larger energy interval before we can expect any real improvement over this sort of accuracy.

IV. RELATION BETWEEN THE $I_t=0$ AND $I_t=1$ ABSORPTIVE PARTS AT INTERMEDIATE ENERGIES

In general it is more convenient to consider states of definite I_t rather than I_s, since fewer Regge poles then come into Eq. (12). If we use Figs. 1(b) and 4, as discussed in the preceding section, and assume semilocal linear duality, we can also obtain a simple relation between the $I_t=0$ and $I_t=1$ absorptive parts. To see how this arises let us first consider the special case where the horizontal exchanges in Fig. 4 are elementary pions. In that case, $a^{RR}_{\rm Reg}$ is related to the product of two t-channel $\pi\pi\to\pi\pi$ amplitudes T (see Fig. 5), each of which has the general Lovelace-Veneziano

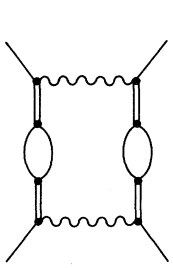

FIG. 4. Unitarity diagram giving $a^{RR}_{\rm Reg}$.

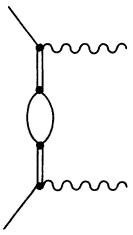

FIG. 5. Particle-Regge "scattering amplitude" which can be used to construct Fig. 4.

isospin structure[11]

$$T^{I_t=0}=\tfrac{3}{2}[F(t,s)+F(t,u)]-\tfrac{1}{2}F(s,u), \tag{16}$$

$$T^{I_t=1}=F(t,s)-F(t,u), \tag{17}$$

$$T^{I_t=2}=F(s,u), \tag{18}$$

if we wish to satisfy crossing, Bose statistics, and exchange degeneracy. In the Lovelace-Veneziano model, F would be given by Eq. (6), although in general it could be any function with Mandelstam-type singularities.

If we do in fact assume that F is given by Eq. (6), we find that $T^{I_t=2}$ vanishes for $t=0$ and continues to be small for a wide range of t around this point. We can thus drop the $F(s,u)$ term in Eq. (16). The s-absorptive parts of T then satisfy the relation

$$\mathrm{Im}\,T^{I_t=0}\simeq\tfrac{3}{2}\mathrm{Im}\,T^{I_t=1}. \tag{19}$$

If we now use the fact that a_{Reg}^{RR} can be calculated from the absorptive part of T alone,[14] we obtain from Figs. 4 and 5

$$[a_{\mathrm{Reg}}^{RR}]_{I_t=0}=(\tfrac{3}{2})^2[a_{\mathrm{Reg}}^{RR}]_{I_t=1}. \tag{20}$$

At first sight, it might appear that this result depends critically on the specific form of F given by Eq. (6) since this is what we used to show that $\mathrm{Im}\,T^{I_s=2}\simeq 0$. However the vanishing of $\mathrm{Im}\,T^{I_s=2}$ follows more generally if we require Eq. (1) to hold semilocally. Since the $I_s=2$ state is exotic, so that $A_{\mathrm{Reg}}=0$, this requires that all resonances must cancel each other out, at least when we average over all those having the same mass.

Equation (20) can be expected to continue to hold even if the horizontal lines in Figs. 4 and 5 are replaced by any $I=1$ system, since the "amplitude" in Fig. 5 can again be written in the form given by Eqs. (16)–(18) if we wish to satisfy crossing, Bose statistics, and exchange degeneracy, and since semilocal linear duality would once again lead to a negligible $F(s,u)$ term. By using Eq. (20) we can calculate certain results without ever having to make any detailed assumptions about the exchanges in Fig. 2(b).

To calculate $a_{\mathrm{Reg}}^{\pi\pi}$ we again use Eq. (4). For $I_s=1$ and $I_s=2$ we take Eqs. (9) and (13) for the integrand, as before. For $I_s=0$ the Lovelace-Veneziano model gives

$$T_{\mathrm{Reg}}^{I_s=0}(s,t)=\beta(t)e^{-i\pi\alpha(t)}(\tfrac{3}{2}-\tfrac{1}{2}e^{i\pi\alpha(t)})\nu^{\alpha(t)}+\beta(u)e^{-i\pi\alpha(u)}(\tfrac{3}{2}-\tfrac{1}{2}e^{i\pi\alpha(u)})\nu^{\alpha(u)}. \tag{21}$$

Once again, if we use the fact that the first and second terms in Eqs. (8) and (21) dominate only near the forward and backward directions, respectively, and so have little interference between them, we obtain

$$|T_{\mathrm{Reg}}^{I_s=0}(s,t)|^2\simeq\tfrac{5}{2}|F(t,u)|^2, \tag{22}$$

where we have also made the additional approximation of replacing the small $\frac{1}{2}e^{i\pi\alpha(t)}$ and $\frac{1}{2}e^{i\pi\alpha(u)}$ terms in Eq. (21) by their values at $t=0$ and $u=0$, respectively – a somewhat stronger version of forward and backward dominance. If we now apply crossing we obtain

$$[a_{\mathrm{Reg}}^{\pi\pi}(s,0)]_{I_t=1}\simeq 7[a_{\mathrm{Reg}}^{\pi\pi}(s,0)]_{I_t=0}\simeq\frac{7}{2}\frac{1}{2q\sqrt{s}}\int_{-4q^2}^{0}dt|F(t,u)|^2. \tag{23}$$

Again, in practice, Eq. (23) was only used to evaluate $a_{\mathrm{Reg}}^{\pi\pi}$ at $s=s_0$. Equation (10) was then used to evaluate it elsewhere.

V. CALCULATION OF $b_P(0)$ AND $\bar{\beta}$ USING A SEMILOCAL VERSION OF TYPE-B DUALITY

We will now apply Eq. (12) in the region of the g resonance, using the relations of the preceding section. In the $I_t=0$ and $I_t=1$ states, we have

$$\int_{s_0}^{s_1}[b_P(0)\nu^{\alpha_P(0)}+b_f(0)\nu^{\alpha(0)}]ds=\int_{s_0}^{s_1}[a_{\mathrm{Reg}}^{\pi\pi}(s,0)+a_{\mathrm{Reg}}^{RR}(s,0)]_{I_t=0}ds \tag{24}$$

and

$$\int_{s_0}^{s_1}b_\rho(0)\nu^{\alpha(0)}\,ds=\int_{s_0}^{s_1}[a_{\mathrm{Reg}}^{\pi\pi}(s,0)+a_{\mathrm{Reg}}^{RR}(s,0)]_{I_t=0}ds. \tag{25}$$

As in Sec. II we will take $(s_0,s_1)=(4m_\rho^2,6m_\rho^2)$. The latter value happens to coincide with the $N\bar{N}$ threshold and means that we do not have to worry about the possible importance of the $N\bar{N}$ channel in our calculation. The values of b_ρ and b_f could be extracted directly from the Lovelace-Veneziano model. We have found it more convenient, however, to use the fact that Eq. (1) is a good approximation for this model, so that we can write

$$\int_{s_0}^{s_1}b_\rho(t)\nu^{\alpha(t)}ds=\int_{s_0}^{s_1}\mathrm{Im}F(t,s)ds \tag{26}$$

and

$$\int_{s_0}^{s_1}b_f(t)\nu^{\alpha(t)}ds\simeq\int_{s_0}^{s_1}\tfrac{3}{2}\mathrm{Im}F(t,s)ds, \tag{27}$$

where we have also used Eqs. (16) and (17). From Eq. (6), $\mathrm{Im}F$ is a sum of δ functions in the s variable.

Since the $a_{\mathrm{Reg}}^{\pi\pi}$ are given by Eqs. (10) and (23), the remaining unknowns in Eqs. (24) and (25) are $b_P(0)$ and a_{Reg}^{RR}. Now in a specific model, such as the one-pion-exchange model of Ref. 3, the latter quantity could be calculated explicitly. Since we

are trying to be as model-independent as possible, we will, instead, simply eliminate it between Eqs. (24) and (25) by using the general result (20). This then gives us the sum rule

$$\int_{s_0}^{s_1} b_P(0)\nu^{\alpha_P(0)}ds = (\tfrac{3}{2})^2\int_{s_0}^{s_1}[b_\rho(0)\nu^{\alpha(0)} - a^{\pi\pi}_{\mathrm{Reg}}(s,0)]_{I_t=1}ds$$

$$-\int_{s_0}^{s_1}[b_f(0)\nu^{\alpha(0)} - a^{\pi\pi}_{\mathrm{Reg}}(s,0)]_{I_t=0}ds\,. \tag{28}$$

If we now take $\alpha_P(0)=1$ and use Eqs. (10), (23), (26), and (27) we are led immediately to a simple expression for $b_P(0)$ in terms of $\bar{\beta}$. Relating this to σ_{tot} through the optical theorem we finally obtain

$$\sigma_{\mathrm{tot}} = (10.1\bar{\beta} + 17.3\bar{\beta}^2)\ \mathrm{mb}\,. \tag{29}$$

The actual magnitudes of σ_{tot} and $\bar{\beta}$ can be calculated by combining Eq. (29) with Eq. (15). With (15a) we obtain $\sigma_{\mathrm{tot}}=10.3$ mb and $\bar{\beta}=0.53$, whereas with (15b) we have $\sigma_{\mathrm{tot}}=13.5$ mb and $\bar{\beta}=0.64$. Experimentally, the πp and pp total cross sections, together with factorization, give $\sigma_{\mathrm{tot}}\simeq 15$ mb, while a ρ width[13] of 125 MeV corresponds to $\bar{\beta}=0.6$ in the Lovelace-Veneziano model.

Finally we can combine the above results with the g-meson $\pi\pi$ elasticity $r=0.57\bar{\beta}$ obtained from Type-A duality in Sec. II. Thus, with $\bar{\beta}=0.53$ we have $r=0.30$ while $\bar{\beta}=0.64$ gives $r=0.36$.

VI. AN ALTERNATIVE CALCULATION OF THE $I_t=0$ FORWARD REGGE RESIDUES

In the calculations of the preceding sections our duality averages were taken over fairly small intervals. In addition, rather strong versions of exchange degeneracy and the Freund-Harari hypothesis were used in inferring the properties of the lower-lying trajectories. Although this leads to simplifications in the calculations, it might be desirable to see what happens if we weaken or modify some of these procedures. We shall see that this can lead to a certain improvement in some of our results.

One way of extending our duality interval is to use a conventional finite-energy sum rule (FESR)[15,5]

$$\int_u^{s_1} ds\sum_i b_i(t)\nu^{\alpha_i(t)+n} = \int_u^{s_1} ds\,A(s,t)\nu^n, \tag{30}$$

where the lower limit corresponds to $\nu=0$. This sum rule only assumes average Regge behavior for $s>s_1$. Now since nonlinear duality may not be valid at very low energies,[10] we probably should not set $A\approx a^{\pi\pi}_{\mathrm{Reg}}$ for $s<4m_\rho{}^2$. Instead, we shall take

$$A(s,t) = A_{\mathrm{res}}(s,t), \quad s<s_0 \tag{31a}$$

$$= \sum_c a^c_{\mathrm{Reg}}(s,t), \quad s>s_0, \tag{31b}$$

with $s_0=4m_\rho{}^2$. Equations (30) and (31) can therefore be thought of as a hybrid of linear and nonlinear duality, and arise naturally if we use an FESR technique to solve a multiperipheral integral equation.[2,3] Strictly speaking, we should also include a background term in Eq. (31a). Since this is a low-energy interval, however, we shall neglect it as a first approximation. For the upper limit in Eq. (30) we shall take $s_1=9m_\rho{}^2$, the highest energy at which we can still neglect the effect of Fig. 3.

If we write out Eqs. (30) and (31) for $I_t=0$ and $I_t=1$ and again use Eq. (20) to eliminate the contribution of a^{RR}_{Reg} we obtain the sum rule

$$\left[\int_u^{s_1} ds\sum_i b_i(t)\nu^{\alpha_i(t)+n}\right]_{I_t=0} = \tfrac{3}{2}\int_u^{s_0} ds\,\mathrm{Im}F(t,s)\nu^n + \int_{s_0}^{s_1} ds\{\tfrac{9}{4}\,\mathrm{Im}F(t,s) + [a^{\pi\pi}_{\mathrm{Reg}}(s,t)]_{I_t=0} - \tfrac{9}{4}[a^{\pi\pi}_{\mathrm{Reg}}(s,t)]_{I_t=1}\}\nu^n, \tag{32}$$

where we have taken $A_{\mathrm{res}}=\mathrm{Im}T$, with T given by Eqs. (16) and (17). We have also used Eq. (26) but with s_0 replaced by u, so that this $I_t=1$ linear duality average is taken over a wider interval than before. So far, we have not had to assume exchange degeneracy or the Freund-Harari hypothesis. Instead of doing so, we will consider two different models for the left-hand side of Eq. (32). The first is to assume that it can be represented by a single effective trajectory, while the second is that it is dominated by a P with $\alpha_P(0)=1$ and a P' or f with $\alpha_f(0)=\frac{1}{2}$, but not necessarily with $b_f=\frac{3}{2}b_\rho$, as would be required by the strong form of exchange degeneracy. The two remaining trajectory parameters in each case will be determined by applying Eq. (32) with $n=0$ and $n=1$.

If we have only one effective output trajectory, Eq. (32) can be used to calculate $b(0)$ and $\alpha(0)$ for that trajectory. Using Eqs. (10), (23), and (6), with $\bar{\beta}=0.6$ (see Sec. V), we obtain an intercept $\alpha(0)=0.81$. This, not surprisingly, lies between the intercepts for the physical f and P trajectories

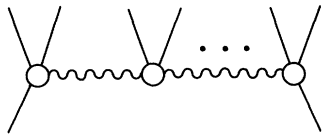

FIG. 6. A Regge generalization of a general ABFST production amplitude.

and must be some kind of average between them. Another quantity we can easily calculate in the one-trajectory case is the multiplicity $\langle N\rangle$ for pion production at high energies, *if* we assume that it is given by the multiperipheral graphs of Fig. 6. We then have

$$\langle N\rangle = 2g\,\frac{\partial\alpha(0)}{\partial g}\,\ln s, \tag{33}$$

where $\sqrt{g}$ is a strength parameter which we associate with each of the blobs in these diagrams. This means that we must associate g with $A_{\rm res}$, $\sqrt{g}$ with Fig. 2(a), and g with Fig. 2(b). Equations (30)–(33) then yield $\langle N\rangle = 1.02\ln s$. The experimental value is $\langle N\rangle = (1.05\pm 0.15)\ln s$.[16]

If we next put in the actual P and f into the left-hand side of Eq. (32) we obtain $b_P(0) = 0.0064\pi$, which corresponds to $\sigma_{\rm tot} = 13.7$ mb, and $b_f(0) = 0.039\pi$. The latter value is somewhat smaller than the one given by the Lovelace-Veneziano model, which gives $b_f(0) = 0.066\pi$. On the other hand, it agrees somewhat better with experiment. If we calculate the ratio

$$R_{\pi\pi} = [b_P(0)/b_f(0)]\Lambda^{\alpha_P(0)-\alpha_f(0)}, \tag{34}$$

for example, where $\Lambda = 1$ GeV^2, we obtain $R_{\pi\pi} \simeq 1.2$, whereas the Lovelace-Veneziano prediction with the same $b_P(0)$ would be $R_{\pi\pi}\simeq 0.7$. Now factorization gives $R_{\pi\pi} = R_{\pi p}{}^2/R_{pp}$, where $R_{\pi p}$ and R_{pp} are the ratio (34) for πp and pp scattering. Moreover there is a large class of models which would give $R_{\pi\pi} = R_{\pi p} = R_{pp}$.[17] Experimentally,[18] we have $R_{\pi p} = 1.4$ and $R_{pp}\approx 1.1$, which agrees fairly well with our calculated value of $R_{\pi\pi} = 1.2$.

VII. NEGATIVE-MOMENT SUM RULES

Instead of using positive-moment FESR's, such as those given by Eq. (30), we could also use negative-moment sum rules.[19] The simplest such sum rule for $I_t = 0$ is obtained by simply evaluating an ordinary fixed-t dispersion relation at $\nu = 0$. This gives

$$\int_u^{s_1} ds\left[A(s,t) - \sum_i b_i(t)\nu^{\alpha(t)}\right]\nu^{-1} = [T(s,t)]_{\nu=0}. \tag{35}$$

For $I_t = 1$ one can also write an equation of this form, but then T is no longer the true amplitude, but rather the partial-wave sum of the even-J unphysical "amplitudes" obtained by making a Froissart-Gribov continuation from the odd-J amplitudes. Higher-moment sum rules can be obtained by again taking a fixed-t dispersion relation and evaluating the derivatives of T at $\nu = 0$. But in all such relations we have to know something about T as well as the absorptive part.

If we write out Eqs. (35) and (31) for $I_t = 0$ and $I_t = 1$ and once again use Eq. (20) to eliminate the contribution of $a_{\rm Reg}^{RR}$, we obtain a negative-moment version of Eq. (32):

$$\left[\int_u^{s_1} ds\sum_i b_i(0)\nu^{\alpha i(0)-1}\right]_{I_t=0} = -\tfrac{1}{2}\pi[T(s,0)]_{\nu=0}^{I_t=0} + \tfrac{3}{2}\int_u^{s_0} ds\,{\rm Im}F(0,s)\nu^{-1} + \int_{s_0}^{s_1} ds\left\{\tfrac{9}{4}\,{\rm Im}F(0,s) + [a_{\rm Reg}^{\pi\pi}(s,0)]_{I_t=0} - \tfrac{9}{4}[a_{\rm Reg}^{\pi\pi}(s,0)]_{I_t=1}\right\}\nu^{-1}, \tag{36}$$

where again, we have taken $A_{\rm res} = {\rm Im}T$, with T given by Eqs. (16) and (17). We have also used the fact that the Lovelace-Veneziano model itself satisfies negative-moment relations like (35); in the $I_t = 1$ state, which does not have any Pomeranchuk complications, we can thus write

$$\int_u^{s_1} ds[{\rm Im}F(s,t) - b_\rho(t)\nu^{\alpha(t)}]\nu^{-1} = [T(s,t)]_{\nu=0}^{I_t=1}, \tag{37}$$

where we have used Eq. (17). To evaluate T on the left-hand side of Eq. (36) we shall use PCAC. This predicts that

$$[T(s,0)]_{\nu=0}^{I_t=0} \simeq 0 \tag{38}$$

if we treat the pion mass as being negligible.[11]

Let us first check to see how well the two-trajectory solution obtained in Sec. VI satisfies Eq. (36). With this solution we obtain a value of 3.02π for the left-hand side and a value of 2.90π for the right-hand side, so that the sum rule is satisfied fairly well. Alternatively, we could require it to be satisfied exactly, along with the $n = 0$ version of Eq. (32). This then gives $b_f(0) = 0.029\pi$ and $b_P(0) = 0.0073\pi$, which corresponds to $\sigma_{\rm tot} = 15.6$ mb and $R_{\pi\pi} = 1.8$. These values are at least roughly the same as those obtained in the previous section.

Actually it is not even necessary to use PCAC at all to obtain Eq. (38), if we take $\alpha_0 \simeq 0.5$, a value which follows either from experiment or from an assumption of exact semilocal linear duality, as

we have seen in Sec. II. The $I_t=2$ amplitude, which does not have any Pomeranchuk contribution, must then have a zero at $\nu\simeq 0$, $t=0$, as can be seen from Eq. (18). But crossing symmetry requires that

$$T^{I_t=0}=\tfrac{5}{2}T^{I_t=2} \tag{39}$$

at $s=t=u$.[20] Since this point is essentially the same as $\nu=0$, $t=0$, at least if we treat the square of the pion mass as being negligible on the scale set by α'^{-1}, we conclude that $T^{I_t=0}$ must also have a zero at this point.

VIII. POSSIBLE IMPROVEMENTS AND EXTENSIONS

The most obvious way of improving our results would be to go to higher energies. This is particularly true of Type-B duality, which, as we have seen, does not use a resonance description at all, and involves approximating one relatively smooth function by another. Of course, such a duality is always likely to work best at energies where the resonances have already smoothed out. Even at lower energies it may work better if the partial waves containing resonances are explicitly subtracted out. However, the calculations of Sec. V and VI both gave reasonable results, even though Eq. (2b) was applied in a region where resonances are still important. Moreover, these results did not differ too much from each other, even though the energy intervals involved were quite different. This suggests that our approach should converge fairly rapidly as we go up in energy.

Although Type-A duality is probably the simplest way of calculating certain quantities, such as the total width of the g, it should perhaps be emphasized that all such quantities could also be calculated from the Type-B form. But since the latter only gives partial widths, we would then have to consider more than one process simultaneously. In the case of the g, for example, we would also have to consider unitary diagrams like Figs. 1(b) or 4 for the $\pi\pi\to RR$ process. It might be interesting to see how the resulting sum of partial widths compares with the total width obtained in Sec. II.

In the calculations of the present paper we have made extensive use of the Lovelace-Veneziano model. The main function of this model, however, was to provide a convenient framework for incorporating linear duality and exchange degeneracy for the $\pi\pi$ scattering amplitude. Any other model capable of doing this would be equally acceptable. Indeed, recent investigations of dip structure seem to show that the resonance distribution in a Veneziano-type model may not be sufficiently peripheral to be consistent with the data at intermediate energies.[21] It may therefore be desirable to repeat some of our calculations in the context of a peripheral-resonance model. Preliminary calculations seem to indicate that the resulting Pomeranchukon residue is not too different from the one obtained in Sec. V, at least in the forward direction.

Instead of considering two-body scattering amplitudes we could also look at particle-Regge (or even Regge-Regge amplitudes). This kind of "amplitude" is actually needed if we are to apply Eq. (2b) at higher energies, where diagrams such as Fig. 3 or Fig. 6 would come in. Recently a number of authors have written down finite-energy sum rules for Regge-particle "scattering,"[22] which arises naturally from the Mueller analysis of inclusive reactions.[23] As in ordinary scattering amplitudes, linear duality does not give us any information on Pomeranchukon residues in such amplitudes, but Type-B nonlinear duality presumably should.

ACKNOWLEDGMENT

The author would like to thank Professor P. G. O. Freund for several very helpful comments, and, in particular, for pointing out the possibility of testing the accuracy of some of our assumptions in the domain of low-energy K^+p and pp scattering.

APPENDIX: $I_t=0$ TRAJECTORY FOR $t>0$

Since Eq. (32) applies at any moderate value of t it might be interesting to see what it has to say about the $I_t=0$ trajectory in the physical region of the t channel. Now the main complication here comes from $a^{\pi\pi}_{\mathrm{Reg}}$, which is difficult to evaluate for $t\neq 0$. But this term comes from Fig. 1(b), which is an Amati-Fubini-Stanghellini-type diagram, with an asymptotic behavior of $\nu^{2\alpha(t/4)-1}$ to within logarithms. We might therefore expect its contribution to become progessively less important in

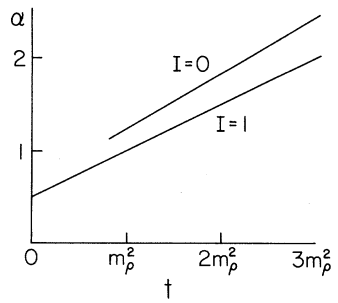

FIG. 7. Effective $I_t=0$ Regge trajectory calculated from Eq. (32) with $a^{\pi\pi}_{\mathrm{Reg}}=0$ in the one-trajectory case, and the Lovelace-Veneziano $I_t=1$ trajectory.

Eq. (32) as we increase t, at least relative to the other terms in the (s_0, s_1) interval, which have a $\nu^{\alpha_i(t)}$ behavior. Actually, even at $t=0$ the results are not affected all that much if we drop $a_{\mathrm{Reg}}^{\pi\pi}$. With a single effective trajectory on the left-hand side of Eq. (28), for example, the intercept is only shifted from 0.81 to 0.69. We will therefore simply set $a_{\mathrm{Reg}}^{\pi\pi}=0$.

For $t>0$ we again have the choice of choosing only one effective trajectory or trying to put in both the P and the f on the left-hand side of Eq. (32). However, in practically all models which are currently in favor, the P is relatively flat compared with the f. For sufficiently large t the latter will thus give the dominant contribution in Eq. (32).[24] For $t \gtrsim m_\rho^2$ we can therefore safely drop the contribution of P in this equation.

If we now impose Eq. (32) for $n=0$, 1 with F given by Eq. (6), we obtain the $I_t=0$ trajectory shown in Fig. 7. This is compared with the $I_t=1$ trajectory, which, since there is no Pomeranchuk complication, continues to be given by the Lovelace-Veneziano model. We see that exchange degeneracy is broken somewhat, although not very drastically. In particular the slopes of the trajectories are now somewhat different, although the inclusion of a finite $a_{\mathrm{Reg}}^{\pi\pi}$ is likely to flatten the $I_t=0$ curve somewhat and lead perhaps to better agreement with exchange degeneracy. The position of the f^0 resonance which we obtain is now at $s=2.3m_\rho^2$, which should be compared with the Lovelace-Veneziano value of $s\simeq 3.0m_\rho^2$ and the experimental value of $s\simeq 2.7m_\rho^2$. Perhaps it even represents some kind of average of the f^0 and the $\mu_N(1080)$ resonance,[13] which has been seen at $s\simeq 2.0m_\rho^2$.

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*Supported in part by the U. S. Atomic Energy Commission and by an Alfred P. Sloan Foundation Research Fellowship.

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