S matrix for quantum charged massive scalar particles on Schwarzschild black holes

N. E. Firsova
Institute for Problems of Mechanical Engineering, Russian Academy of Sciences, Saint Petersburg 199178, Russia
(Received 23 November 1998; published 21 May 1999)

We study the scattering problem arising when considering the contribution of the topologically inequivalent configurations of the massive complex scalar field on Schwarzschild black holes to Hawking radiation. [S0556-2821(99)06310-9]

PACS number(s): 04.20.Jb, 04.70.Dy, 14.80.Hv

Recently there arose an interest in studying topologically inequivalent configurations (TICs) of various fields on 4D black holes since TICs might give marked additional contributions to the quantum effects in 4D black hole physics, for instance, such as Hawking radiation [1–3].

As discussed in Refs. [1–3], TICs exist owing to the high nontriviality of the standard topology of 4D black hole spacetimes which is of the $\mathbb{R}^2\times\mathbb{S}^2$ form. The high nontriviality of the given topology consists in the fact that over it there exists a huge (countable) number of nontrivial real and complex vector bundles of any rank $N>1$ (for complex ones for $N=1$ too) and also a countable number of so-called Spin^c structures. Physically the appearance of TICs should be obliged to the natural presence of the whole family of Dirac monopoles on a black hole, and as a result of the interaction with them, one or another field splits into TICs.

As a result, there arises a nontrivial problem to take in theory into account the possibilities connected with the existence of TICs. The first step here is to study the contribution of TICs for a complex scalar field. This was the main features of what was done in Refs. [2,3] in the massless case for Schwarzschild and Reissner-Nordström metrics. The expression of luminosity regarding Hawking radiation for any TIC, however, contains an element of the S matrix connected with some potential barrier surrounding the black hole and which effectively arises for the quantum scalar particle leaving black hole; so one should solve some scattering problem on the whole axis with the given potential. But the potentials mentioned depend on both the type of black hole and the particle masses and are sufficiently complicated since they can be described only in an implicit form. Therefore, for the physical results to be obtained one needs to apply numerical methods. But before computing one should have information about the existence and qualitative behavior of the mentioned S matrix in a form convenient to computation.

In the present paper we consider the S matrix which arises if we study Hawking radiation for the TICs of a massive complex scalar field on Schwarzschild black holes. We write down the black hole metrics (using the ordinary set of local coordinates t,r,ϑ,φ) in the form

$$ds^2=a\,dt^2-a^{-1}dr^2-r^2(d\vartheta^2+\sin^2\vartheta\, d\varphi^2), \tag{1}$$

with $a=1-r_g/r$, $r_g=2M$, and M is a black hole mass.

Throughout the paper we employ the system of units with $\hbar=c=G=1$, unless explicitly stated otherwise. Finally, we shall denote by $L_2(F)$ the set of the modulo square integrable complex functions on any manifold F furnished with an integration measure.

As was discussed in [1] TICs of a complex scalar field on the chosen class of black holes are conditioned by the availability of a countable number of complex line bundles over the $\mathbb{R}^2\times\mathbb{S}^2$ topology underlying 4D black hole physics. Each TIC corresponds to sections of a complex line bundle E while the latter can be characterized by its Chern number $n\in\mathbb{Z}$ (set of integers). A TIC with $n=0$ can be called *untwisted*, while the rest of the TICs with $n\neq 0$ should be reffered to as *twisted*. Using the fact that all the line bundles mentioned can be trivilized over the chart of local coordinates (t,r,ϑ,φ) covering almost the whole manifold $\mathbb{R}^2\times\mathbb{S}^2$ one can obtain a suitable wave equation on the given chart for TIC ϕ with mass μ_0 and Chern number $n\in\mathbb{Z}$ in the form

$$(\Box_n+\mu_0^2)\phi=0, \tag{2}$$

where the form of the conforming wave operator $\Box_n$ depends on the gauge choice for connection A_μ (vector potential for the corresponding Dirac monopole) in the line bundle with Chern number n. We shall consider here two gauges which are most interesting from the physical point of view (see below).

As was shown in [1], Eq. (2) has in $L_2(\mathbb{R}^2\times\mathbb{S}^2)$ a complete set of solutions of the form

$$f_{\omega ml}=(1/r)\,e^{i\omega t}e^{im\varphi}Y_{nlm}(\vartheta,\varphi)R_{\omega l}(r),$$
$$l=|n|,|n|+1,\ldots,\quad |m|\leqslant l, \tag{3}$$

where the explicit form of the *monopole (spherical) harmonics* $Y_{nlm}(\vartheta,\varphi)$ can be found in [1–3].

As to the functions $R_{\omega l}(r)$, denoting $k=r_g\omega$, $y(x)=r/r_g$ and introducing the functions $\psi(x,k,l)=R_{\omega(k)l}(r(x))$, where $y(x)$ is a function reverse to $x(y)=y+\ln(y-1)$ (i.e., $-\infty<x<\infty$, $1\leqslant y<\infty$), we shall obtain that $\psi(x,k,l)$ obeys the Schrödinger-like equation

$$[d^2/dx^2+(k^2-\mu^2)]\psi(x,k,l)=q(x,l)\psi(x,k,l), \tag{4}$$

$$q(x,l)=-\frac{\mu^2}{y(x)}+\left(1-\frac{1}{y(x)}\right)\left[\frac{N}{y^2(x)}+\frac{1}{y^3(x)}\right], \tag{5}$$

where $\mu=\mu_0 r_g$ and $N=l(l+1)$ or $N=l(l+1)-n^2$ (in both cases $l=|n|,|n|+1,\ldots$) in dependence on the gauge choice.

Under the circumstances the luminosity $L(n)$ with respect to the Hawking radiation for TICs of massive complex scalar fields with Chern number n and mass μ_0 is (in usual units)

$$L(n)=A\sum_{l=|n|}^{\infty}(2l+1)c_l,\quad c_l=\int_{\mu/\mu_{pl}^2}^{\infty}\frac{\Gamma(k,l)kdk}{e^{8\pi k}-1},\tag{6}$$

where

$$A=\frac{1}{2\pi\hbar}\left(\frac{\hbar c^3}{GM}\right)^2\approx 0.273\,673\times 10^{50}\ \text{erg s}^{-1}M^{-2}$$

(M in g), while $\mu_{pl}^2=c\hbar/g$ is the Planck mass square.

To find the barrier transparency

$$\Gamma(k,l)=|s_{11}(k,l)|^2,\tag{7}$$

where s_{11} is a transition coefficient for Eq. (4), we should consider the corresponding scattering problem on the whole x axis. As will be seen below, the potential (5) at $\mu_0\neq 0$ is not integrable as $x\to+\infty$ and we cannot calculate s_{11} according to the receipt for the $\mu_0=0$ case [2]. As a result, the mentioned scattering problem for Eq. (4) should be regularized. This regularization and calculation of $\Gamma(k,l)$ is the subject of the present paper.

It is clear that for the total luminosity L of a black hole with respect to the Hawking radiation concerning the complex scalar field with a mass μ_0 to be obtained one should sum up over all n:

$$L=\sum_{n\in\mathbb{Z}}L(n)=L(0)+2\sum_{n=1}^{\infty}L(n),\tag{8}$$

since $L(-n)=L(n)$.

It is not complicated to check that as $x\to-\infty$ (i.e., $r\to r_g$ or $y\to 1$) Eqs. (4),(5) have the form

$$\left(\frac{d^2}{dx^2}+k^2\right)\psi(x,k,l)=q^-(x,l)\psi(x,k,l),\tag{9}$$

$$q^-(x,l)=\left[1-\frac{1}{y(x)}\right]\left[\mu^2+\frac{N}{y^2(x)}+\frac{1}{y^3(x)}\right].\tag{10}$$

Hence we see [2] that $q^-(x,l)_{x\to-\infty}=O(e^x)$ and

$$\int_{-\infty}^{b}|q^-(x,l)|dx<C(b),\tag{11}$$

where $C(b)$ is a constant depending on b.

As $x\to+\infty$, we can rewrite Eqs. (4),(5) in the form

$$\left[\frac{d^2}{dx^2}+\left(k^2-\mu^2+\frac{\mu^2}{x}\right)\right]\psi(x,k,l)=q^+(x,l)\psi(x,k,l),\tag{12}$$

$$q^+(x,l)=\mu^2\left[\frac{1}{x}-\frac{1}{y(x)}\right]+\left[1-\frac{1}{y(x)}\right]\left[\frac{N}{y^2(x)}+\frac{1}{y^3(x)}\right].\tag{13}$$

Under this situation

$$\int_{b}^{+\infty}|q^+(x,l)|dx<C(b).\tag{14}$$

To consider the scattering problem at $x\to+\infty$, we should first study the homogenous equation for Eq. (12):

$$\left[\frac{d^2}{dx^2}+\left(k^2-\mu^2+\frac{\mu^2}{x}\right)\right]\psi_0^+(x,k)=0.\tag{15}$$

Transforming it we obtain, for

$$u(z,k)=\psi_0^+\left(\frac{z}{2ik^+(k)}\right),\quad k^+(k)=\sqrt{k^2-\mu^2},\tag{16}$$

the Whittaker equation [4]

$$u''_{zz}(z,k)+\left[-\frac{1}{4}-\frac{i\tilde{\mu}}{z}\right]u(z,k)=0,\quad \tilde{\mu}=\frac{\mu^2}{2k^+}=\frac{\mu^2}{2\sqrt{k^2-\mu^2}}.\tag{17}$$

A couple of the Whittaker functions $W_{i\tilde{\mu},1/2}(-z)$, $W_{-i\tilde{\mu},1/2}(z)$ forms a fundamental system of solutions for Eq. (17) because they have asymptotic behavior as follows [4]:

$$W_{i\tilde{\mu},1/2}(-z)=e^{z/2}(-z)^{i\tilde{\mu}}[1+O(z^{-1})],$$
$$|\arg(-z)|<\pi,\quad z\to\infty,\tag{18}$$

$$W_{-i\tilde{\mu},1/2}(z)=e^{-z/2}z^{-i\tilde{\mu}}[1+O(z^{-1})],$$
$$|\arg(-z)|<\pi,\quad z\to\infty.\tag{19}$$

Hence the general solution of Eq. (17) is

$$u(z,k)=C_1^+W_{i\tilde{\mu},1/2}(-z)+C_2^+W_{-i\tilde{\mu},1/2}(z).$$

It means that the general solution of Eq. (15) has the form

$$\psi_0^+(x,k)=C_1^+W_{i\tilde{\mu},1/2}(-2ik^+x)+C_2^+W_{i\tilde{\mu},1/2}(2ik^+x)\tag{20}$$

and

$$W_{i\tilde{\mu},1/2}(-2ik^+x)=e^{ik^+x}|2k^+x|^{i\tilde{\mu}}e^{\pi\tilde{\mu}/2}\times\left[1+O\left(\frac{1}{|k^+x|}\right)\right],\quad x\to\infty,\tag{21}$$

$$W_{-i\tilde{\mu},1/2}(2ik^+x)=e^{-ik^+x}\frac{1}{|2k^+x|^{i\tilde{\mu}}}e^{\pi\tilde{\mu}/2}\times\left[1+O\left(\frac{1}{|k^+x|}\right)\right],\quad x\to\infty.\tag{22}$$

We also introduce the functions

$$w_{\pm i\tilde{\mu},1/2}(\pm z)=W_{\pm i\tilde{\mu},1/2}(\pm z)e^{-\pi\tilde{\mu}/2},$$

which are more convenient. They form the fundamental system of solutions of Eq. (15) as well. We may write the general solution of Eq. (15) as follows:

$$\psi_0^+(x,k)=c_1^+w_{i\tilde{\mu},1/2}(-2ik^+x)+c_2^+w_{-i\tilde{\mu},1/2}(2ik^+x)\tag{23}$$

and

$$w_{i\tilde{\mu},1/2}(-2ik^+x)=e^{ik^+x}e^{i\tilde{\mu}\ln|2k^+x|}[1+O(|k^+x|^{-1})],\quad x\to\infty, \tag{24}$$

$$w_{-i\tilde{\mu},1/2}(2ik^+x)=e^{-ik^+x}e^{-i\tilde{\mu}\ln|2k^+x|}[1+O(|k^+x|^{-1})],\quad x\to\infty. \tag{25}$$

The functions $w_{-i\tilde{\mu},1/2}(z)$, $w_{i\tilde{\mu},1/2}(-z)$ are complex conjugate, i.e., $\overline{w_{i\tilde{\mu},1/2}(-2ik^+x)}=w_{-i\tilde{\mu},1/2}(2ik^+x)$, and their Wronskian is equal to

$$[w_{-i\tilde{\mu},1/2}(2ik^+x),w_{i\tilde{\mu},1/2}(-2ik^+x)]=2ik^+. \tag{26}$$

We denote $\psi^-(x,k,l)$ the Jost-type solution of Eq. (9) obeying the condition

$$\psi^-(x,k,l)=e^{-ikx}+o(1),\quad x\to-\infty. \tag{27}$$

Varying the constants according to the Lagrange method, as usual, we obtain the integral equation equivalent to the problem (9), (27):

$$\psi^-(x,k,l)=e^{-ikx}+\frac{1}{k}\int_{-\infty}^{x}\sin[k(x-t)]\times q^-(t,k,l)\psi^-(t,k,l)dt. \tag{28}$$

The convergence of the series of approximations for this equation follows from Eq. (11) as usual (see, e.g., [5]).

Differentiating Eq. (28) we obtain an expression for the derivative:

$$(\psi^-)'_x(x,k,l)=-ike^{-ikx}+\int_{-\infty}^{x}\cos[k(x-t)]\times q^-(t,k,l)\psi^-(t,k,l)dt.$$

A couple of functions $\psi^-(x,k,l)$, $\overline{\psi^-(x,k,l)}$ is the fundamental system of solutions for Eq. (9) and

$$[\psi^-(x,k,l),\overline{\psi^-(x,k,l)}]=2ik. \tag{29}$$

We denote $\psi^+(x,k^+,l)$ the Jost-type solution of Eq. (12) obeying the condition

$$\psi^+(x,k^+,l)=w_{i\tilde{\mu},1/2}(-2ik^+x)+o(1),\quad x\to+\infty. \tag{30}$$

Varying the constants in Eqs. (12), (23) according to the Lagrange method, we obtain the integral equation equivalent to the problem (12), (30):

$$\begin{aligned}\psi^+(x,k^+,l)=&\,w^-_{i\tilde{\mu},1/2}(-2ik^+x)\\&+\frac{1}{2ik^+}\int_{x}^{+\infty}[w_{-i\tilde{\mu},1/2}(2ik^+t)w_{i\tilde{\mu},1/2}(-2ik^+x)\\&-w_{i\tilde{\mu},1/2}(-2ik^+t)w_{-i\tilde{\mu},1/2}(2ik^+x)]\\&\times q^+(t,l)\psi^+(t,k^+,l)dt.\end{aligned} \tag{31}$$

The existence of the solution of this equation follows from Eq. (14) as usual. Differentiating Eq. (31) we obtain an expression for the derivative:

$$\begin{aligned}(\psi^+)'_x(x,k^+,l)=&-2ik^+w'_{i\tilde{\mu},1/2}(-2ik^+x)\\&-\int_{x}^{+\infty}[w_{-i\tilde{\mu},1/2}(2ik^+t)w'_{i\tilde{\mu},1/2}(2ik^+x)\\&+w_{i\tilde{\mu},1/2}(-2ik^+t)w'_{-i\tilde{\mu},1/2}(2ik^+x)]\\&\times q^+(t,l)\psi^+(t,k^+,l)dt.\end{aligned}$$

The pair of functions $\psi^+(x,k^+,l)$, $\overline{\psi^+(x,k^+,l)}$ is the fundamental system of solutions for Eq. (12) and

$$[\psi^+(x,k^+,l),\overline{\psi^+(x,k^+,l)}]=-2ik^+. \tag{32}$$

We write now the decomposition of the solution $\psi^+(x,k^+,l)$ to the fundamental system $\psi^-(x,k,l)$, $\overline{\psi^-(x,k,l)}$ and the solution $\psi^-(x,k,l)$ to the pair $\psi^+(x,k^+,l)$, $\overline{\psi^+(x,k^+,l)}$. We obtain

$$\begin{aligned}\psi^-(x,k,l)&=c_{11}(k,l)\psi^+(x,k^+,l)+c_{12}(k,l)\overline{\psi^+(x,k^+,l)},\\ \psi^+(x,k^+,l)&=c_{21}(k,l)\overline{\psi^-(x,k,l)}+c_{22}(k,l)\psi^-(x,k,l).\end{aligned} \tag{33}$$

We call the matrix $\mathbf{C}=\{c_{ij}\}$ *the transition matrix* for Eq. (4). Let us explore its elements. From Eqs. (29), (32), (33) it follows that

$$\begin{aligned}c_{11}(k,l)&=\frac{1}{2ik^+}[\overline{\psi^+(x,k^+,l)},\psi^-(x,k,l)],\\ c_{12}(k,l)&=\frac{1}{2ik^+}[\psi^-(x,k,l),\psi^+(x,k^+,l)],\\ c_{21}(k,l)&=\frac{1}{2ik}[\psi^-(x,k,l),\psi^+(x,k^+,l)],\\ c_{22}(k,l)&=\frac{1}{2ik}[\psi_+(x,k^+,l),\overline{\psi^-(x,k,l)}].\end{aligned} \tag{34}$$

Let us investigate the properties of the elements c_{ij}. For this purpose we put Eqs. (33) one into another. Then we get

$$\begin{aligned}&c_{11}(k,l)c_{22}(k,l)+c_{12}(k,l)\overline{c_{21}(k,l)}=1,\\ &c_{22}(k,l)c_{11}(k,l)+c_{21}(k,l)\overline{c_{12}(k,l)}=1,\\ &c_{11}(k,l)c_{21}(k,l)+c_{12}(k,l)c_{22}(k,l)=0,\\ &c_{21}(k,l)\overline{c_{11}(k,l)}+c_{22}(k,l)c_{12}(k,l)=0.\end{aligned} \tag{35}$$

It follows from Eqs. (34) that

$$\gamma(k)c_{11}(k,l)=-\overline{c_{22}(k,l)},\quad \gamma(k)c_{12}(k,l)=c_{21}(k,l), \tag{36}$$

where $\gamma(k)=k^+/k$. From Eqs. (35), (36) we also get

$$\begin{aligned}\gamma(k)[|c_{12}(k,l)|^2-|c_{11}(k,l)|^2]&=1,\\ \gamma^{-1}(k)[|c_{21}(k,l)|^2-|c_{22}(k,l)|^2]&=1.\end{aligned}$$

These equations can be rewritten in the form

$$\frac{1}{\gamma(k)}\frac{1}{|c_{12}(k,l)|^2}+\left|\frac{c_{11}(k,l)}{c_{12}(k,l)}\right|^2=1,$$

$$\gamma(k)\frac{1}{|c_{21}(k,l)|^2}+\left|\frac{c_{22}(k,l)}{c_{21}(k,l)}\right|^2=1. \qquad (37)$$

We introduce now solutions $\Psi^+(x,k^+,l)$, $\Psi^-(x,k,l)$ of Eq. (4) satisfying the following conditions:

$$\Psi^+(x,k,l)=\begin{cases} e^{ikx}+s_{12}(k,l)e^{-ikx}+o(1), & x\to-\infty,\\ s_{11}(k,l)w_{-i\tilde{\mu},1/2}(-2ik^+x)+o(1), & x\to+\infty,\end{cases}$$

$$\Psi^-(x,k,l) =\begin{cases} s_{22}(k,\alpha,l)e^{-ikx}+o(1), \quad x\to-\infty,\\ w_{i\tilde{\mu},1/2}(2ik^+x)+s_{21}(k,l)w_{-i\tilde{\mu},1/2}(-2ik^+x)+o(1),\\ \quad x\to+\infty.\end{cases}$$

As a result, from Eq. (33) we obtain

$$\Psi^+(x,k^+,l)=\overline{\psi^-(x,k,l)}+s_{12}(k,l)\psi^-(x,k,l) =s_{11}(x,k,l)\psi^+(x,k^+,l),$$

$$\Psi^-(x,k,l)=s_{22}(k,l)\psi^-(x,k,l)=\overline{\psi^+(x,k^+,l)} +s_{21}(k,l)\psi^+(x,k^+,l).$$

In the similar way, from Eq. (33) we get

$$\Psi^-(x,k,l)=\frac{\psi^-(x,k,l)}{c_{12}(k,l)}, \quad s_{21}(k,l)=\frac{c_{11}(k,l)}{c_{12}(k,l)},$$

$$s_{22}(k,l)=\frac{1}{c_{12}(k,l)}$$

and

$$\Psi^+(x,k^+,l)=\frac{\psi^+(x,k^+,l)}{c_{21}(k,l)},$$

$$s_{12}(k,l)=\frac{c_{22}(k,l)}{c_{21}(k,l)}, \quad s_{11}(k,l)=\frac{1}{c_{21}(k,l)}.$$

For example, for the coefficient $s_{11}(k,l)$ which appears in Eq. (7) for the barrier transparency $\Gamma(k,l)$, we have a relation applicable to numerical calculations:

$$s_{11}(k,l)=2ik/[\psi^-(x,k,l),\psi^+(x,k^+,l)],$$

where the Wronskian can be computed numerically with the help of integral equations (28),(31).

According to Eqs. (35) we get the unitarity relations for the matrix $S=\{s_{ij}\}$:

$$\gamma^{-1}(k)|s_{22}(k,l)|^2+|s_{21}(k,l)|^2=1,$$

$$\gamma(k)|s_{11}(k,l)|^2+|s_{12}(k,l)|^2=1.$$

It is not difficult, as usual, to show that the asymptotic behavior is

$$s_{12}(k,l)=O(k^{-1}), \quad s_{21}(k,l)=O(k^{-1})$$

and

$$\Gamma(k,l)=|s_{11}(k,l)|^2=1+O(k^{-1}).$$

Though for studying the Hawking radiation one needs only s_{11}, the other elements of the *S* matrix obtained can be important in a number of the problems within 4D black hole physics, for instance, when studying vacuum polarization near black holes for TICs. However, the convergence of the series in Eqs. (6),(8) over *l* will be discussed elsewhere since it requires knowledge of more exact asymptotics for $|s_{11}|^2$.

We considered one of the scattering problems which are encountered in 4D black hole physics. The relations obtained can be employed when numerically calculating the Hawking radiation luminosity for actual charged massive scalar particles, for example, for $\pi^\pm$ and $K^\pm$ mesons. Other *S* matrices will emerge when both the type of field and the type of black hole vary (for example, for Reissner-Nordström [6] or Kerr [7] black holes). As was mentioned, the elements of the corresponding *S* matrices are the important ingredients when calculating miscellaneous quantum effects connected with black holes. We hope, therefore, to continue a strict study of a number of the problems mentioned within the framework of our further investigations.

The author is thankful to Yu. Goncharov for useful discussions.

[1] Yu. P. Goncharov and J. V. Yarevskaya, Mod. Phys. Lett. A **9**, 3175 (1994); **10**, 1813 (1995); Yu. P. Goncharov, Nucl. Phys. **B460**, 167 (1996); Int. J. Mod. Phys. A **12**, 3347 (1997).
[2] Yu. P. Goncharov and N. E. Firsova, Int. J. Mod. Phys. D **5**, 419 (1996).
[3] Yu. P. Goncharov and N. E. Firsova, Nucl. Phys. **B486**, 371 (1997).
[4] E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis* (Cambridge University Press, Cambridge, England, 1973).
[5] K. Chadan and P. C. Sabatier, *Inverse Problems in Quantum Scattering Theory* (Springer-Verlag, Berlin, 1977).
[6] N. E. Firsova, Int. J. Mod. Phys. D **7**, 509 (1998).
[7] Yu. P. Goncharov, Phys. Lett. B **398**, 32 (1997); Pis'ma Zh. Eksp. Teor. Fiz. **67**, 1021 (1998) [JETP Lett. **67**, 1082 (1998)]; Mod. Phys. Lett. A **13**, 1495 (1998).