Simulating four-dimensional simplicial gravity using degenerate triangulations

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We extend a model of four-dimensional simplicial quantum gravity to include *degenerate* triangulations in addition to the combinatorial triangulations traditionally used. Relaxing the constraint that every 4-simplex is uniquely defined by a set of five distinct vertices, we allow triangulations containing multiply connected simplexes and distinct simplexes defined by the same set of vertices. We demonstrate numerically that including degenerated triangulations substantially reduces the finite-size effects in the model. In particular, we provide strong numerical evidence for an exponential bound on the entropic growth of the ensemble of degenerate triangulations, and show that a discontinuous crumpling transition is already observed on triangulations of volume $N_4 \approx 4000$. [S0556-2821(99)01812-3]

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Discretized models of four-dimensional Euclidean quantum gravity, known as simplicial gravity or dynamical triangulations, have received ample attention in recent years, the hope being that in a suitable scaling limit they might provide a sensible nonperturbative definition of quantum gravity. In the simplicial gravity approach the integration over metrics is replaced by a sum over an ensemble of triangulations constructed by all possible gluings of equilateral 4-simplexes into closed (piece-wise linear) simplicial manifolds (see, e.g., Refs. [1,2]). The regularized Euclidean Einstein-Hilbert action is particularly simple; it can be taken to depend on only two coupling constants, μ and κ, related to the cosmological and the inverse Newton's constants. The coupling constants are conjugate to the volume—the number of 4-simplexes—and the number of triangles in a given triangulation respectively. The regularized grand-canonical partition function thus becomes

$$Z(\mu,\kappa)=\sum_{T\in\mathcal{T}}\frac{1}{C_T}e^{-\mu N_4+\kappa N_2}. \tag{1}$$

The sum is over all distinct triangulations $T\in\mathcal{T}$, N_i is the number of i-simplexes in a triangulation T and C_T denotes its symmetry factor—the number of equivalent labeling of the vertices.

Extensive numerical simulations have established that the model Eq. (1) has a strong-coupling (small κ) crumpled phase and a weak-coupling (large κ) elongated phase, separated by a discontinuous phase transition. In the crumpled phase the internal geometry collapses and is dominated by a novel *singular* structure—two singular vertices connected to an extensive fraction of the total volume and joined by a subsingular edge [3]. The elongated phase, on the other hand, is dominated by branched polymer like triangulations, i.e., bubbles glued together *via* small necks into a treelike structure.

In Eq. (1), $\mathcal{T}$ denotes a suitable ensemble of triangulations included in the partition function. Different ensembles are defined by imposing various restrictions on how the simplexes are glued together. Provided this leads to a well-defined partition function, and as long as the difference is only at the level of discretization, one expects different choices of $\mathcal{T}$ to lead to the same continuum theory in the thermodynamic limit. This is known to be true in two dimensions where models of simplicial gravity corresponding to different choices of $\mathcal{T}$ are soluble as matrix models [5]. Even in two dimensions, however, for the partition function to be convergent the topology of the triangulations included in the sum Eq. (1) must be fixed, regardless of the ensemble used. As the same most likely is true in higher dimensions, in this paper we only consider triangulations of fixed spherical topology.

All simulations of four-dimensional simplicial gravity have, as of yet, used an ensemble of *combinatorial* triangulations $\mathcal{T}_C$. In a combinatorial triangulation every D-simplex is uniquely defined by a set of $(D+1)$ distinct vertices—it is said to be combinatorially unique. In this letter we study a larger ensemble of *degenerate* triangulations $\mathcal{T}_D$ where we relax this constraint and allow distinct simplexes to be defined by the same set of vertices. This includes two simplexes with more than one face in common. We do, however, retain the restriction that every 4-simplex is defined by a set of five distinct vertexes, i.e., we exclude degenerate simplexes. Clearly $\mathcal{T}_C \subset \mathcal{T}_D$. This corresponds to an ensemble of restricted degenerate triangulations as defined in Ref. [6].

The benefits of using a larger ensemble of triangulations are well known from simulations of two-dimensional simplicial gravity which have demonstrated that less restricted the triangulations are translates into smaller finite-size effects [7]. Recently the same observation has been made in three dimensions [6]. As simulations of four-dimensional simplicial gravity are notoriously time-consuming, primarily due to the large volumes needed to observe any ''true'' infinite-volume behavior, any reduction in the finite-size effects is of great practical importance.

In this paper we show that including degenerates triangulations in simulations of four-dimensional simplicial gravity likewise leads to reduced finite-size effects. This reduction is most pronounced in the crumpled phase of the model where the free-energy density of the canonical (fixed volume) ensemble—the pseudo-critical cosmological constant $\mu_c(N_4)$—converges very rapidly to an infinite-volume value. This in turn implies an exponential bound on the entropic growth of the ensemble of degenerate triangulations, something that has been the subject of some controversy in the past for combinatorial triangulations [8,9]. Although we observe qualitatively the same phase structure as with the model Eq. (1) restricted to combinatorial triangulations, there are some dissimilarities. A discontinuous phase transition, separating the elongated and the crumpled phases, is already observed on triangulations of relatively modest size, $N_4 \approx 4000$, compared to combinatorial triangulations where a volume of $N_4 \approx 32.000$ is needed. And while the crumpled phase is still dominated by a singular structure, for degenerate triangulations this corresponds to a *gas* of subsingular vertices rather than to only two singular vertexes.

We have simulated the model Eq. (1) using degenerate triangulations on volumes up to 32.000 4-simplexes using Monte Carlo methods. As customary we work in a quasicanonical ensemble of spherical manifolds with almost fixed N_4:

$$Z(\mu,\kappa;\overline{N}_4)=\sum_{N_4} e^{-\mu N_4-\delta(N_4-\overline{N}_4)^2}\Omega_{N_4}(\kappa), \tag{2}$$

where $\Omega_{N_4}(\kappa)=\sum_{T\in\mathcal{T}(N_4)}\exp(\kappa N_2)$ is the canonical partition function. As there do not exist ergodic volume conserving local moves, hence the canonical ensemble cannot be simulated directly, we must allow the volume to fluctuate. The quadratic potential term added to the action ensures, for an appropriate choice of δ, that these fluctuations are small.

In the simulations the triangulation space is explored using a set of local geometric changes, the (p,q) moves. In a (p,q) move, where $p=D+1-q$, a $(q-1)$ subsimplex in the triangulation is replaced by its ''dual'' $(p-1)$ subsimplex. For combinatorial triangulations the (p,q) moves are known to be ergodic for $D\leqslant 4$ [10]. To demonstrate that the same holds true for degenerate triangulations we observe that, just as in three dimensions [6], every set of combinatorially equivalent simplexes, or subsimplexes, can be made distinct by a finite sequence of the (p,q) moves. Thus every degenerate triangulation can be reduced to a combinatorial one. In addition the local nature of the (p,q) moves prohibits the creation of pseudo-manifolds in the simulations, i.e., triangulations containing vertexes with a neighborhood not homeomorphic to the D ball.

From a practical point of view simulating degenerate triangulations is actually simpler than simulating their combinatorial counterpart as one avoids the nonlocal manifold checks necessary to exclude combinatorially equivalent simplexes. For combinatorial triangulations these checks are the most time-consuming part of the simulations [11]. This simplification of is particularly beneficial in the crumpled phase where the singular structure dominates; in this phase we observe a tenfold reduction in the effective autocorrelation times (measured in ''real'' time) when using degenerate instead of combinatorial triangulations. In the branched polymer phase, on the other hand, the autocorrelation times appear comparable for the two ensembles.

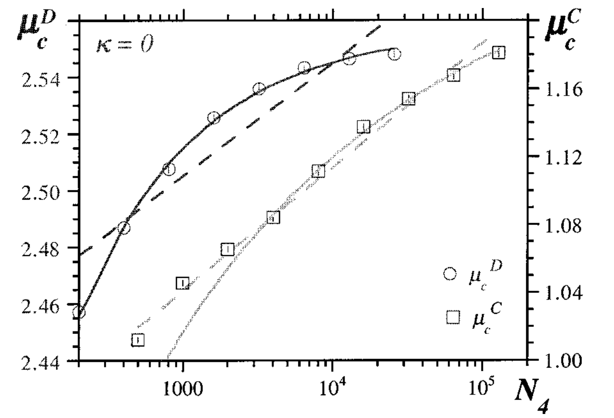

FIG. 1. The pseudo-critical cosmological constant $\mu_c^D(N_4)$ for an ensemble of degenerate triangulations, together with fits assuming a power-law convergence, Eq. (3) (solid curve), or a logarithmic divergence, Eq. (4) (dashed curve). Also included are the corresponding values, $\mu_c^C(N_4)$, for an ensemble of combinatorial triangulations.

The real benefit of using degenerate triangulations is the reduction of geometric finite-size effects. This reduction is most striking for the volume dependence of the pseudo-critical cosmological constant, $\mu_c^D(N_4)$, which we shown in Fig. 1. For comparison we also include the corresponding values $\mu_c^C(N_4)$, for combinatorial triangulations. For degenerate triangulations we observe a rapid convergence to an infinite volume value $\overline{\mu}$. This can be quantified by comparing the fit of $\mu_c^D(N_4)$ to two different functional forms: a weak power-law convergence,

$$\mu_c(N_4)=\overline{\mu}+\frac{b}{N_4^{\gamma}}, \tag{3}$$

or a logarithmic divergence,

$$\mu_c(N_4)=\overline{\mu}+b' \log N_4 . \tag{4}$$

The fit parameters and the quality of the fits are shown in Table I. In contrast to combinatorial triangulations for degenerate triangulations there is no comparison in the quality of the fits; the latter, which corresponds to a divergent partition function Eq. (1), is ruled out by a $\chi^2/(\mathrm{DOF})\approx 117$. For combinatorial triangulations, on the other hand, it is difficult to use the quality of the fit of $\mu_c^C(N_4)$ to either Eq. (3) or Eq. (4) to distinguish between those two scenarios (see, e.g., Ref. [9]).

TABLE I. The parameters in the fit of the pseudo-critical cosmological constant $\mu_c^D(N_4)$ to Eqs. (3) and (4), respectively. Measurements on volumes $N_4=400$ to 25.600 are included in the fits.

	$\overline{\mu}$	γ	χ^2/DOF
Eq. (3)	2.556(3)	0.55(5)	[3.8]
Eq. (4)	2.385(4)		[117]

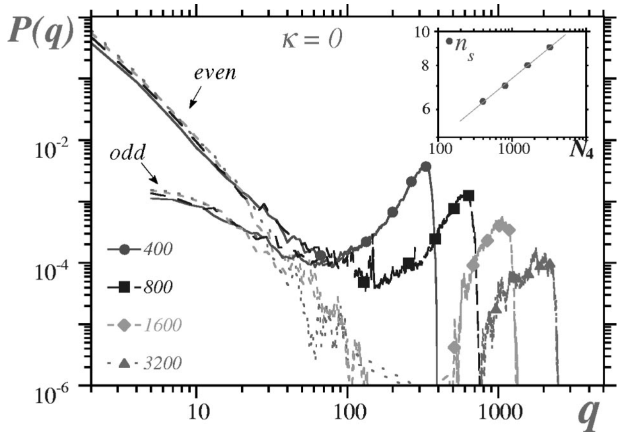

FIG. 2. The (normalized) distributions of local volumes, $P(q)$, for degenerate triangulations. This is for $400 \leqslant N_4 \leqslant 3200$, and for $\kappa = 0$. (Inset) The number, n_s, of subsingular vertices versus volume.

The importance of this result is that it provides strong numerical evidence for an exponential bound on the entropic growth of the ensemble of degenerate triangulations as a function of volume—a necessary condition for a well-defined partition function Eq. (1). And, as $\mathcal{T}_C \in \mathcal{T}_D$, this implies an exponential bound on the number of combinatorial triangulations as well.

The origin of the large finite-size effects present in simulations with combinatorial triangulations lies in the nature of the quantum geometry in the crumpled phase. As stated, the partition function is dominated by triangulations characterized by two singular vertexes connected by a sub-singular edge. A singular vertex has a local volume q—the number of 4-simplexes containing that vertex—which grows linearly with the volume of the manifold, while the subsingular edge has a local volume which grows like N_4^{α}, $\alpha \approx 2/3$ [3]. However, for combinatorial triangulations this singular structure only dominates on large enough volumes, on small volumes triangulations with only one singular vertex have larger entropy. This results in a cross-over behavior in the fractal structure at $N_4 \approx 1000$, as can be observed in Fig. 1.

For degenerate triangulation the crumpled phase is likewise dominated by a singular structure. This is evident from the probability distribution $P(q)$ of the local volumes which contains an isolated peak in the tail. This is shown in Fig. 2 for $\kappa = 0$. However, the distribution $P(q)$ differs in two respects from the corresponding distribution measured on combinatorial triangulations [3].

(a) The peak corresponds to not just two singular vertices but rather to several subsingular vertices, i.e., vertices with local volumes that scale like N_4^{α}, $\alpha < 1$. A rough estimate yields $\alpha \approx 0.9$. The number of these sub-singular vertices, n_s, increases logarithmically with the volume as is shown in the inset in Fig. 2. This suggests that the crumpled phase is dominated by a *gas* of subsingular vertices.

(b) For each volume, $P(q)$ effectively separates into two distinct distributions depending on whether the local volume is even or odd.

It is not clear though how much significance should be attached to this difference in the singular structure. Because of the collapsed nature of the internal geometry it is unlikely that any sensible continuum limit exists in the crumpled phase, hence there is no reason to expect identical scaling behavior for the two different ensembles. The details of the discretization may still be important in the thermodynamic limit for $\kappa < \kappa_c$. It is, however, worth noticing that for degenerate triangulations we do not observe any change in the fractal structure as the volume is increased as for combinatorial triangulations.

Additional evidence of a collapsed intrinsic geometry in the crumpled phase comes from the (absence of) volume scaling of the simplex-simplex distribution $s(r)$, i.e., the number of simplexes at a geodesic distance r from a marked simplex. Using the scaling ansatz $s(r) = N_4^{1-1/d_H} F(x)$, where $x = r/N_4^{1/d_H}$ [12,13], we tried to collapse distributions $s(r)$ measured on different volumes onto a single scaling curve. This was though not possible with an ''acceptable'' collapse [with $\chi^2/(\mathrm{DOF})$ of order unity]; moreover, the estimate of the fractal dimension d_H appeared to increase with the volume. From this we conclude that $d_H = \infty$ in the crumpled phase.

We have also investigated the phase structure of the model for nonzero values of the inverse Newton's constant κ. As for combinatorial triangulations we observe a phase transition to an elongated phase at $\kappa_c \approx 1.7$. To establish the nature of the phase transition we have studied the Monte Carlo time series of the energy density, $n_0 = N_0/N_4$, in the critical region. We show an example of one such time series

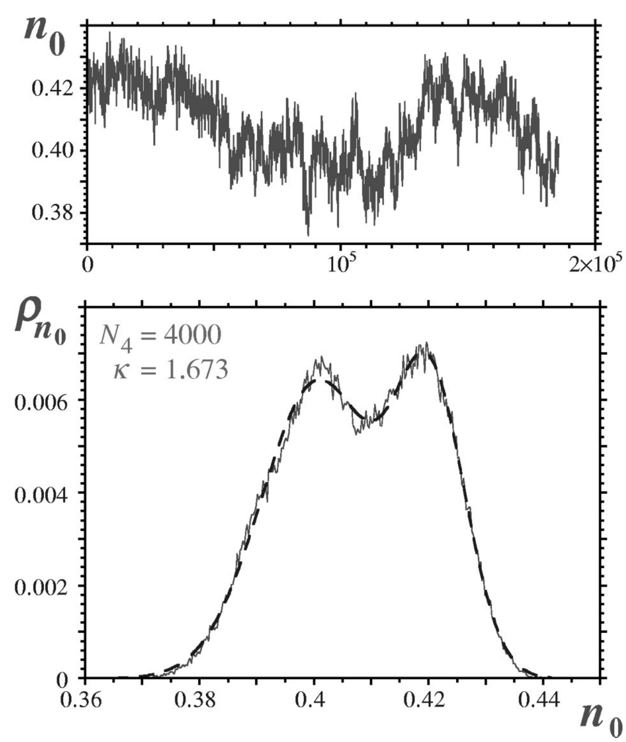

FIG. 3. (Top) A MC time series of the energy density, $n_0 = N_0/N_4$, for $N_4 = 4000$ and $\kappa = 1.673$. (Bottom) The corresponding histogram together with a fit to a function composed of two Gaussian peaks (dashed line).

in Fig. 3 together with the corresponding histogram. This is for volume $N_4 = 4000$ and $\kappa = 1.673$. The histogram shows a clear double-peak structure characteristic of a discontinuous phase transition. Note that in order to observe the corresponding two-state signal using the combinatorial ensemble, triangulations of volume $N_4 \approx 32.000$ are needed [4].

For $\kappa > \kappa_c$ the model is in an elongated or branched polymer phase. This we have established by measuring the fractal dimensions d_H and the spectral dimension d_s for $\kappa = 2$. The former is determined from the scaling of the simplex-simplex distribution, the latter from the return probability of a walker on the dual graph, $p(t) \sim t^{-d_s/2}$ [13]. Including measurements on volumes $N_4 = 400$ to 1600, we get $d_H = 1.9(2)$ and $d_s = 1.32(5)$, in excellent agreement with $d_H = 2$ and $d_s = 4/3$ as expected for branched polymer.

In this paper we have demonstrated that including degenerate triangulations in simulations of four-dimensional simplicial gravity has many potential advantages over the model restricted to combinatorial triangulations. This agrees with the same observations previously made in both two and three dimensions. The chief benefit is the reduction in geometric finite-size effects mainly due to an enlarged ensemble—with a larger triangulation space the infinite-volume fractal structure is more easily approximated on small volumes. The most important result presented in this letter is a strong numerical evidence for an exponential bound on the entropic growth of the canonical partition function Ω_{N_4}. A more practical result is the observation of a discontinuous phase transition on triangulations of relatively modest size.

A natural extension of the work presented in this paper is to investigate how the phase structure is affected if the model Eq. (1) is changed either by using a modified measure [14] or by adding matter fields [15]. For combinatorial triangulations it has recently been observed that this can substantially alter the phase structure and, for a suitable modification, a new *crinkled* phase appears [16]. If this observed phase structure corresponds to a genuine change in the continuum behavior of the model Eq. (1), one expects on basis of universality that it should be independent of the ensemble of triangulations used.

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[1] F. David, ''Simplicial quantum gravity and random lattices,'' Lectures given at Les Houches Summer School on Gravitation and Quantizations, Session LVII, Les Houches, France, 1992, hep-th/9303127.
[2] J. Ambjørn, B. Durhuus, and T. Jonsson, *Quantum Geometry: A Statistical Field Theory Approach* (Cambridge University Press, Cambridge, England, 1997).
[3] T. Hotta, T. Izubuchi, and J. Nishimura, Prog. Theor. Phys. **94**, 263 (1995); S. Catterall, G. Thorleifsson, J. Kogut, and R. Renken, Nucl. Phys. **B468**, 263 (1996); S. Catterall, R. Renken, and J. Kogut, Phys. Lett. B **416**, 274 (1998).
[4] P. Bialas, Z. Burda, A. Krzywicki, and B. Petersson, Nucl. Phys. **B472**, 293 (1996).
[5] E. Brezin, C. Itzykson, G. Parisi, and J. B. Zuber, Commun. Math. Phys. **59**, 35 (1978).
[6] G. Thorleifsson, Nucl. Phys. **B538**, 278 (1999).
[7] J. Ambjørn, G. Thorleifsson, and M. Wexler, Nucl. Phys. **B439**, 187 (1995); M. Bowick, S. Catterall, and G. Thorleifsson, Phys. Lett. B **391**, 305 (1997).
[8] S. Catterall, J. Kogut, and R. Renken, Phys. Rev. Lett. **72**, 4062 (1994); J. Ambjørn and J. Jurkiewicz, Phys. Lett. B **335**, 355 (1994); B. Brugmann and E. Marinari, *ibid.* **349**, 35 (1995).
[9] S. Catterall, J. Kogut, R. Renken, and G. Thorleifsson, Phys. Lett. B **366**, 72 (1996).
[10] M. Gross and S. Varsted, Nucl. Phys. **B378**, 367 (1992).
[11] S. Catterall, Comput. Phys. Commun. **87**, 409 (1995).
[12] S. Catterall, G. Thorleifsson, M. Bowick, and V. John, Phys. Lett. B **354**, 56 (1995).
[13] J. Ambjørn, J. Jurkiewicz, and Y. Watabiki, Nucl. Phys. **B454**, 313 (1995).
[14] B. Brugmann and E. Marinari, Phys. Rev. Lett. **70**, 1908 (1993).
[15] J. Ambjørn, Z. Burda, J. Jurkiewicz, and C. Kristjansen, Acta Phys. Pol. B **23**, 991 (1992).
[16] S. Bilke, Z. Burda, A. Krzywicki, B. Petersson, J. Tabaczek, and G. Thorleifsson, Phys. Lett. B **418**, 266 (1998); **432**, 279 (1998).