Black hole lasers

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High frequency dispersion does not alter the low frequency spectrum of Hawking radiation from a single black hole horizon, whether the dispersion entails subluminal or superluminal group velocities. We show here that in the presence of an inner horizon as well as an outer horizon the superluminal case differs dramatically however. The negative energy partners of Hawking quanta return to the outer horizon and stimulate more Hawking radiation if the field is bosonic or suppress it if the field is fermionic. This process leads to exponential growth or damping of the radiated flux and correlations among the quanta emitted at different times, unlike in the usual Hawking effect. These phenomena may be observable in condensed matter black hole analogues that exhibit ‘‘superluminal’’ dispersion. [S0556-2821(99)07510-4]

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I. INTRODUCTION

Recent work has shown that Hawking radiation is highly insensitive to modifications of the short distance physics of the quantum field. In these models linear fields are considered, and the field equation is modified at high wave vectors in some preferred frame, yielding a nonlinear dispersion relation $\omega(k)$ relating frequency to wave vector. Models with both subluminal [1–3] and superluminal [4,5] group velocities at high wave vectors have been studied, including lattice black hole spacetimes [6] (which have subluminal dispersion). The picture that emerges from these studies is that the thermal Hawking spectrum is very robust for black holes with temperature much less than the energy scale of the new physics. Although short distance physics does modify this spectrum, the modifications are so slight at the frequencies of interest that they seem well nigh impossible to observe.

We have found a dramatic exception to this rule however. If there is both an outer and an inner horizon, and if the dispersion is superluminal, then the Hawking process for a bosonic field is self-amplifying and the radiated flux grows exponentially in time, while for a fermionic field the process is self-attenuating. What happens is that the negative energy partner of a Hawking particle, after falling to the inner horizon, ‘‘bounces’’ and returns to the outer horizon on a superluminal trajectory, where it either stimulates or suppresses more Hawking radiation in the bosonic or fermionic case respectively. This secondary radiation is not only different than the usual Hawking flux, but it is correlated to the prior radiation. In the bosonic case the process continues to amplify at least until the back reaction becomes important.

Charged black holes have inner horizons, but astrophysical ones would lose their charge very rapidly; so it is difficult to imagine how this runaway Hawking effect could ever be observed for real black holes. Even so, it provides an interesting theoretical laboratory in which to explore the effects of short distance physics. Moreover, it is conceivably relevant to string theory, and it might be observable in a condensed matter analogue of a black hole. Let us briefly indicate these ideas in turn.

In spite of many points of close agreement between the physics of near extremal D-branes and black holes, a glaring discrepancy persists. If a radiating near extremal D-brane state is maintained at fixed energy by a constant influx of energy in a pure state, then the entropy in the radiation will be constant and there will be correlations in the radiation that emerges at different times. For a black hole, on the other hand, the usual Hawking process leads to uncorrelated thermal radiation for all time. The effects of superluminal dispersion invalidate the usual Hawking picture because the negative energy partners return to the event horizon. If there is something analogous to the superluminal dispersion of our model in string theory, then perhaps that could eliminate the discrepancy between the string and black hole pictures. This may not be so farfetched. String theory is, after all, non-local in some sense, and there is some evidence [7] suggesting that it supports superluminal effects.

A condensed matter analogue—Unruh's sonic black hole [8,1,9]—was the original stimulus for the development of the dispersive models. In this model, a sonic horizon occurs where the flow velocity of an inhomogeneous fluid exceeds the speed of sound. Although it seems unlikely that this situation can be experimentally realized for a low temperature superfluid, there are variations of the idea that might be realizable, involving quasiparticles other than phonons in different systems. For example, this may occur for fermion quasiparticles in rotating superfluid vortex cores with gap nodes such as ^{3}He-*A* or *d*-wave superconductors [10], or in moving ^{3}He-*A* textures [11]. In both these examples there are both inner and outer horizons. Moreover, the quasiparticle dispersion relation is ‘‘relativistic’’ sufficiently near a gap node, and the group velocity increases (i.e. becomes ‘‘superluminal’’) as the difference between the momentum and the gap node increases; so the effective field theory has ‘‘superlumi-

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nal'' dispersion. Thus it is not inconceivable that the phenomena discussed here may someday be observable.

This paper is organized as follows. In Sec. II the superluminal dispersion model for both bosons and fermions is discussed. The propagation of wave packets in the black hole spacetime with inner and outer horizons is analyzed qualitatively in Sec. III and the implications for the amplification or suppression and the correlations in the Hawking radiation are drawn in Sec. IV. Section V renders the previous discussion quantitative by using explicit wave packet solutions (derived in the Appendix) to find expressions for the number and correlations between the radiated quanta. Open issues concerning the boundary conditions on the quantum state and the gravitational back reaction are discussed briefly in Sec. VI.

We use units with $\hbar = c = 1$ and metric signature $(+---)$.

II. SUPERLUMINAL DISPERSION MODEL

A 2-dimensional model suffices to illustrate the essential physics. We assume that the spacetime metric is static, and therefore [12] coordinates can be chosen (at least locally) so that the line element takes the form

$$ds^2 = dt^2 - [dx - v(x)dt]^2. \tag{2.1}$$

A special case is the line element of the t-r subspace of the Reissner-Nordström black hole spacetime in Painlevé-Gullstrand coordinates, where $v(r) = -\sqrt{2GM/r - Q^2/r^2}$. [These coordinates cover the black hole interior down to where $v(r)=0$, at $r = Q^2/2GM$.] More generally, we consider any $v(x)$ which is negative, vanishes as $x \to +\infty$, and is greater than -1 except between inner and outer horizons, located at x_i and x_o, where $v(x_{i,o}) = -1$.

A. Boson field

We adopt a linear field theory with higher spatial derivatives included in the action in order to provide a superluminal dispersion relation. In this section we restrict to the case of a real bosonic field. The case of a Majorana fermion field will be discussed in Sec. II B. The action for the field is given by

$$S_\phi = \frac{1}{2}\int d^2x\{[(\partial_t + v\partial_x)\phi]^2 + \phi\hat{F}(\partial_x)\phi\}. \tag{2.2}$$

In the ordinary relativistic action one has $\hat{F}(\partial_x) = \partial_x^2$. In this paper we take

$$\hat{F}(\partial_x) = \partial_x^2 - \frac{1}{k_0^2}\partial_x^4. \tag{2.3}$$

To motivate this action we note that the black hole defines a preferred frame, the frame of freely falling observers. In the Painlevé-Gullstrand coordinate system, $(\partial_t + v\partial_x)$ is the unit tangent to free-fall world lines that start from rest at infinity, and ∂_x is its unit, outward pointing normal. Our action comes from modifying the derivative operator, only along the unit normal ∂_x, by the addition of higher derivative terms which become important only when the wavelength is of order $1/k_0$ or shorter. We will assume that this length scale of ''new physics'' is much shorter than the length scale of the metric (2.1), i.e. $k_0 \gg |v'/v|$. [In particular, we assume $k_0 \gg \kappa$, where $\kappa = |v'(x_{i,o})|$ is the surface gravity of the horizon.] The idea is that the microstructure of spacetime, or of a condensed matter analogue, might give rise to such higher derivative terms in the effective action. The choice (2.3) is just the generic form for the lowest order such term that is reflection invariant and produces superluminal group velocities.

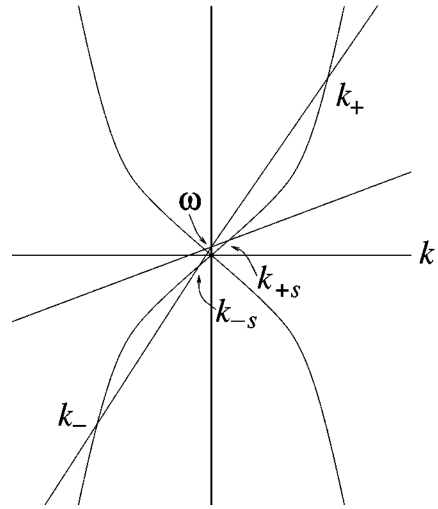

FIG. 1. Plot of $(\omega + |v|k)$ (for one value of ω and two values of v) and $F(k)$ as functions of k. The intersection points of the curves are the allowed wave vector roots of the dispersion relation (2.6).

The action (2.2),(2.3) produces the equation of motion

$$(\partial_t + \partial_x v)(\partial_t + v\partial_x)\phi = \partial_x^2\phi - \frac{1}{k_0^2}\partial_x^4\phi. \tag{2.4}$$

To derive the dispersion relation for this equation we look for solutions of the form

$$\phi(t,x) = \exp\left(-i\omega t + i\int^x k(x')\,dx'\right) \tag{2.5}$$

where $k(x)$ is a position dependent wave vector. Substituting this ansatz into the equation of motion (2.4) and neglecting derivatives of $v(x)$ and $k(x)$ results in the dispersion relation

$$(\omega - vk)^2 = F^2(k) \tag{2.6}$$

where

$$F^2(k) = k^2 + k^4/k_0^2. \tag{2.7}$$

The group velocity in the free-fall frame is dF/dk; so wave packets with $k \ll k_0$ propagate near the speed of light, whereas wave packets with $k \gtrsim k_0$ propagate superluminally.

The dispersion relation (2.6) is a fourth order polynomial equation in the wave vector k, and so it has four solutions for k at given values of ω and v. The nature of these roots is revealed by a graphical method. In Fig. 1 we plot the straight line $(\omega+|v|k)$ for one value of ω (satisfying $0<\omega\ll k_0$) and two values of v, and the curve $\pm F(k)$, as functions of k. [We define $F(k)$ as the *positive* square root of Eq. (2.7).] The intersection points are the allowed real wave vector roots to the dispersion relation. When $|v|<1$ there are only two *real* roots [corresponding to the two roots for the ordinary dispersion relation with $F_{ord}(k)=\pm k$], the other two being complex. The positive wave vector is denoted k_{+s}. When $|v|\gtrsim 1+\frac{3}{2}(\omega/k_0)^{2/3}\approx 1$, on the other hand, all four roots are real, with one positive and three negative. The positive wave vector is denoted k_+ in this case and, in decreasing magnitude, the negative wave vectors are denoted k_- and k_{-s} respectively (the other negative wave vector corresponds to an ingoing wave that plays no role in this paper, and so we do not give it a name). These roots are labeled in Fig. 1.

The dispersion relation plot in Fig. 1 is also quite convenient for tracing the motion of wave packets in the background spacetime. The coordinate group velocity $v_g=dx/dt$ of a wave packet centered on a given wave vector is given by

$$v_g=\frac{d\omega}{dk}=-|v|\pm\frac{dF}{dk}, \tag{2.8}$$

where $\pm dF/dk$ is the group velocity in the free-fall frame. Thus at any wave vector v_g is just the slope of the $\pm F(k)$ curve minus the slope of the straight line $(\omega+|v|k)$. For all four types of wave vectors $k_{\pm s,\pm}$ of interest to us, $\pm dF/dk$ is positive; hence the sign of v_g is determined by which of the two slopes is larger, something that is easily read from the figure. For $\omega>0$, the group velocity for k_{+s} and $k_\pm$ is positive, whereas for k_{-s} it is negative.

When generalized to a complex scalar field, the action (2.2) is invariant under constant phase transformations of the field. This implies the existence of a conserved current j^μ. The integral of the time component j^0 over a spatial slice serves as a conserved inner product when evaluated on complex solutions to the equation of motion (2.4). For the metric (2.1), this inner product takes the form

$$(f,g)=i\int dx[f^*(\partial_t+v\partial_x)g-g(\partial_t+v\partial_x)f^*], \tag{2.9}$$

where $f(t,x)$ and $g(t,x)$ are solutions to Eq. (2.4).

Two classes of complex solutions to the field equation (2.4) are of special interest for quantization. The first are the positive free-fall frequency wave packets. They can be written as sums of solutions satisfying

$$(\partial_t+v\partial_x)f(t,x)=-i\omega' f(t,x) \tag{2.10}$$

where $\omega'>0$. The second are the positive Killing frequency wave packets. These are sums of solutions of the form $e^{-i\omega t}\varphi(x)$ where $\omega>0$. A positive free-fall frequency wave packet confined to a constant $v(x)$ interval at one time necessarily has a positive norm under Eq. (2.9), as does a positive Killing frequency wave packet confined to a region where $v(x)=0$ (where the Killing frequency coincides with the free-fall frequency). Since the norm is conserved, it is positive at all times if it is at one time, even when the wave packet does not remain in an interval of constant or vanishing $v(x)$. Note that if the wavelength is small compared to the scale of variations of $v(x)$, then a positive free-fall frequency wave packet will have positive norm even if $v(x)$ is not constant.

To quantize the field we assume that $\hat\phi(t,x)$ is a self-adjoint operator solution to the field equation that satisfies the canonical commutation relations. We define the annihilation operator $a(f)$ associated to a normalized complex solution to the wave equation $f(t,x)$ by

$$a(f)\equiv(f,\hat\phi). \tag{2.11}$$

The commutation relations for the field operator are equivalent to the relations

$$[a(f),a^\dagger(g)]=(f,g) \tag{2.12}$$

for all f and g. If $f(t,x)$ is a positive norm solution, then $a(f)$ behaves as an annihilation operator. If $f(t,x)$ is a negative norm solution, $f^*(t,x)$ has positive norm; so $a(f)=-a^\dagger(f^*)$ behaves as a creation operator.

B. Fermion field

For simplicity we consider two-dimensional massless Majorana fermions. Following the conventions of [13], the action in a general curved spacetime is given by[1]

$$S_\psi=\frac{i}{2}\int d^2x\sqrt{-g}\,\overline{\psi}\Gamma^\mu\partial_\mu\psi \tag{2.13}$$

where $\Gamma^\mu=\Gamma^a e_a^\mu$ and e_a^μ is the zweibein. We take the flat space gamma matrices as

$$\Gamma^0=\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix},\quad \Gamma^1=\begin{pmatrix}0 & i\\ i & 0\end{pmatrix}. \tag{2.14}$$

Decomposing the spinor ψ as

$$\psi=\begin{pmatrix}\psi_+\\ \psi_-\end{pmatrix} \tag{2.15}$$

and expanding the action in the metric (2.1) using the zweibein $(e_0,e_1)=(\partial_t+v\partial_x,\partial_x)$ we find

$$S_\psi=\frac{i}{2}\int d^2x\{\psi_+[\partial_t+(1+v)\partial_x]\psi_+ + \psi_-[\partial_t-(1-v)\partial_x]\psi_-\}. \tag{2.16}$$

In this form it is clear that ψ_+ and ψ_- do not mix. Furthermore, at infinity, where $v(x)=0$, ψ_+ is right-moving while ψ_- is left-moving. We therefore drop ψ_- in the remainder as it plays no role in the Hawking radiation calculation.

[1] In higher dimensions there would be a spin connection term as well. In two dimensions it is easy to show that this term vanishes identically.

Following the same motivation described in Sec. II, we now modify the action for ψ_+ by subtracting the higher derivative term $k_0^{-2}\psi_+\partial_x^3\psi_+$, obtaining the action

$$S_\psi=\frac{i}{2}\int d^2x\{\psi_+[\partial_t+(1+v)\partial_x-k_0^{-2}\partial_x^3]\psi_+\}. \quad (2.17)$$

Varying with respect to ψ_+ results in the equation of motion

$$(\partial_t+v\partial_x+\partial_x v/2+\partial_x-k_0^{-2}\partial_x^3)\psi_+=0. \quad (2.18)$$

Substituting $\psi_+(t,x)=\exp[-i\omega t+i\int^x k(x')dx']$ into the equation of motion and dropping derivatives of $k(x)$ and $v(x)$ results in the dispersion relation

$$\omega-vk=k+k^3/k_0^2. \quad (2.19)$$

This is the same (up to the coefficient of the k^3 term and higher order terms) as the branch of the scalar field dispersion relation corresponding to positive group velocity in the free-fall frame given in Eq. (2.6) and displayed in Fig. 1. The classification of scalar wave packet types in Sec. II A therefore applies to fermion wave packets as well. In particular, the higher derivative term leads to superluminal propagation at large wave vectors.

To quantize the field we assume that $\hat{\psi}_+(t,x)$ is a self-adjoint operator solution to the field equation that satisfies the canonical anti-commutation relations $\{\hat{\psi}_+(t,x),\hat{\psi}_+(t,x')\}=\delta(x,x')$. The conserved inner product is the integral of the time component of the conserved current associated with phase invariance of the action (2.17) (generalized to complex fermions), and takes the form

$$\langle\psi_1,\psi_2\rangle=\int dx\,\psi_1^*\psi_2. \quad (2.20)$$

We define the annihilation operator $b(f)$ associated with a normalized complex solution to the wave equation $f(t,x)$ by

$$b(f)\equiv\langle f,\hat{\psi}_+\rangle. \quad (2.21)$$

The anti-commutation relations for the field operator are then equivalent to the relations

$$\{b(f),b^\dagger(g)\}=\langle f,g\rangle \quad (2.22)$$

for all f and g. We represent the operators $b(f)$ on the fermionic Fock space generated by positive free-fall frequency solutions to the equation of motion (2.18). If $f(t,x)$ is a positive free-fall frequency solution, then $b(f)$ behaves as an annihilation operator on this space. If $f(t,x)$ is a negative free-fall frequency solution, then $f^*(t,x)$ has positive free-fall frequency; so $b(f)=b^\dagger(f^*)$ behaves as a creation operator.

III. WAVE PACKET PROPAGATION

In this section we give a qualitative analysis of the role of the inner horizon in modifying the Hawking radiation. This analysis will exploit a WKB description of wave packet propagation, allowing for non-WKB ‘‘mode conversion’’ in the vicinity of the horizons. The analysis applies equally well for the bosonic and fermionic quantum fields. Scattering of waves on account of the background curvature of the metric (2.1) is negligible as long as the radius of curvature is much greater than $1/k_0$. For small wave vectors, $k\ll k_0$, this is because the wave equation is approximately conformally invariant and the metric (like any two-dimensional metric) is conformally flat. For large wave vectors, $k\gtrsim k_0$, it is because the wavelength is much smaller than the radius of curvature.

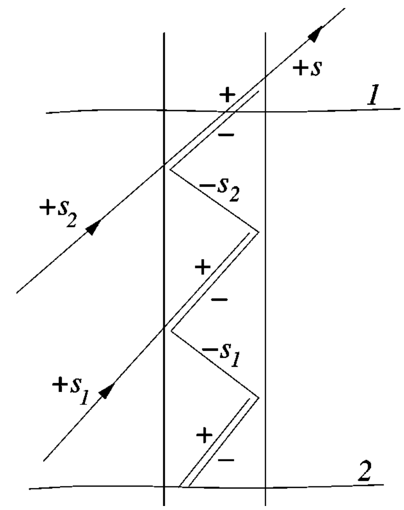

FIG. 2. Spacetime sketch of the evolution of an outgoing k_{+s} wave packet backward in time. The end of a line indicates a wave packet destroyed by mode conversion, while a continuous line indicates that the wave vector evolves continuously on the dispersion curve.

We begin outside the outer horizon with a low frequency outgoing wave packet peaked around a wave vector of type k_{+s} (see Fig. 1), and we follow this wave packet backwards in time. A sketch of what we find is given in Fig. 2. The final wave packet (i.e. the one we begin with) is labeled $+s$ in Fig. 2. This packet has positive group velocity and therefore is right-moving, as can be seen from the graph of the dispersion relation (Fig. 1). Following this packet backward in time it moves toward the black hole and blueshifts. The Killing frequency ω is conserved; so the increase in the wave vector can be seen from Fig. 1 by increasing the slope of the straight line while keeping the intercept fixed. As the wave vector grows, the group velocity increases in the free-fall frame, and so the packet becomes superluminal and crosses the horizon (backward in time), becoming a packet with wave vectors of type k_+ (see Fig. 2).

The wave packet inside the horizon also has a k_- component, which is not obvious if we simply follow continuously along the dispersion curve. In fact, the WKB approximation breaks down near the horizon, and ‘‘mode conversion’’ from the positive wave vector to the negative wave vector, negative free-fall frequency, branch of the dispersion relation occurs. This is easily shown analytically, and is made plausible by the fact that, around the horizon, the straight line of Fig. 1 nearly coincides with a large portion of the curved line of the dispersion curve, thus allowing other wave vectors to become mixed in. The dispersion relation allows wave vectors of types k_+, k_-, and k_{-s} in between the horizons; however, only the first two are right moving, whereas the last type is left moving. Since our final wave packet is by assumption purely outgoing outside the horizon, there can be no k_{-s} component generated here. The k_+ and k_- wave packets are labeled $+$ and $-$ in Fig. 2. In this figure the end of a line indicates a wave packet destroyed by mode conversion, while a continuous line indicates that the wave vector evolves continuously on the dispersion curve.

The k_+ and k_- packets propagate backward in time toward the inner horizon where they both undergo partial mode conversion. The group velocity of the k_+ packet remains positive around the inner horizon and therefore it can cross, becoming a k_{+s} packet, labeled $+s_2$ in Fig. 2. As before, though, there is also some mode conversion from the positive to the negative wave vector branch of the dispersion relation, and a left-moving k_{-s} packet ($-s_2$ in Fig. 2) is generated which propagates backward in time back toward the outer horizon. The k_- packet on the other hand cannot cross the inner horizon on the negative wave vector branch because its group velocity drops to zero at the horizon. Indeed the group velocity goes through zero and becomes negative; so the k_- packet turns around and propagates back toward the outer horizon as a k_{-s} packet still on the negative wave vector branch. In addition, some mode conversion from the negative to the positive wave vector branch of the dispersion relation occurs at the inner horizon. Therefore part of the k_- packet does cross the horizon as a k_{+s} packet and is superposed with the k_{+s} packet that evolved from the k_+ packet.

The k_{+s} packet inside the inner horizon continues propagating to the left backward in time. The k_{-s} packet however returns to the outer horizon, near which its group velocity drops to zero. Again, partial mode conversion to the positive wave vector branch occurs; so the k_{-s} packet evolves backward in time to a pair of k_+ and k_- packets which are heading back to the inner horizon. This is now almost the same situation we started with, since the original k_{+s} packet also evolved into a pair of k_+ and k_- packets between the horizons (although with a different relative weight). The analysis given above thus tells us qualitatively what happens when they reach the inner horizon, namely, the same thing as happened before. The general pattern that emerges is shown in Fig. 2.

We have so far discussed the history of an outgoing k_{+s} wave packet followed backward in time. It is also instructive to look at the *future* evolution of a k_{-s} wave packet in between the horizons, since the negative energy partner of a Hawking particle is such a wave packet. This evolution can be inferred by the same sort of analysis just given or simply by time and space reversal of that analysis, and is shown in Fig. 3.

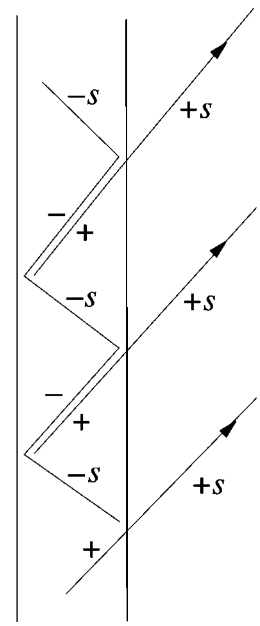

FIG. 3. Spacetime sketch of the trajectory of a Hawking particle and its partner forward in time. The end of a line indicates a wave packet created by conversion, while a continuous line indicates that the wave vector evolves continuously on the dispersion curve.

IV. PARTICLE CREATION: ORIGIN OF THE AMPLIFICATION OR SUPPRESSION OF HAWKING RADIATION

The amount of particle creation in an outgoing positive frequency wave packet ψ is indicated by the expectation value of the number operator $N(\psi)=a^\dagger(\psi)a(\psi)$. To determine this expectation value an initial quantum state must be specified. Let us define an in-Hilbert space on some spacelike surface as the Fock space generated by positive free-fall frequency wave packets on that surface. The corresponding ground state is then annihilated by annihilation operators of these wave packets. We shall suppose the initial state is such a free-fall ground state associated with a given surface Σ. Decomposing $\psi=\psi^+ + \psi^-$ into its positive and negative free-fall frequency parts on Σ, the ground state condition implies in the bosonic case that $\langle N(\psi)\rangle = -(\psi^-,\psi^-)$ and in the fermionic case $\langle N(\psi)\rangle = \langle \psi^-,\psi^-\rangle$.

Suppose we choose Σ as surface 1 in Fig. 2, i.e., a surface that cuts through the k_+ and k_- packets first produced by propagating the k_{+s} packet back in time. Then the number expectation value for the k_{+s} packet is just (minus) the norm of the k_- packet. In [4,5] this was shown (for bosons) to be thermal at the Hawking temperature, for wave packets with Killing frequencies ω satisfying $\kappa \lesssim \omega \ll k_0$. That is, the standard Hawking effect occurs even in the presence of superluminal dispersion, if there is only a single horizon.

When there is also an inner horizon, the particle creation depends very much on which surface is used to define the initial ground state. If we impose the ground state condition on the earlier surface 2 in Fig. 2, instead of surface 1, the occupation number for the final k_{+s} packet is no longer thermal. The norm of the negative frequency part of the wave packet on surface 2 is determined not just by the final passage across the outer horizon, but also by the mode conversion processes at the inner and outer horizons.

As the time between the initial ground state and the final outgoing wave packet grows, there is an exponential amplification or suppression in the occupation number of the final wave packet in the boson and fermion cases respectively. To see why, note that the k_{-s} packet denoted $-s_2$ in Fig. 2 evolves into the orthogonal k_{+s} and k_{-s} packets denoted $+s_1$ and $-s_1$ respectively; hence the norms are related by

$$\|-s_2\|^2=\|+s_1\|^2+\|-s_1\|^2 \tag{4.1}$$

where $\|f\|^2$ stands for (f,f) in the bosonic case and $\langle f,f\rangle$ in the fermionic case.

Consider first the bosonic case. A $k_{+s}(k_{-s})$ packet has positive (negative) free-fall frequency and therefore positive (negative) norm under Eq. (2.9); so it follows from Eq. (4.1) that $\|-s_1\|^2$ is larger in magnitude than $\|-s_2\|^2$. Continuing into the past this process repeats, and for each ''bounce'' between the horizons the norm of the wave packet between the horizons grows by some fixed multiple, resulting in exponential growth of both $\|-s_n\|^2$ and $\|+s_n\|^2$.[2] Since the negative frequency part of this wave packet determines the number of created particles in the final outgoing wave packet, that number will grow exponentially in time between the initial surface Σ and the emergence of the outgoing wave packet ψ. Viewed forward in time, the Hawking effect is a self-amplifying process since the negative energy partners of the Hawking particles return to the event horizon (in the form of a pair of k_+ and k_- packets) and stimulate the emission of more radiation and more partners. The wave packet trajectories associated with this forward in time picture are shown in Fig. 3.

For a fermionic field, the above discussion is modified only by the fact that all wave packets have positive norm; so Eq. (4.1) implies that $\|-s_1\|^2$ is *smaller* in magnitude than $\|-s_2\|^2$. This means that the number of created particles will be exponentially *damped* in time. In effect, the allowed states between the horizons for the negative energy partners of the Hawking particles become filled, cutting off further pair creation.

One further important point can be extracted from this analysis. Since a single particle and partner wave packet pair evolves to a sequence of outgoing wave packets as shown in Fig. 3, the states of all these outgoing wave packets will be *correlated*. This can also be seen from the backwards in time picture. It is clear from Fig. 2 that successive outgoing wave packets will have past histories that partly overlap, in particular on the initial ground state surface; so there will be correlations between the quanta emitted from the horizon at different times. These correlations are in sharp contrast to the usual Hawking effect which produces uncorrelated thermal radiation. The information loss that is normally associated with the correlations between Hawking quanta and their partners is largely eliminated, since an unending sequence of Hawking quanta is coherently correlated to the same partner degrees of freedom.

[2]It is perhaps surprising to have exponential growth in time when there are no imaginary frequency solutions to the dispersion relation (2.6). There is no contradiction however, since these exponentially growing wave packet solutions cannot be Fourier transformed in time and so need not be expressible as superpositions of time-independent mode solutions.

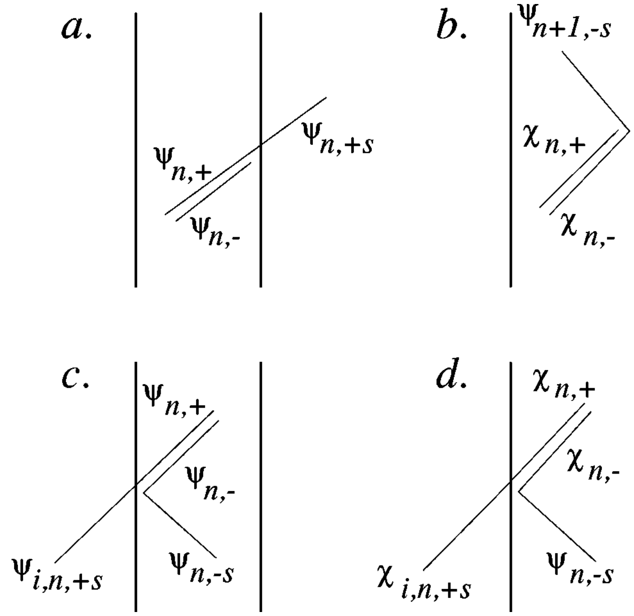

FIG. 4. Spacetime sketches of the local wave packet evolutions given by (5.1a)–(5.1d) respectively.

V. QUANTITATIVE ANALYSIS

The qualitative analysis of the previous section will now be sharpened by explicitly constructing the wave packet solutions discussed there. This will allow us to quantify the amount of amplification, suppression, and correlation of the black hole radiation. In the first two subsections we treat only the bosonic case, and in the last subsection we discuss the fermionic case.

A. Wave packet solutions

The basic idea applied here is to patch together local wave packet solutions with the aid of ''evolution formulas.'' The derivation of these evolution formulas is discussed in the Appendix of this paper. They are derived using connection formulas for time-independent mode functions which are obtained by matching WKB solutions to near-horizon approximations. Forming wave packets with the mode functions we then obtain the evolution formulas for the wave packets.

Evolution formulas are needed for two different boundary conditions at both the inner and outer horizons, corresponding to the spacetime diagrams in Figs. 4a–4d. Using the notation ''$f\rightarrow g$'' to denote ''f evolves to g'' (forward in time), the evolution formulas about the outer horizon are

$$\psi_{n,+}+\psi_{n,-}\rightarrow\psi_{n,+s} \tag{5.1a}$$

$$\chi_{n,+}+\chi_{n,-}\rightarrow\psi_{n+1,-s}\,, \tag{5.1b}$$

while about the inner horizon they are

$$\psi_{i,n,+s}+\psi_{n,-s}\rightarrow\psi_{n,+}+\psi_{n,-} \tag{5.1c}$$

$$\chi_{i,n,+s}+\psi_{n,-s}\rightarrow\chi_{n,+}+\chi_{n,-}\,, \tag{5.1d}$$

where all packets have been left unnormalized in order to keep the formulas as simple as possible. The evolution formulas given here are preferred for evolving packets backwards in time. Following the same techniques described in the Appendix evolution formulas more conducive to evolving wave packets forward in time can be derived. The $+$, $-$, $+s$ and $-s$ subscripts denote which type of wave vector the packet is peaked about. $\psi_{n,+s}$ lies outside the outer horizon, while $\psi_{i,n,+s}$ and $\chi_{i,n,+s}$ both lie inside the inner horizon. The subscript n is a sort of time variable. Translation of n by one unit has the effect of translating the wave packet in time by a certain amount and also distorting the wave packet. Note that in the evolution formula at the outer horizon (5.1b) n increases by one unit on the $\psi_{n+1,-s}$ wave packet. The evolution formulas (5.1a)–(5.1d) are basically scattering solutions about a black hole event horizon with (5.1a)–(5.1b) corresponding to scattering the $\psi_{n,+s}$ and $\psi_{n,-s}$ packets off the outer horizon backward in time (see Figs. 4a and 4b respectively) and (5.1c)–(5.1d) corresponding to scattering the resulting combinations of $+$ and $-$ packets off the inner horizon backward in time as well (see figures 4c and 4d respectively).

To construct the wave packet solution with final data consisting of a k_{+s} packet outside the outer horizon we start with the local solution (5.1a) (Fig. 4a). This clearly is not a global solution since $\psi_{n,\pm}$ do not solve the equation of motion (2.4) about the inner horizon. The combination may be replaced by Eq. (5.1c) which does however. This results in the evolution formula

$$(\psi_{i,n,+s}+\psi_{n,-s})\rightarrow\psi_{n,+s}. \tag{5.2}$$

The evolution (5.2) is also not a global solution to the equation of motion (2.4) however since $\psi_{n,-s}$ is not a solution about the outer horizon. Using Eq. (5.1b) to evolve $\psi_{n,-s}$ about the outer horizon, followed by Eq. (5.1d) to evolve the resulting wave packets $\psi_{n-1,+}$ and $\psi_{n-1,-}$ about the inner horizon, we obtain an evolution formula that can be iterated indefinitely:

$$(\chi_{i,n-1,+s}+\psi_{n-1,-s})\rightarrow\psi_{n,-s}. \tag{5.3}$$

Beginning with Eq. (5.2) and iterating Eq. (5.3) $(n-m)$ times yields

$$\left(\psi_{i,n,+s}+\sum_{j=1}^{n-m}\chi_{i,n-j,+s}+\psi_{m,-s}\right)\rightarrow\psi_{n,+s}. \tag{5.4}$$

In this manner we can evolve the final wave packet back to the spacelike surface where the initial ground state boundary condition is defined. Solutions of the form (5.4) correspond to those used in the qualitative discussion of Sec. III, which are depicted in Fig. 2 [wherein the first k_{-s} packet has been traded for a $k_{\pm}$ pair using (5.1b)].

If the horizons are not sufficiently widely separated, then intermediate wave packets that arise between the initial and final packets of Eq. (5.4) will overlap with the initial and final wave packets, thus complicating the analysis of particle creation. We can avoid such complications by constructing a different solution. Setting n equal to m in Eq. (5.2) and subtracting from Eq. (5.4), we obtain

$$\left(\psi_{i,n,+s}+\sum_{j=1}^{n-m}\chi_{i,n-j,+s}-\psi_{i,m,+s}\right)\rightarrow(\psi_{n,+s}-\psi_{m,+s}). \tag{5.5}$$

This solution corresponds to sending $(n-m)$ $\chi_{i,k,+s}$ packets and a pair of $\psi_{i,k,+s}$ packets into the inner horizon and getting a pair of ψ_{+s} packets out of the outer horizon.

B. Particle creation

We can now compute the average number of particles in the wave packet[3] $\hat{\psi}_{n,+s}$ and the correlations between emitted particles for different values of the time index n. To begin with let us evaluate the occupation number $\langle 0|N(\hat{\psi}_{0,+s})|0\rangle$ of the first outgoing packet after the initial state condition is assumed. Then the only evolution formula we need is Eq. (5.1a), with $n=0$. The annihilation operator for a normalized wave packet f is given by $a(f)=(f,\hat{\phi})$, Eq. (2.11). Taking the inner product of Eq. (5.1a) with the quantum field $\hat{\phi}$, and using the ground state conditions

$$a(\hat{\psi}_{0,+})|0\rangle=0=a(\hat{\psi}^{*}_{0,-})|0\rangle, \tag{5.6}$$

we obtain

$$\langle 0|N(\hat{\psi}_{0,+s})|0\rangle=-\frac{(\psi_{0,-},\psi_{0,-})}{(\psi_{0,+s},\psi_{0,+s})}=\frac{1}{\omega_u-\omega_l}\int_{\omega_l}^{\omega_u}d\omega\,\frac{1}{e^{2\pi\omega/\kappa}-1}. \tag{5.7}$$

[The norms (A30) were used in the last equality.] This is just the Planck distribution at the Hawking temperature $T_H=\kappa/2\pi$, as was shown previously [4,5] for a superluminal dispersive field theory in the case that there is just one horizon. It holds for $\kappa\lesssim\omega\ll k_0$.

We would have obtained a different result for $\langle 0|N(\hat{\psi}_{0,+s})|0\rangle$ had we replaced the initial conditions (5.6) with, for example,

$$a(\hat{\psi}_{i,0,+s})|0\rangle=0=a(\hat{\psi}^{*}_{0,-s})|0\rangle. \tag{5.8}$$

Indeed, from Eq. (5.2) with $n=0$, we find that if Eq. (5.8) holds, the occupation number of $\hat{\psi}_{0,+s}$ is given by

$$\langle 0|N(\hat{\psi}_{0,+s})|0\rangle=-\frac{(\psi_{0,-s},\psi_{0,-s})}{(\psi_{0,+s},\psi_{0,+s})}=\frac{1}{\omega_u-\omega_l}\int_{\omega_l}^{\omega_u}d\omega\times\frac{2\{1-\cos[\theta_+(\omega)-\theta_-(\omega)]\}}{e^{2\pi\omega/\kappa}+e^{-2\pi\omega/\kappa}-2}. \tag{5.9}$$

[3] We use a caret to denote normalized wave packets.

The phase angles $\theta_\pm(\omega)$ are defined implicitly in Eq. (A18), and the norms (A30) were used in the last equality of Eq. (5.9). This differs from the thermal result (5.7).

It is not yet clear to us what is the ''correct'' initial condition on the quantum state of the field. To determine this would require following the evolution of the field state as the black hole (or condensed matter black hole analogue) forms. It does seem however that the conditions $a(\hat{\psi}_{i,k,+s})|0\rangle=0$ are likely to hold, while the remaining specification of the state remains to be determined. Fortunately these conditions alone suffice to determine the rate of growth of the number of particles emitted and the correlations between them.

In order to find the number and correlations in the radiation for $n>0$ we use the solution (5.5). We take $n>m\gg 1$ so that the intermediate wave packets that entered the construction of Eq. (5.5) will not have any support on the initial surface, and we assume the ground state conditions

$$a(\hat{\psi}_{i,k,+s})|0\rangle=0=a(\hat{\chi}_{i,k,+s})|0\rangle \tag{5.10}$$

for $k\geq m$. Taking the inner product of Eq. (5.5) with the field operator $\hat{\phi}$ and using conditions (5.10) we obtain

$$a(\hat{\psi}_{n,+s})|0\rangle=\sqrt{\frac{(\psi_{m,+s},\psi_{m,+s})}{(\psi_{n,+s},\psi_{n,+s})}}a(\hat{\psi}_{m,+s})|0\rangle, \tag{5.11}$$

from which it follows that

$$\langle 0|a^\dagger(\hat{\psi}_{k,+s})a(\hat{\psi}_{n,+s})|0\rangle =\frac{(\psi_{m,+s},\psi_{m,+s})}{\sqrt{(\psi_{k,+s},\psi_{k,+s})(\psi_{n,+s},\psi_{n,+s})}}\langle 0|N(\hat{\psi}_{m,+s})|0\rangle. \tag{5.12}$$

In particular taking $k=n$ we obtain

$$\langle 0|N(\hat{\psi}_{n,+s})|0\rangle=\frac{(\psi_{m,+s},\psi_{m,+s})}{(\psi_{n,+s},\psi_{n,+s})}\langle 0|N(\hat{\psi}_{m,+s})|0\rangle. \tag{5.13}$$

The norm of $\psi_{n,+s}$, Eq. (A30), is given by

$$(\psi_{n,+s},\psi_{n,+s})=4\pi\int_{\omega_l}^{\omega_u}d\omega\left(1+\frac{1-\cos(\theta_+-\theta_-)}{2\sinh^2(\pi\omega/\kappa)}\right)^{-n} \tag{5.14}$$

where we have used Eq. (A21). This decreases monotonically with n except for at most a discrete set of frequencies for which $\theta_+(\omega)=\theta_-(\omega)+2\pi k$ for some integer k. [For these frequencies T_1, Eq. (A21), vanishes; so according to Eq. (A20) the corresponding mode is a bound state trapped between the horizons.] Therefore the particle creation in $\hat{\psi}_{n,+s}$ increases monotonically with n, diverging as $n\to\infty$. In particular, if the wave packets are narrowly peaked about a frequency ω, Eq. (5.13) yields

$$\frac{\langle 0|N(\hat{\psi}_{n,+s})|0\rangle}{\langle 0|N(\hat{\psi}_{m,+s})|0\rangle}=\left(1+\frac{1-\cos(\theta_+-\theta_-)}{2\sinh^2(\pi\omega/\kappa)}\right)^{n-m}, \tag{5.15}$$

which grows exponentially with $n-m$.

A measure of the correlation between emitted particles is given by

$$C(m,n):=\frac{\langle 0|a^\dagger(\hat{\psi}_{m,+s})a(\hat{\psi}_{n,+s})|0\rangle}{[\langle 0|N(\hat{\psi}_{m,+s})|0\rangle\langle 0|N(\hat{\psi}_{n,+s})|0\rangle]^{1/2}} =1, \tag{5.16}$$

independent of the difference $n-m$. This should be contrasted with the correlation obtained when $\hat{\phi}$ satisfies the ordinary wave equation

$$C(m,n)=(\hat{\psi}_{m,+s},\hat{\psi}_{n,+s}), \tag{5.17}$$

which is nonvanishing only to the extent that these wave packets are not orthogonal. As $n-m$ grows the overlap of these wave packets and hence the correlation (5.17) go to zero, whereas the correlation (5.16) remains.

Finally let us estimate the time between the successive particle emissions (see Fig. 3), i.e., the difference in times when successive $\psi_{n,+s}$ packets (A22) cross a fixed x coordinate. The trajectory of the packets is given approximately by the condition of stationary phase,

$$t=\frac{d}{d\omega}\mathrm{Arg}[A_n(\omega)\phi_{+s}(x,\omega)], \tag{5.18}$$

and therefore the time Δt between the nth and $(n+1)$th packets crossing the coordinate x is given approximately by

$$\Delta t\approx\frac{d}{d\omega}\mathrm{Arg}(T_3) =\frac{d}{d\omega}[\gamma+\mathrm{Arg}(e^{-\pi\omega/\kappa}e^{-i\theta_+}-e^{\pi\omega/\kappa}e^{-i\theta_-})] \tag{5.19}$$

where we have substituted for A_n using Eq. (A23), and T_3 is given by Eq. (A21).

Using the results given in the Appendix the ω-dependence of the phase factors $\gamma(\omega)$ and $\theta_\pm(\omega)$ can be computed. Rather than carrying out this calculation—which we can only do explicitly for any particular $v(x)$ in some approximation anyway—let us make a rough estimate. The interval Δt is determined by the time it takes a wave packet to ''bounce'' back and forth between the horizons. If $v(x)+1$ is of order unity between the horizons, then using the group velocity of the k_{-s} and $k_\pm$ waves one finds that this bounce time is of order a, the coordinate distance between the horizons.

C. Fermionic case

In this section we briefly describe the differences in the quantitative analysis of the fermion case. The derivation of the wave packet solutions for fermions parallels that given for the scalar field in Sec. V A and yields an evolution formula very similar to Eq. (5.5). The final expression for the average value of the number operator is identical in form to Eq. (5.13); however, the norm $N_{n,+s}$ now increases monotonically with n so that the number expectation value decreases exponentially in n. This is to be expected since unlike the scalar case, the conserved norm $\langle f,f\rangle$, Eq. (2.20), is positive definite. The effect of this crucial difference is that instead of exponential growth of Hawking radiation we now get exponential decay.

VI. DISCUSSION

We left the question of the ''correct'' initial condition unanswered. For a condensed matter black hole it should be straightforward to deduce this by following the state of the field as the ''black hole'' forms. It seems fairly clear that the $\psi_{i,+s}$ wave packets inside the inner horizon will be in their ground states. What is less clear is the state of the wave packets ψ_+, ψ_-^* and ψ_{-s}^* between the horizons. For a real black hole—if one wants to entertain the possibility of superluminal dispersion—the same may be true. The $\psi_{i,+s}$ wave packets inside the inner horizon arise from ingoing waves that scatter around or through the central singularity of a Reissner-Nordström black hole in the manner discussed in [14]. Since the region inside the inner horizon is static, it would seem plausible that these are in their ground state as well.

Another issue we have not touched is that of the gravitational back reaction to the radiation studied here. In the bosonic case the exponential growth of the number of negative energy Hawking partners between the horizons would surely rapidly entail a strong gravitational reaction. In the fermionic case, the exponential suppression of radiation leads quickly to a state with no radiation at all. This is hard to reconcile with the usual picture in which Hawking radiation is a robust consequence of a general ''well-behaved'' state near the horizon. One would expect that although the negative energy states of the Hawking partners in the ergoregion between the horizons become filled, there is not all that much energy in these states (since the partners at late times are the same as the partners at earlier times due to the ''bounce'' between the horizons) so the back reaction should be limited. If so, then why does the Hawking radiation not continue? The answer, it would seem, is that although the state is reasonably well-behaved in terms of energy density, it has peculiar features in just those modes relevant to the Hawking effect.

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APPENDIX: WAVE PACKET SOLUTIONS

In this appendix we explain how the evolution formulas (5.1a)–(5.1d) for wave packet solutions are derived with the help of the results of [5]. We treat only the bosonic case, although the fermionic case is essentially identical.

The wave packet evolution formulas are inferred from connection formulas for mode solutions to the field equation (2.4) of the form

$$u(t,x)=e^{-i\omega t}\phi(x,\omega), \tag{A1}$$

where $\phi(x,\omega)$ satisfies the ordinary differential equation (ODE)

$$-\phi^{(iv)}(x)+[1-v^2(x)]\phi''(x)+2v(x)[i\omega-v'(x)]\phi'(x)$$
$$-i\omega[i\omega-v'(x)]\phi(x)=0. \tag{A2}$$

In [5] such solutions were constructed for a black hole spacetime with a single horizon. The basic technique used was to find approximate solutions to Eq. (A2) using the WKB approximation away from the horizon, and to match these solutions across the horizon by comparing to the near horizon solution obtained by the method of Laplace transforms.

1. Outer horizon connection formulas

Assuming that the horizon is located at $x=0$, and that the metric in the vicinity of the horizon is given by

$$v(x)\approx-1+\kappa x, \tag{A3}$$

the analysis of [5] leads to the following two connection formulas:

$$K(e^{\pi\omega/2\kappa}\phi_+ + e^{-\pi\omega/2\kappa}\phi_-)\leftrightarrow\phi_{+s} \tag{A4a}$$

$$-\phi_{-s}+K(e^{-\pi\omega/2\kappa}\phi_+ + e^{\pi\omega/2\kappa}\phi_-)\leftrightarrow 0, \tag{A4b}$$

where

$$K=[\omega/2\sinh(\pi\omega/\kappa)]^{1/2}. \tag{A5}$$

The notation ''$\phi_1(x)\leftrightarrow\phi_2(x)$'' denotes that the approximate WKB solution $\phi_1(x)$ behind the horizon connects to the approximate WKB solution $\phi_2(x)$ outside the horizon. The modes $\phi_\pm$, $\phi_{\pm s}$ are approximate WKB solutions to Eq. (A2) and are given by[4]

$$\phi_\pm(x)\approx C_\pm[v(x)^2-1]^{-3/4}\exp\left(i\int_{-\epsilon}^{x}ds\,k_\pm(v(s),\omega)\right) \tag{A6}$$

[4]We have changed notation slightly from that in [5]. We have added the lower limit of integration $\pm\epsilon$ to the integrals appearing in the exponents and consequently the coefficients $C_{\pm,\pm s}$ acquire some ϵ dependence to compensate. Furthermore, a factor of i appearing in the matching formulas of [5] has been absorbed in ϕ_- and the phase of N as defined in [5] has been absorbed into $\phi_{\pm s}$. We have also renamed the ϕ_{-m} solution in [5] as ϕ_{-s} here.

$$\phi_{\pm s}(x) \approx C_{\pm s} \exp\left(i \int_{-\epsilon}^{x} ds\, k_{\pm s}(v(s),\omega) \right), \tag{A7}$$

where the approximate WKB wave vectors are given by

$$k_{\pm} \approx \pm k_0 \sqrt{v^2-1} + \omega v/(1-v^2), \tag{A8}$$

$$k_{\pm s} \approx \omega/(1+v), \tag{A9}$$

provided we assume $\omega \ll k_0$ and choose $|x|, \epsilon \gg (\omega/k_0)^{2/3}/\kappa$. The WKB solutions are only valid for $|x| \gg \kappa^{-1/3} k_0^{-2/3}$. The coefficients $C_{\pm,\pm s}$ are necessary to match these WKB solutions to the near-horizon Laplace transform solutions. They can be determined by comparing the Laplace transform solutions given in [5] to the matching formulas (A4a)–(A4b) with the WKB modes (A6),(A7) evaluated in the small x limit.[5] We find that the coefficients are given by

$$C_{\pm} = i^{(1\mp 1)/2} \exp\left(\mp i \frac{2}{3} \sqrt{2\kappa/k_0} (k_0 \epsilon)^{3/2} \right) \times \exp\left(-i \frac{\omega}{2\kappa} \ln(2\kappa\epsilon) \right), \tag{A10}$$

$$C_{\pm s} = \exp\left(i \frac{\omega}{\kappa} \ln(k_0 \epsilon) + i \frac{\pi}{4} - i \arg[\Gamma(1+i\omega/\kappa)] \right). \tag{A11}$$

2. Inner horizon connection formulas

In the case of a black hole with both inner and outer horizons, the connection formulas Eq. (A4a)–(A4b) remain valid locally about each horizon (after some slight modifications to be discussed presently), but the solutions are no longer global. Assume that the outer horizon is located at $x_o = 0$, with $v(x)$ taking the same form as given by Eq. (A3), and that the inner horizon is at $x_i = -a$ with $v(x)$ near the inner horizon taking the form

$$v(x) \approx -1 - \kappa(x+a). \tag{A12}$$

(We assume that the surface gravity of the inner horizon has the same magnitude as that of the outer horizon to simplify the results. There is no difficulty however in allowing the surface gravities to be different.) Then the connection formulas (A4a)–(A4b) are valid for Eq. (A2) locally about the outer horizon, where the notation ''$\phi_1(x) \leftrightarrow \phi_2(x)$'' now denotes that $\phi_1(x)$ is valid between the horizons and $\phi_2(x)$ is valid outside the outer horizon.

To find the ''local'' mode solutions about the inner horizon we reexpress the mode equation (A2) in terms of the new coordinate $y := -(x+a)$. The resulting y-equation is the complex conjugate of the x-equation (A2), with $v(x)$ replaced by $\tilde{v}(y) := v(-y-a)$. We denote the WKB mode solutions to this y-equation by $\tilde{\phi}_{\pm}$ and $\tilde{\phi}_{\pm s}$. Since we have chosen the surface gravities to have the same magnitude, $\tilde{v}(y)$ takes the same form near the inner horizon as $v(x)$ does near the outer horizon (A3). Therefore the mode solutions near the inner horizon are the complex conjugates of those about the outer horizon, with x replaced by y; so they satisfy the same connection formulas

$$K(e^{\pi\omega/2\kappa} \tilde{\phi}_+ + e^{-\pi\omega/2\kappa} \tilde{\phi}_-) \leftrightarrow \tilde{\phi}_{i,+s} \tag{A13a}$$

$$-\tilde{\phi}_{-s} + K(e^{-\pi\omega/2\kappa} \tilde{\phi}_+ + e^{\pi\omega/2\kappa} \tilde{\phi}_-) \leftrightarrow 0. \tag{A13b}$$

The notation ''$\tilde{\phi}_1(y) \leftrightarrow \tilde{\phi}_2(y)$'' now denotes that $\tilde{\phi}_1(y)$ is valid between the horizons and $\tilde{\phi}_2(y)$ is valid inside the inner horizon. We have included a subscript ''i'' in $\tilde{\phi}_{i,+s}(y)$ to make it clear that this solution is valid only inside the inner horizon.

The WKB solutions $\tilde{\phi}_{\pm,-s}(y(x))$ are in fact the same functions of x, up to ω-dependent phases, as the WKB solutions $\phi_{\pm,-s}(x)$ respectively. To see this, note that the WKB modes $\phi_{\pm,-s}(x)$ given in Eqs. (A6),(A7) all take the form[6]

$$\phi(x) = C(\omega) f(v(x)) \exp\left(i \int_{-\epsilon}^{x} ds\, k(v(s),\omega) \right) \tag{A14}$$

where C is x-independent and f and k are real functions of $v(x)$. Since, as discussed above, the x and y equations are related by substituting $v(x) \to \tilde{v}(y) = v(-y-a)$ and complex conjugating, the WKB modes $\tilde{\phi}_{\pm,-s}(y)$ are given by

$$\tilde{\phi}(y) = C^*(\omega) f(\tilde{v}(y)) \exp\left(-i \int_{-\epsilon}^{y} ds\, k(\tilde{v}(s),\omega) \right). \tag{A15}$$

Using $y = -x-a$ and $\tilde{v}(y) = v(x)$, and changing the integration variable inb Eq. (A15) to $s' = -s-a$, we obtain

$$\tilde{\phi}(-x-a) = C^*(\omega) f(v(x)) \exp\left(i \int_{-a+\epsilon}^{x} ds'\, k(v(s'),\omega) \right), \tag{A16}$$

which differs from Eq. (A14) only by an ω-dependent phase factor, i.e.,

$$\tilde{\phi}(-x-a) = \left[\frac{C^*(\omega)}{C(\omega)} \exp\left(i \int_{-a+\epsilon}^{-\epsilon} ds\, k(v(s),\omega) \right) \right] \phi(x). \tag{A17}$$

We shall not attempt to compute these phase factors, but rather shall assume the generic form

[5] x cannot be arbitrarily small however because the WKB approximation breaks down as $x \to 0$. In [5] it was shown that the WKB and Laplace transform approximate solutions are both valid when $\kappa^{-1/3} k_0^{-2/3} \ll |x| \ll \kappa^{-1}$, and therefore the matching can be done in this range. We also choose ϵ to satisfy the same inequality.

[6] We have dropped the $\pm,-s$ subscripts on $\phi(x)$, C, f, and k.

$$\tilde{\phi}_{\pm}(-x-a)=e^{i\theta_{\pm}(\omega)}\phi_{\pm}(x),$$
$$\tilde{\phi}_{-s}(-x-a)=e^{i\gamma(\omega)}\phi_{-s}(x), \tag{A18}$$

where $\theta_{\pm}, \gamma$ also depend on $v(x), k_0$ and a (the coordinate distance between the horizons) but do not depend on ϵ as long as it is chosen large enough so that the WKB approximation holds.

Using the phase relations (A18) the inner horizon connection formulas (A13a),(A13b) can be reexpressed in terms of the same linear combinations of $\phi_{\pm}$ appearing in the outer horizon formulas (A4a),(A4b):

$$K(e^{\pi\omega/2\kappa}\phi_{+}+e^{-\pi\omega/2\kappa}\phi_{-})+e^{i\gamma}T_1\phi_{-s}\leftrightarrow T_2\tilde{\phi}_{i,+s} \tag{A19}$$

$$K(e^{-\pi\omega/2\kappa}\phi_{+}+e^{\pi\omega/2\kappa}\phi_{-})+T_3\phi_{-s}\leftrightarrow T_1\tilde{\phi}_{i,+s} \tag{A20}$$

with

$$T_1=\omega^{-1}K^2(e^{-i\theta_+}-e^{-i\theta_-})$$
$$T_2=\omega^{-1}K^2(e^{\pi\omega/\kappa}e^{-i\theta_+}-e^{-\pi\omega/\kappa}e^{-i\theta_-})$$
$$T_3=\omega^{-1}K^2e^{i\gamma}(e^{-\pi\omega/\kappa}e^{-i\theta_+}-e^{\pi\omega/\kappa}e^{-i\theta_-}). \tag{A21}$$

3. Wave packet evolution formulas

We can now form wave packets from the modes and use the mode connection formulas to obtain wave packet evolution formulas. To keep the latter simple, the ω-dependence of the coefficients in the connection formulas is built into the definition of the wave packets as follows. Define

$$\psi_{n,+s}=\int_{\omega_l}^{\omega_u}\frac{d\omega}{\sqrt{\omega}}A_n\,e^{-i\omega t}\phi_{+s}$$
$$\psi_{n,-s}=\int_{\omega_l}^{\omega_u}\frac{d\omega}{\sqrt{\omega}}B_n\,e^{-i\omega t}\phi_{-s}$$
$$\psi_{n,\pm}=\int_{\omega_l}^{\omega_u}\frac{d\omega}{\sqrt{\omega}}C_{n,\pm}\,e^{-i\omega t}\phi_{\pm}$$
$$\chi_{n-1,\pm}=\int_{\omega_l}^{\omega_u}\frac{d\omega}{\sqrt{\omega}}D_{n-1,\pm}\,e^{-i\omega t}\phi_{\pm}$$
$$\psi_{i,n,+s}=\int_{\omega_l}^{\omega_u}\frac{d\omega}{\sqrt{\omega}}E_n\,e^{-i\omega t}\tilde{\phi}_{i,+s}$$
$$\chi_{i,n-1,+s}=\int_{\omega_l}^{\omega_u}\frac{d\omega}{\sqrt{\omega}}F_{n-1}\,e^{-i\omega t}\tilde{\phi}_{i,+s}, \tag{A22}$$

where the coefficients $A,\dots,F$ depend on ω, and the mode functions ϕ depend on both ω and x.

With these definitions the evolution formulas about the outer horizon (5.1a),(5.1b) follow immediately from the outer horizon connection formulas (A4a),(A4b) provided we choose $C_{n\pm}=K\exp(\pm\pi\omega/2\kappa)\,A_n$ and $D_{n,\pm}=-K\exp(\mp\pi\omega/2\kappa)\,B_{n+1}$. Similarly, the inner horizon evolution formula (5.1c) follows provided $B_n=e^{i\gamma}T_1A_n$ and $E_n=T_2A_n$, while Eq. (5.1d) requires $F_n=-T_1B_{n+1}$ and $B_n=-T_3B_{n+1}$. The solution (up to an undetermined overall constant) is given by

$$A_n=(-T_3)^{-n}$$
$$B_n=e^{i\gamma}T_1(-T_3)^{-n}$$
$$C_{n,\pm}=Ke^{\pm\pi\omega/2\kappa}(-T_3)^{-n}$$
$$D_{n,\pm}=-Ke^{\mp\pi\omega/2\kappa}e^{i\gamma}T_1(-T_3)^{-n-1}$$
$$E_n=T_2(-T_3)^{-n}$$
$$F_n=-e^{i\gamma}(T_1)^2(-T_3)^{-n-1}. \tag{A23}$$

4. Norm of the wave packets

The wave packets defined in Eqs. (A22) are not normalized. Their norms can be determined as follows. A generic one of these wave packets has the form

$$\psi=\int_{\omega_l}^{\omega_u}\frac{d\omega}{\sqrt{\omega}}G\,e^{-i\omega t}\phi, \tag{A24}$$

which has the norm [cf. Eq. (2.9)]

$$(\psi,\psi)=i\int dx\int\frac{d\omega}{\sqrt{\omega}}\int\frac{d\omega'}{\sqrt{\omega'}}\{G_\omega^*G_{\omega'}e^{i(\omega-\omega')t}\times[\phi_\omega^*(\partial_t+v\partial_x)\phi'_\omega-\phi'_\omega(\partial_t+v\partial_x)\phi_\omega^*]\}. \tag{A25}$$

The norm is conserved under time evolution, and so it suffices to evaluate it at any one time.

The key assumption we need in order to evaluate the norm is that at some time the wave packet is confined to a constant v region. This is certainly the case for the $+s$ wave packets, since they are outgoing and eventually reach the asymptotic region. If the region between the horizons is large and has a large constant velocity region, then it may similarly hold for the $-s$ and $\pm$ wave packets as well. Alternatively, these wave packets spend some time squeezed near the horizon, with wavelengths much smaller than the length scale over which $v(x)$ changes (but, in the case of k_{-s}, still much longer than k_0^{-1}; so we can nevertheless use the small k approximation). If the wave packet is contained in a constant v region, then for the purposes of evaluating the norm, we can imagine this region to extend to infinity in both directions. The fixed ω mode equation (A2) in a constant v region has solutions $\phi_\omega=C_\omega\exp(ikx)$, where $\omega-vk=\pm F(k)$ with $F(k)$ given by Eq. (2.7). Matching to the WKB modes (A6),(A7) we see that $|C_\omega| = \{1,(v^2-1)^{-3/4}\}$ for the $\pm s$ modes and $\pm$ modes respectively. Thus we have

$$\int dx\, \phi^*_{\omega'}\phi_\omega = 2\pi\{1,(v^2-1)^{-3/4}\}\left|\frac{d\omega}{dk}\right|\delta(\omega'-\omega). \tag{A26}$$

Using Eq. (A26) in Eq. (A25) yields

$$(\psi,\psi) = 4\pi\{1,(v^2-1)^{-3/2}\}\int d\omega\, |G_\omega|^2 \frac{\pm F}{\omega}\left|\frac{d\omega}{dk}\right|. \tag{A27}$$

Using the small and large k approximations for $k_{\pm s}$ and $k_\pm$ respectively, we find that

$$\{1,(v^2-1)^{-3/2}\}\frac{\pm F}{\omega}\left|\frac{d\omega}{dk}\right| = \pm\{1,\omega^{-1}\} \tag{A28}$$

respectively. Thus, finally,

$$(\psi,\psi) = \pm 4\pi\int d\omega\, |G_\omega|^2\{1,\omega^{-1}\}. \tag{A29}$$

With Eq. (A29) and the coefficients (A23) for the wave packets (A22) we obtain the norms needed in Sec. V B:

$$(\psi_{0,-},\psi_{0,-}) = -4\pi\int_{\omega_l}^{\omega_u} d\omega\, \frac{1}{e^{2\pi\omega/\kappa}-1},$$

$$(\psi_{n,+s},\psi_{n,+s}) = 4\pi\int_{\omega_l}^{\omega_u} d\omega\, |T_3|^{-2n},$$

$$(\psi_{0,-s},\psi_{0,-s}) = -4\pi\int_{\omega_l}^{\omega_u} d\omega\, |T_1|^2$$

$$= -4\pi\int_{\omega_l}^{\omega_u} d\omega\, \frac{2\{1-\cos[\theta_+(\omega)-\theta_-(\omega)]\}}{e^{2\pi\omega/\kappa}+e^{-2\pi\omega/\kappa}-2}. \tag{A30}$$

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