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Thermodynamics of Reissner–Nordström–anti-de Sitter black holes in the grand canonical ensemble

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The thermodynamical properties of the Reissner–Nordström–anti-de Sitter black hole in the grand canonical ensemble are investigated using York's formalism. The black hole is enclosed in a cavity with a finite radius where the temperature and electrostatic potential are fixed. The boundary conditions allow one to compute the relevant thermodynamical quantities, e.g., thermal energy, entropy, and charge. The stability conditions imply that there are thermodynamically stable black hole solutions, under certain conditions. By taking the boundary to infinity, and leaving the event horizon and charge of the black hole fixed, one rederives the Hawking-Page action and Hawking-Page specific heat. Instantons with negative heat capacity are also found.
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I. INTRODUCTION

The path-integral approach to the thermodynamics of black holes was originally developed by Hawking *et al.* [1–3]. In this approach the thermodynamical partition function is computed from the path integral in the saddle-point approximation, thus obtaining the thermodynamical laws for black holes.

In the path-integral approach we can use the three different ensembles: microcanonical, canonical, and grand canonical. Because of difficulties related to the stability of the black hole in the canonical ensemble, the microcanonical ensemble was originally considered [3,4]. However, further developments by York *et al.* [5–8] allowed us to define the canonical ensemble. Effectively, by carefully defining the boundary conditions, one can obtain the partition function of a black hole in thermodynamical equilibrium. This approach was further developed to include other ensembles [10], and to study charged black holes in the grand canonical ensemble [11] and black holes in asymptotically anti–de Sitter spacetimes [12,9,13]. This approach was also applied to black holes in two [14] and three [12] dimensions.

In York's formalism the black hole is enclosed in a cavity with a finite radius. The boundary conditions are defined according to the thermodynamical ensemble under study. Given the boundary conditions and imposing the appropriate constraints, one can compute a reduced action suitable for doing black hole thermodynamics [11,15]. Evaluating this reduced action at its stable stationary point one obtains the corresponding classical action, which is related to a thermodynamical potential. In the canonical ensemble this thermodynamical potential corresponds to the Helmholtz free energy, while for the grand canonical ensemble the thermodynamical potential is the grand canonical potential [2,11]. From the thermodynamical potential one can compute all the relevant thermodynamical quantities and relations [16].

Some controversy has appeared related to the boundary conditions chosen in this formalism [12,17,18]. More precisely, Hawking and Page [17,18] fix the Hawking temperature of the black hole (i.e., the temperature defined so that the respective Euclidean metric has no conical singularity at the horizon), while York *et al.* [5,11,12] fix the local temperature at a finite radius, where the boundary conditions are defined. For asymptotically flat spacetimes the two formalisms coincide, because at infinity the local temperature is equal to the Hawking temperature. On the contrary, for asymptotically anti–de Sitter spacetimes the two procedures disagree, since the local temperature is redshifted to zero at infinity and is equal to the Hawking temperature only in the region where spacetime has a flat metric. Louko and Winters-Hilt [13] have studied the thermodynamics of the Reissner–Nordström–anti-de Sitter black hole fixing a renormalized temperature at infinity that corresponds to the same procedure used in Refs. [17,18]. In this paper we have chosen to follow York's formalism [5,11,12] and study the thermodynamics of the Reissner–Nordström–anti-de Sitter black hole fixing the local temperature at finite radius.

We find that the two procedures give some identical results, e.g., in both procedures the Hawking-Bekenstein formula for the entropy [19,20] is obtained. In addition, by that taking the boundary to infinity, and leaving the event horizon and charge of the black hole fixed, we rederive the Hawking-Page action and Hawking-Page specific heat from York's formalism. However, the value for the energy at infinity differs depending on which procedures one uses. In Ref. [13] it was found that the energy at infinity is equal to the mass of the black hole, a result that does not hold here. These results conform with the similarities and differences found for the Schwarzschild–anti-de Sitter black hole in Refs. [17,12].

In Sec. II we briefly introduce York's formalism. In Sec. III we compute the reduced action for the Reissner–Nordström–anti-de Sitter black hole and evaluate its thermodynamical quantities. In Sec. IV we analyze the black hole solutions. In Sec. V we study the local and global stability of these solutions. The limit where the boundary is taken to infinity mentioned above is studied in Sec. VI. Finally some special cases are briefly referred in Sec. VII.

II. THE ACTION

The Euclidean Einstein-Maxwell [3] is given by

$$I=-\frac{1}{16\pi}\int_{\mathcal{M}}d^4x\,\sqrt{g}(R-2\Lambda)+\frac{1}{8\pi}\int_{\partial\mathcal{M}}d^3x\,\sqrt{h}K$$
$$-\frac{1}{16\pi}\int_{\mathcal{M}}d^4x\,\sqrt{g}F_{\mu\nu}F^{\mu\nu}-I_{\rm subtr}, \tag{1}$$

where $\mathcal{M}$ is a compact region with boundary $\partial\mathcal{M}$, R is the scalar curvature, Λ the cosmological constant, g the determinant of the Euclidean metrics, K the trace of the extrinsic curvature of the boundary $\partial\mathcal{M}$, h is the determinant of the Euclidean induced metrics on the boundary, $F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu$ is the Faraday tensor, and $I_{\rm subtr}$ is an arbitrary term that can be used to define the zero of the energy as will be seen later.

In order to set the nomenclature we follow [11] in this section. We consider a spherical symmetric static metric of the form [11]

$$ds^2=b^2d\tau^2+a^2dy^2+r^2d\Omega^2, \tag{2}$$

where a, b, and r are functions only of the radial coordinate $y\in[0,1]$. The Euclidean time τ has period 2π. The event horizon, given by $y=0$, has radius $r_+=r(0)$ and area $A_+=4\pi r_+^2$. The boundary is given by $y=1$ and at this boundary the thermodynamical variables defining the ensemble are fixed. The boundary is a two sphere with area $A_B=4\pi r_B^2$, where $r_B=r(1)$. We will consider the grand canonical ensemble, where heat and charge can flow in and out through the boundary to maintain a constant temperature $T\equiv T(r_B)$ and electrostatic potential $\phi\equiv\phi(r_B)$ at the boundary. We impose a black hole topology to the metric (2), by using the conditions, $b(0)=0$, $b'/a|_{y=0}=1$ and $(r'/a)^2|_{y=0}=0$, see Ref. [11].

Evaluating the action (1) for the metric (2),

$$I=\frac{1}{2}\int_0^{2\pi}d\tau\int_0^1 dy\left(-2\frac{rb'r'}{a}-\frac{br'^2}{a}-ab+\Lambda abr^2\right)$$
$$-\frac{1}{2}\int_0^{2\pi}d\tau\frac{(br^2)'}{a}\bigg|_{y=0}-\frac{1}{2}\int_0^{2\pi}d\tau\int_0^1 dy\,\frac{r^2}{ab}A_\tau'^2-I_{\rm subtr}. \tag{3}$$

In order to obtain the reduced action one uses the proper constraints. For the gravitational part of the action I_g given in Eq. (3), the constraint used is the Hamiltonian constraint [11,15]

$$G^\tau{}_\tau+\Lambda g^\tau{}_\tau=8\pi T^\tau{}_\tau. \tag{4}$$

In addition, for the matter fields part of the action we use Maxwell equations $F^{\mu\nu}{}_{;\nu}=0$.

The thermodynamical quantities and relations are obtained from the ''classical action'' $\tilde{I}$ (defined as the reduced action evaluated at its locally stable stationary points) using the well known relation between the ''classical action'' and the thermodynamical potential

$$\tilde{I}=\beta F. \tag{5}$$

Here F is the grand canonical potential since we are considering the grand canonical ensemble. All the thermodynamical quantities can be obtained from F using the classical thermodynamical relations (see, for example, Ref. [16]).

III. THE REISSNER–NORDSTRÖM–ANTI-de SITTER BLACK HOLE

The Reissner–Nordström–anti-de Sitter black hole in the grand canonical ensemble is obtained using a negative cosmological constant Λ and the boundary conditions $T\equiv T(R_B)$ and $\phi\equiv\phi(r_B)$, where r_B is the boundary radius of the spherical cavity, T the temperature at the boundary and ϕ the electrostatic potential difference between the horizon and the boundary. Instead of T we can also use its inverse β.

The reduced action for Reissner–Nordström–anti-de Sitter black hole is obtained from the Euclidean Einstein-Hilbert-Maxwell action I given in Eq. (1). For simplicity we split the action in two terms $I=I_g+I_m$, where I_g is the gravitational term and I_m the matter field term. To obtain the reduced action we use the Hamiltonian constraint (4) and the Maxwell equations.

The evaluation of I_m is identical to the case $\Lambda=0$ and can therefore be found in Ref. [11]:

$$I_m=-\tfrac{1}{2}\beta e\phi, \tag{6}$$

where e is the electrical charge of the black hole and ϕ is the difference of potential between $y=0$ and $y=1$. To evaluate the gravitational term I_g (3) we use as mentioned above the constraint (4):

$$G^\tau{}_\tau+\Lambda g^\tau{}_\tau=8\pi T^\tau{}_\tau. \tag{7}$$

The component of the Einstein tensor $G^\tau{}_\tau$ for the metric (2) is

$$G^\tau{}_\tau=\frac{r'^2}{a^2r^2}-\frac{1}{r^2}+\frac{2r''}{a^2r}-\frac{2a'r'}{a^3r}. \tag{8}$$

The stress-energy tensor component $T^\tau{}_\tau$ is given by

$$T^\tau{}_\tau=\frac{1}{8\pi}\left(\frac{A_\tau{}'}{ab}\right)^2=-\frac{1}{8\pi}\frac{e^2}{r^4}. \tag{9}$$

Substituting Eqs. (9) and (8) in Eq. (7) we obtain

$$\Lambda=\left(\frac{A_\tau{}'}{ab}\right)^2-\frac{r'^2}{a^2r^2}+\frac{1}{r^2}-\frac{2r''}{a^2r}+\frac{2a'r'}{a^3r}. \tag{10}$$

Rearranging terms in Eq. (10) and using Eq. (9) we obtain

$$\frac{1}{r^2 r'}\left\{\left[r\left(\frac{r'^2}{a^2}-1\right)\right]'+\frac{e^2 r'}{r^2}+\Lambda r^2 r'\right\}=0. \tag{11}$$

Integrating and simplifying the previous equation yields

$$\left(\frac{r'}{a}\right)^2=1-\frac{2M}{r}+\frac{e^2}{r^2}+\alpha^2 r^2, \tag{12}$$

where $2M$ is an integration constant and $\alpha^2=-\Lambda/3$. The integration constant $2M$ can be evaluated using the black hole topology condition $(r'/a)^2|_{y=0}=0$:

$$2M=r_+ +\frac{e^2}{r_+}+\alpha^2 r_+^3. \tag{13}$$

This is the known relation between the Arnowitt-Deser-Misner (ADM) mass of the Reissner–Nordström–anti-de Sitter black hole and its event horizon radius.

Substituting Eq. (10) in Eq. (3) yields

$$\begin{aligned} I_g^*=&\frac{1}{2}\int_0^{2\pi}d\tau\int_0^1 dy\left(-2\frac{rb'r'}{a}-2\frac{br'^2}{a}\right.\\ &\left.-2\frac{brr''}{a}+2\frac{bra'r'}{a^2}-\frac{r^2}{ab}\;A_\tau'^2\right)\\ &-\frac{1}{2}\int_0^{2\pi}d\tau\frac{(br^2)'}{a}\bigg|_{y=0}-I_{\mathrm{subtr}}\\ =&-\int_0^{2\pi}d\tau\int_0^1 dy\left(\frac{brr'}{a}\right)'\\ &-\frac{1}{2}\beta e\phi-\frac{1}{2}\int_0^{2\pi}d\tau\frac{b'r^2}{a}\bigg|_{y=0}-I_{\mathrm{subtr}}, \end{aligned} \tag{14}$$

where we have used the topology conditions given in Sec. II and to evaluate the term in A_τ, we used Eq. (6), since this term is identical to I_m.

The first term after the second equality in Eq. (14) can be evaluated by integrating and substituting Eqs. (12) and (13). The respective third term is integrated and using the topology conditions gives $-\pi r_+^2$. Following this procedure, we obtain

$$I_g^*=-\beta r_B f(r_B;r_+,e,\alpha)-\tfrac{1}{2}\beta e\phi-\pi r_+^2-I_{\mathrm{subtr}}, \tag{15}$$

where the inverse temperature at the boundary β is given by the proper length of the time coordinate at the boundary $\beta\equiv T^{-1}=\int_0^{2\pi}b(1)d\tau=2\pi b(1)$ and

$$f(r_B;r_+,e,\alpha)=\sqrt{1-\frac{r_+}{r_B}-\frac{e^2}{r_+r_B}-\alpha^2\frac{r_+^3}{r_B}+\frac{e^2}{r_B^2}+\alpha^2 r_B^2}. \tag{16}$$

Adding Eqs. (6) and (15), yields the reduced action

$$I^*=-\beta r_B f(r_B;r_+,e,\alpha)-\beta e\phi-\pi r_+^2-I_{\mathrm{subtr}}. \tag{17}$$

The term I_{subtr} is of the form βE_{subtr}, where E_{subtr} is a constant that does not depend on β or ϕ, since I_{subtr} is an arbitrary term that can be used to fix the zero of the energy but cannot affect other thermodynamical variables [5]. For convenience, we use for the zero of the energy

$$E_{\mathrm{ADS}}=E(r_+=0,e=0)=0, \tag{18}$$

where E_{ADS} is the thermal energy of anti–de Sitter spacetime.

To evaluate I_{subtr}, we compute the thermal energy of the Reissner–Nordström–anti-de Sitter black hole from Eq. (17) and use condition (18). The thermal energy is given by [16]

$$\begin{aligned} E=&F+\beta\left(\frac{\partial F}{\partial\beta}\right)_{\phi,r_B}-\left(\frac{\partial F}{\partial\phi}\right)_{\beta,r_B}\phi=\left(\frac{\partial\tilde I}{\partial\beta}\right)_{\phi,r_B}-\frac{\phi}{\beta}\left(\frac{\partial\tilde I}{\partial\phi}\right)_{\beta,r_B}\\ =&-r_B f(r_B;r_+,e,\alpha)-E_{\mathrm{subtr}}, \end{aligned} \tag{19}$$

where F is the grand canonical potential and we have used Eq. (5). Although the reduced action I^* is not the classical action (therefore we cannot write $I^*=\beta F$), the energy has the form given in Eq. (19). This is because the classical action $\tilde I$ corresponds to the minimum of the reduced action and therefore the equalities $(\partial\tilde I/\partial\beta)_{\phi,r_B}=(\partial I^*/\partial\beta)_{\phi,r_B,r_+,e}$ and $(\partial\tilde I/\partial\phi)_{\beta,r_B}=(\partial I^*/\partial\phi)_{\beta,r_B,r_+,e}$ hold. However, r_+ and e in Eq. (19) are not free parameters, they depend on the boundary conditions (i.e., on the values of β, ϕ, and r_B) and on the cosmological constant. The functions $r_+=r_+(\beta,\phi,r_B,\alpha)$ and $e=e(\beta,\phi,r_B,\alpha)$ are obtained from the equilibrium conditions $\partial I^*/\partial r_+=0$ and $\partial I^*/\partial e=0$ as will be seen later.

Using Eq. (18) on Eq. (19), yields

$$E_{\mathrm{subtr}}=-r_B f_0(r_B;\alpha), \tag{20}$$

where $f_0(r_B;\alpha)=f(r_B;0,0,\alpha)=\sqrt{1+\alpha^2 r_B^2}$. Substituting Eq. (20) in Eq. (17), we finally obtain the reduced action for the Reissner–Nordström–anti-de Sitter black hole

$$I^*=\beta r_B[f_0(r_B;\alpha)-f(r_B;r_+,e,\alpha)]-\beta e\phi-\pi r_+^2. \tag{21}$$

Similarly substituting Eq. (20) in Eq. (19), we obtain its thermal energy

$$E=r_B[f_0(r_B;\alpha)-f(r_B;r_+,e,\alpha)]. \tag{22}$$

The mean value of the charge is

$$Q=-\left(\frac{\partial F}{\partial\phi}\right)_{\beta,r_B}=-\frac{1}{\beta}\left(\frac{\partial\tilde I}{\partial\phi}\right)_{\beta,r_B}=e. \tag{23}$$

The entropy is obtained from

$$S=\beta^2\left(\frac{\partial F}{\partial\beta}\right)_{\phi,r_B}=\beta\left(\frac{\partial\tilde I}{\partial\beta}\right)_{\phi,r_B}-\tilde I=\pi r_+^2, \tag{24}$$

where Eq. (5) was used. Since $A_+/4=\pi r_+^2$, where A_+ is the area of the event horizon, this is the usual Hawking-Bekenstein entropy [19,20].

As mentioned above, the event horizon radius r_+ and electric charge e of the black hole for the given boundary conditions, i.e., β, ϕ, and r_B, are obtained by evaluating the locally stable stationary points of the reduced action with respect to r_+ and e [11]. Effectively, once the values of β, ϕ, and r_B are held fixed by the boundary conditions, then the reduced action is a function of only r_+ and e, i.e., $I^*=I^*(r_+,e)$. The local stability conditions are then (i) $\nabla I^*=0$ and (ii) the Hessian matrix is positive definite. The latter condition corresponds to a condition of dynamical as well as thermodynamical stability [11] and will be discussed in Sec. V. We will start by investigating the first condition.

The condition of stationarity $\nabla I^*=0$ gives

$$\beta=\frac{2\pi}{\kappa}f(r_B;r_+,e,\alpha), \tag{25}$$

where $\kappa=(r_+^2-e^2+3\alpha^2r_+^4)/2r_+^3$ is the surface gravity of the horizon, and

$$\phi=\left(\frac{e}{r_+}-\frac{e}{r_B}\right)f(r_B;r_+,e,\alpha)^{-1}. \tag{26}$$

These are the inverse Hawking temperature and the difference in the electrostatic potential between r_+ and r_B blueshifted from infinity to r_B, respectively.

Inverting these two equations, r_+ and e are obtained as functions of the boundary conditions and the cosmological constant. We define the more convenient variables

$$x\equiv\frac{r_+}{r_B},\quad q\equiv\frac{e}{r_B},\quad \overline{\beta}\equiv\frac{\beta}{4\pi r_B},\quad \overline{\alpha}^2\equiv\alpha^2r_B^2. \tag{27}$$

In these variables (25) and (26) are written

$$\overline{\beta}=x\sqrt{1-x}\sqrt{1-\frac{q^2}{x}+\overline{\alpha}^2(1+x+x^2)}\times\left(1-\frac{q^2}{x^2}+3\overline{\alpha}^2x^2\right)^{-1}, \tag{28}$$

$$\phi=\frac{q}{x}\sqrt{1-x}\left(1-\frac{q^2}{x}+\overline{\alpha}^2(1+x+x^2)\right)^{-1/2}. \tag{29}$$

To invert these equations, we start by inverting Eq. (28) to obtain q;

$$q^2=x^2\phi^2[1+\overline{\alpha}^2(1+x+x^2)](1-x+x\phi^2)^{-1}. \tag{30}$$

Substituting Eq. (30) in Eq. (28) and taking its square, we obtain a seventh degree equation in x with a double root $x=1$ which is not a solution of the initial equations. Getting rid of that root, we obtain a fifth degree equation

$$\begin{aligned}&\overline{\beta}^2(-1+\phi^2+\overline{\alpha}^2\phi^2)^2+4\overline{\alpha}^2\overline{\beta}^2\phi^2(-1+\phi^2+\overline{\alpha}^2\phi^2)x\\&+(-1-\overline{\alpha}^2+6\overline{\alpha}^2\overline{\beta}^2-12\overline{\alpha}^2\overline{\beta}^2\phi^2-6\overline{\alpha}^4\overline{\beta}^2\phi^2\\&+6\overline{\alpha}^2\overline{\beta}^2\phi^4+10\overline{\alpha}^4\overline{\beta}^2\phi^4)x^2+(1-\phi^2-\overline{\alpha}^2\phi^2\\&-12\overline{\alpha}^4\overline{\beta}^2\phi^2+12\overline{\alpha}^4\overline{\beta}^2\phi^4)x^3+\overline{\alpha}^2(9\overline{\alpha}^2\overline{\beta}^2-\phi^2\\&-18\overline{\alpha}^2\overline{\beta}^2\phi^2+9\overline{\alpha}^2\overline{\beta}^2\phi^4)x^4+\overline{\alpha}^2(1-\phi^2)x^5=0.\end{aligned} \tag{31}$$

However, not every solution of this equation corresponds to a physical solution of a black hole. This is because the radius of the event horizon of the Reissner–Nordström–anti-de Sitter must obey the following condition:

$$r_+^2-e^2+3\alpha^2r_+^4\geqslant 0, \tag{32}$$

where the equality defines the extremal Reissner–Nordström–anti-de Sitter black hole.

Comparing Eq. (32) with Eq. (25), yields that β is real and positive. Comparing it with Eq. (26) we obtain the following condition:

$$\phi^2\leqslant\frac{1+3\alpha^2r_+^2}{1+\alpha^2(1+2r_+r_B+3r_+^2)}. \tag{33}$$

In the coordinates given in Eq. (27), the inequality (33) becomes

$$\phi^2\leqslant\frac{1+3\overline{\alpha}^2x^2}{1+\overline{\alpha}^2(1+2x+3x^2)}. \tag{34}$$

This is the condition that the solutions of Eq. (31) must obey in order to represent physical black hole solutions.

Equation (31) has no known analytical solutions. However, its solutions can be numerically computed and presented in graphics. This will be done in the next section.

IV. ANALYSIS OF THE BLACK HOLE SOLUTIONS

In this section we present in graphics an analysis of the solutions of Eq. (31) that obey condition (34). This analysis is done in two steps: (i) first, we analyze Figs. 1–4, that present the solutions x as functions of $\overline{\beta}$ and ϕ, for values $\overline{\alpha}=0$, 0.5, 1, and 5; (ii) afterwards we show in Figs. 5 –9, the regions with zero, one, and two solutions in the space spanned by $\phi\times\overline{\alpha}$ for fixed values of $\overline{\beta}$.

(i) Analysis of Figs. 1–4: $\overline{\alpha}=0$. The solutions for $\overline{\alpha}=0$ are presented in Fig. 1. These are obviously identical to the solutions of the Reissner-Nordström black hole, see Ref. [11]. We can see in Fig. 1 that for fixed ϕ there is a maximum of $\overline{\beta}$, $\overline{\beta}_{\rm max}(\phi)$, so that for $\overline{\beta}>\overline{\beta}_{\rm max}(\phi)$ there are no solutions. For $\overline{\beta}<\overline{\beta}_{\rm max}(\phi)$ one can have two or only one solution depending on the precise value of $\overline{\beta}$. For $\phi=0$ (Schwarzschild) one has always two solutions for $\overline{\beta}<\overline{\beta}_{\rm max}(0)$ $[\overline{\beta}_{\rm max}(0)=2/\sqrt{27}]$. In the limiting case $\overline{\beta}\to 0$

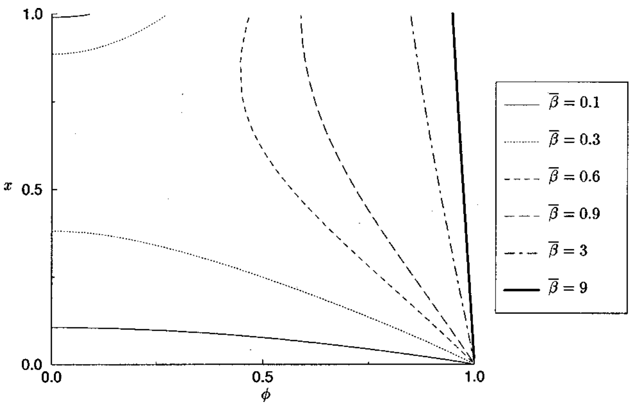

FIG. 1. Solutions of Eq. (31) for $\overline{\alpha}=0$ (Reissner-Nordström) as a function of the electrostatic potential at the boundary ϕ for fixed values of $\overline{\beta}=0.1,0.3,0.6,0.9,3,9$. The stable solutions correspond to the upper branch of the curves. This means that when there are two solutions for given values of $\overline{\beta}$ and ϕ, only the solution with higher value of x is stable.

(i.e., $r_B T\to\infty$), one finds there is a solution with $x=r_+/r_B\to 1$ (as will be seen in Sec. V this is the stable solution), see Ref. [5]. For $\phi=0$, still, and $\overline{\beta}>\overline{\beta}_{\max}(0)$ there are no solutions (see comments at the end of this section). For $0<\phi<1/\sqrt{3}$ one has one or two solutions up to $\overline{\beta}_{\max}(\phi)$, whereas for $\overline{\beta}>\overline{\beta}_{\max}(\phi)$ there are as well no solutions. Finally for $\phi>1/\sqrt{3}$ there is only one solution [again for $\overline{\beta}<\overline{\beta}_{\max}(\phi)$] corresponding to the unstable branch as will be seen in Sec. V. Note that, for $\overline{\alpha}=0$, condition (34) implies that the electrostatic potential has a maximum at $\phi_{\max}=1$. Notice also that in the limit $\overline{\beta}\to\infty(T\to 0)$, the curves in Fig. 1 tend to the curve $\phi=1$, which corresponds to the extremal Reissner-Nordström black hole ($r_+=e$).

The cases $\overline{\alpha}\neq 0$, which are now going to be analyzed require the following prior analysis. As in the case $\overline{\alpha}=0$, for

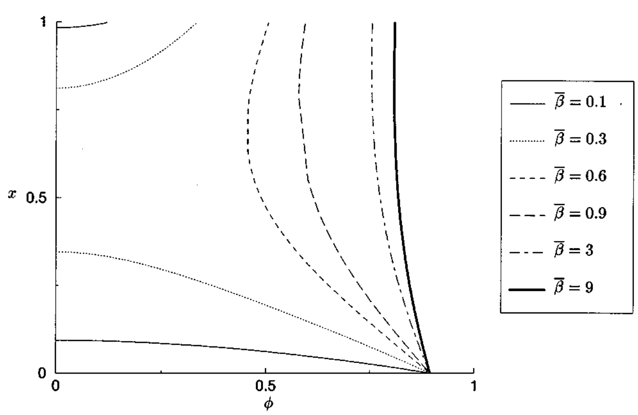

FIG. 2. Solutions of Eq. (31) for $\overline{\alpha}=0.5$ as a function of the electrostatic potential at the boundary ϕ for fixed values of $\overline{\beta}=0.1,0.3,0.6,0.9,3,9$. Notice that $\phi\lesssim 0.89$, as imposed by condition (34). The stable solutions correspond to the upper branch of the curves. This means that when there are two solutions for given values of $\overline{\beta}$ and ϕ, only the solution with higher value of x is stable.

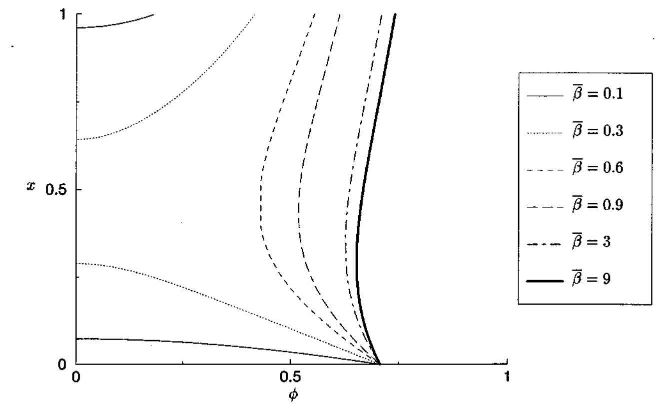

FIG. 3. Solutions of Eq. (31) for $\overline{\alpha}=1$ as a function of the electrostatic potential at the boundary ϕ for fixed values of $\overline{\beta}=0.1,0.3,0.6,0.9,3,9$. The maximum value of ϕ for which there are solutions, $\phi_{\max}\simeq 0.76$ is imposed by condition (34). The stable solutions correspond to the upper branch of the curves. This means that when there are two solutions for given values of $\overline{\beta}$ and ϕ, only the solution with higher value of x is stable.

$\overline{\alpha}\neq 0$ there are solutions at $T=0$ ($\overline{\beta}=\infty$), that correspond to the extremal black holes. This can be analytically verified by replacing $\overline{\beta}=\infty$ in Eq. (31), from where we obtain the equation

$$1-\phi^2-\overline{\alpha}^2\phi^2-2\overline{\alpha}^2\phi^2 x+3\overline{\alpha}^2(1-\phi^2)x^2=0. \qquad (35)$$

Notice this is the equation one obtains taking the equality in condition (34). In fact it corresponds to the condition of extremality of the Reissner–Nordström–anti-de Sitter black hole, which is in agreement with the well known fact that only the extremal black holes have zero temperature.

Equation (35) has at least one solution that verifies $0<x<1$ if

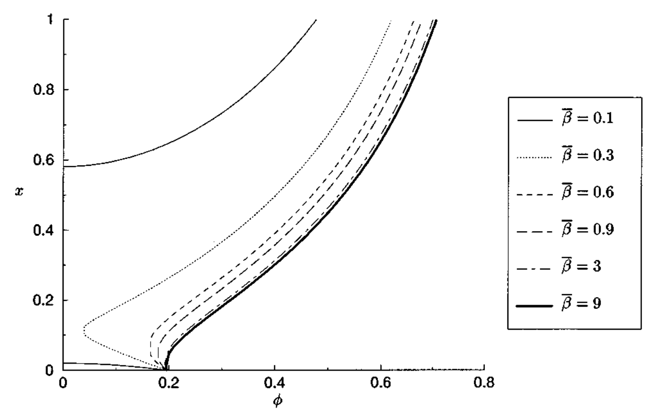

FIG. 4. Solutions of Eq. (31) for $\overline{\alpha}=5$ as a function of the electrostatic potential at the boundary ϕ for fixed values of $\overline{\beta}=0.1,0.3,0.6,0.9,3,9$. The maximum value of ϕ for which there are solutions, $\phi_{\max}\simeq 0.71$, is imposed by condition (34). The stable solutions correspond to the upper branch of the curves. This means that when there are two solutions for given values of $\overline{\beta}$ and ϕ, only the solution with higher value of x is stable.

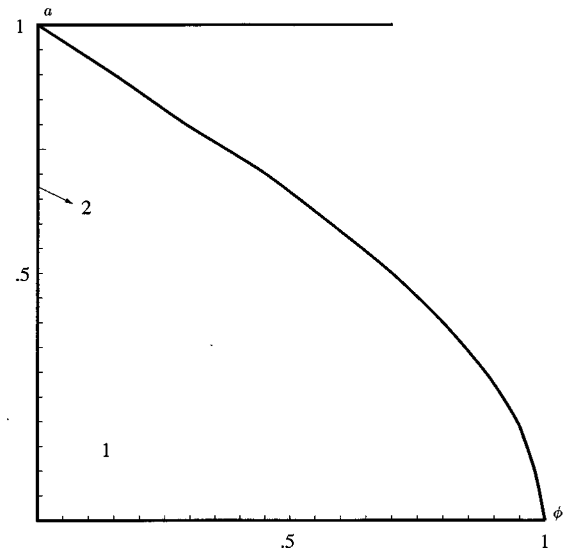

FIG. 5. Number of solutions of Eq. (31) with $\overline{\beta}\rightarrow 0$, in the space $\phi\times a$, where a is given by Eq. (40). There is one black hole solution in the confined region and also for $a=1$ (i.e., infinite cosmological constant), for $\phi<\sqrt{0.5}$. There are two solutions for $\phi=0$, i.e., for the Schwarzschild–anti-de Sitter black hole.

$$\overline{\alpha}^2<\frac{1}{6}\quad \text{and}\quad \frac{1+3\overline{\alpha}^2}{1+6\overline{\alpha}^2}\leqslant\phi^2\leqslant\frac{1}{1+\overline{\alpha}^2}, \tag{36}$$

$$\frac{1}{6}\leqslant\overline{\alpha}^2<\frac{2}{3}\quad \text{and}\quad \frac{3\overline{\alpha}^2+6-\sqrt{9\overline{\alpha}^4+12\overline{\alpha}^2}}{4\overline{\alpha}^2+6}\leqslant\phi^2\leqslant\frac{1}{1+\overline{\alpha}^2},$$

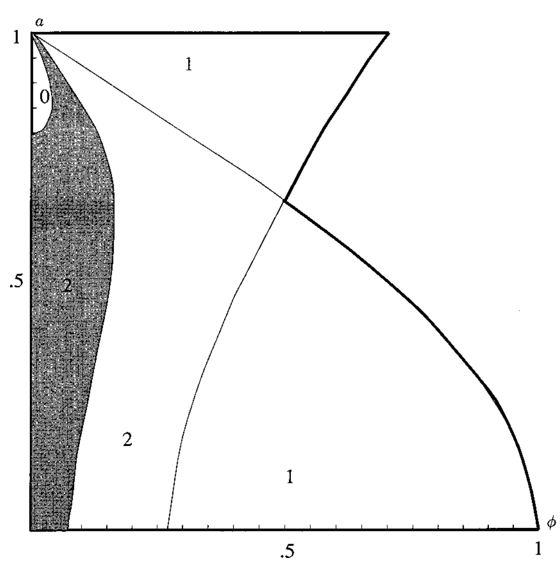

FIG. 6. Number of solutions of Eq. (31) with $\overline{\beta}=0.3$, in the space $\phi\times a$, where a is given by Eq. (40) 0 means that there are zero solutions in this region, i.e., there are no solutions of black holes in thermodynamical equilibrium for this set of values of $\overline{\beta}$, ϕ, and a. 1 means that there is one solution. 2 means that there are two solutions. In the shaded region the stable solutions are not globally stable and therefore represent metastable black holes (see Sec. V B).

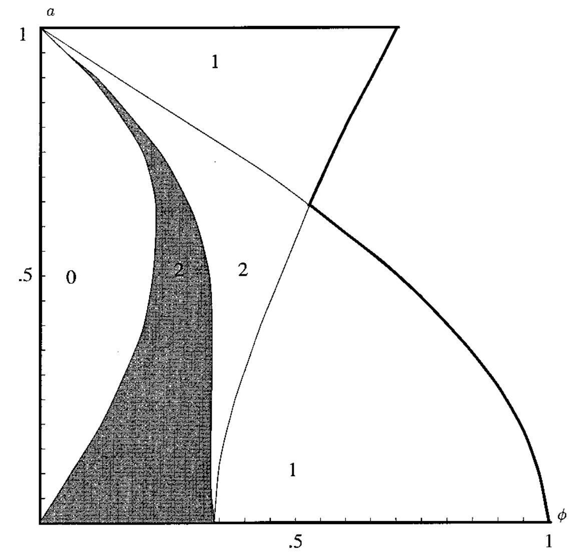

FIG. 7. Number of solutions of Eq. (31) with $\overline{\beta}=2/\sqrt{27}\simeq 0.38$, in the space $\phi\times a$, where a is given by Eq. (40) (see caption of Fig. 6).

$$\overline{\alpha}^2\geqslant\frac{2}{3}\quad \text{and}\quad \frac{3\overline{\alpha}^2+6-\sqrt{9\overline{\alpha}^4+12\overline{\alpha}^2}}{4\overline{\alpha}^2+6}\leqslant\phi^2\leqslant\frac{1+3\overline{\alpha}^2}{1+6\overline{\alpha}^2}.$$

For these values of ϕ and $\overline{\alpha}$, the curves for fixed ϕ presented in the figures reach infinite $\overline{\beta}$. Furthermore, Eq. (35) has two solutions if

$$1/6<\overline{\alpha}^2<2/3\quad \text{and}\quad \frac{3\overline{\alpha}^2+6-\sqrt{9\overline{\alpha}^4+12\overline{\alpha}^2}}{4\overline{\alpha}^2+6}<\phi^2<\frac{1+3\overline{\alpha}^2}{1+6\overline{\alpha}^2}, \tag{37}$$

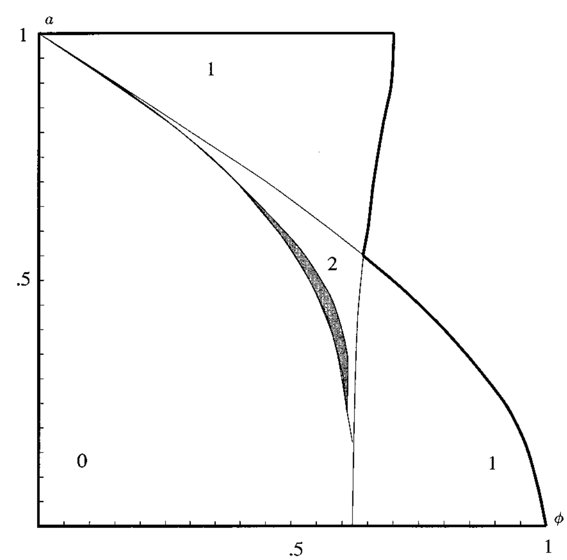

FIG. 8. Number of solutions of Eq. (31) with $\overline{\beta}=1$, in the space $\phi\times a$, where a is given by Eq. (40) (see caption of Fig. 6).

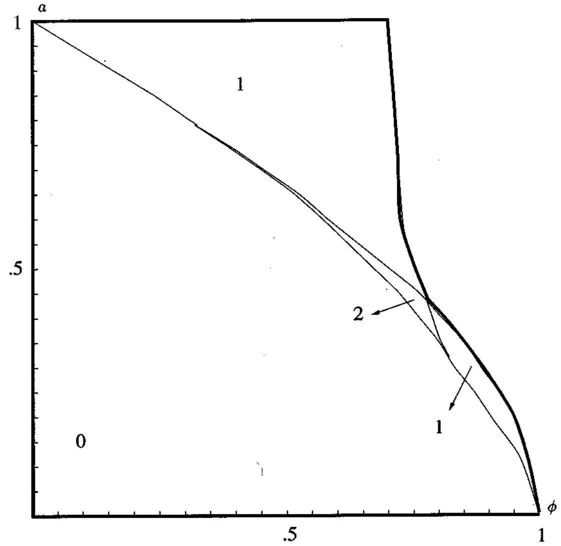

FIG. 9. Number of solutions of Eq. (31) with $\overline{\beta}=\infty$, in the space $\phi\times a$, where a is given by Eq. (40). Notice that for $\overline{\beta}\to\infty$, Eq. (31) becomes Eq. (35) which corresponds to the extremal Reissner–Nordström–anti-de Sitter black hole, see discussion following Eq. (35) (see caption of Fig. 6).

$$\overline{\alpha}^2\geqslant 2/3 \quad\text{and}\quad \frac{3\overline{\alpha}^2+6-\sqrt{9\overline{\alpha}^4+12\overline{\alpha}^2}}{4\overline{\alpha}^2+6}<\phi^2<\frac{1}{1+\overline{\alpha}^2}.$$

$\overline{\alpha}=0.5$. The solutions for $\overline{\alpha}=0.5$ are presented in Fig. 2. This figure presents the same properties mentioned previously for the case $\overline{\alpha}=0$. Comparing Figs. 1 and 2, we verify that the maximum value of $\overline{\beta}$, $\overline{\beta}_{\max}(\phi)$, for which there are solutions is increasing, i.e., there are solutions for slightly lower values of the temperature at $\overline{\alpha}=0.5$ than at $\overline{\alpha}=0$. Furthermore there are solutions for infinite $\overline{\beta}$ in the interval $0.83\lesssim\phi\lesssim 0.89$, see Eq. (36), this can be seen using the curve $\overline{\beta}=9$ in Fig. 2, since it corresponds to a good approximation of infinite $\overline{\beta}$.

Condition (34) implies, for $\overline{\alpha}^2<2/3$,

$$\phi\leqslant\sqrt{\frac{1}{1+\overline{\alpha}^2}}. \tag{38}$$

This condition corresponds to the upper-limit of the interval given in Eq. (36), which in this case is $\phi\simeq 0.89$. Notice that if in Eq. (31) we do $x=0$, we obtain precisely $\phi=1/\sqrt{1+\overline{\alpha}^2}$, as can be seen in Fig. 2.

$\overline{\alpha}=1$. Figure 3 plots the solutions for $\overline{\alpha}=1$. We can see that this figure presents similar properties to the ones obtained for lower values of $\overline{\alpha}$. Using Eq. (36), we can see that for $\phi\lesssim 0.66$ there are solutions only for $\overline{\beta}<\overline{\beta}_{\max}(\phi)$, where $\overline{\beta}_{\max}(\phi)$ is finite and depends on ϕ. On the contrary, the curves for higher values of ϕ reach infinite $\overline{\beta}$. In particular, for $0.66\lesssim\phi\lesssim 0.71$ [see Eq. (37)] there are two solutions at low temperatures (i.e., high $\overline{\beta}$) as can be seen in Fig. 3, since the curve $\overline{\beta}=9$ is representative of the curves with high $\overline{\beta}$. It can be seen that for $\overline{\beta}=9$ there are two solutions for $\phi<\phi_0$, where $\phi_0=1/\sqrt{1+\overline{\alpha}^2}\simeq 0.71$ is the value of ϕ where $x=0$ for every $\overline{\beta}$. There is one solution for $0.71\lesssim\phi\lesssim 0.76$, see Fig. 3, where the upper-limit is imposed by condition (34). In fact, condition (34) imposes, for $\overline{\alpha}^2<2/3$,

$$\phi_{\max}=\sqrt{\frac{1+3\overline{\alpha}^2}{1+6\overline{\alpha}^2}}. \tag{39}$$

Notice this is the upper-limit of the interval given in Eq. (36).

$\overline{\alpha}=5$. Figure 4 presents the solutions for $\overline{\alpha}=5$. Using Eq. (36), we can see that for $\phi\lesssim 0.19$, there are solutions only for $\overline{\beta}<\overline{\beta}_{\max}(\phi)$. For $0.19\lesssim\phi\lesssim 0.71$ there is one solution for high values of $\overline{\beta}$ (consider the curve $\overline{\beta}=9$). For infinite $\overline{\beta}$ this region is $0.195\lesssim\phi\lesssim 0.196$, see Eq. (37). In Fig. 4, we can also see that for $0.19\lesssim\phi\lesssim 0.71$ there is one solution, where the upper-limit is given by Eq. (39). For higher values of $\overline{\alpha}$ there are not new types of solutions and therefore it is not necessary to pursue our analysis.

(ii) Analysis of Figs. 5–9. In order to clarify the disposition of the number of solutions for given values of $\overline{\alpha}$, ϕ, and $\overline{\beta}$, we present, in the space spanned by $\overline{\alpha}$ and ϕ for fixed $\overline{\beta}$, the regions with zero, one, and two solutions. We do this for eleven different values of $\overline{\beta}$, $\overline{\beta}=0,0.3,2/\sqrt{27}\simeq 0.38,1,\infty$, see Figs. 5–9, respectively. In these figures one can see the evolution of the number of solutions as $\overline{\beta}$ increases. To present all possible values of $\overline{\alpha}\epsilon[0,\infty[$, we use in these figures the parameter a instead of $\overline{\alpha}$, a is defined by

$$a=\frac{2}{\pi}\arctan\overline{\alpha}, \tag{40}$$

such that $0\leqslant a\leqslant 1$. It is this variable that appears in the ordinate axis in Figs. 5–9.

Due to condition (34) there are no physically possible solutions on the right-hand side of Figs. 5–9.

An important value of $\overline{\beta}$, first studied by York [5] in connection with the Schwarzschild black hole ($\phi=\overline{\alpha}=0$), is $\overline{\beta}=2/\sqrt{27}\simeq 0.38$, i.e., $\beta=(8\pi/\sqrt{27})r_B$. For higher values of β, lower values of the temperature, there are no black hole solutions. This is a quantum effect and following York [5] can be understood as follows. One can associate a Compton type wavelength λ to the energy k_BT of the thermal particles, by $\lambda=\hbar c/k_BT$, or in Planck units $\lambda=1/T=\beta$. If this Compton wavelength is much larger than the radius r_B of the cavity (or more specifically $\lambda>8\pi/\sqrt{27}r_B\simeq 4.8r_B$) then the thermal particles cannot be confined within the cavity and do not collapse to form a black hole.

By analyzing Figs. 6 and 7, we can see that for nonzero cosmological constant ($\overline{\alpha}\neq 0$) this phenomenon starts at even lower $\overline{\beta}$ (higher T). Indeed using Eq. (28) (with ϕ

$=0$, i.e., $q=0$) we can show that to first order in α^2 ($\alpha^2 r_B^2 \ll 1$), York's criterion for no black hole solutions becomes

$$\lambda=\beta>\frac{8\pi}{\sqrt{27}}r_B\left(1-\frac{5}{18}\alpha^2 r_B^2\right). \tag{41}$$

From Eq. (41) we infer that the role of the negative cosmological constant ($\Lambda=-3\alpha^2$) is to produce an effective cavity radius $r_{\rm eff}=r_B(1-\frac{5}{18}\alpha^2 r_B^2)$ smaller than r_B. Thus for a given temperature, it is more difficult to confine the thermal particles, and harder to form black holes, in accord with the idea that a negative cosmological constant shrinks space.

If we extend our previous first order analysis to include the electrostatic potential ϕ, we obtain

$$\lambda=\beta>\frac{8\pi}{\sqrt{27}}r_B\left(1-\frac{5}{18}\alpha^2 r_B^2+2\phi^2\right). \tag{42}$$

We can see that the electrostatic potential has the opposite effect of the cosmological constant (see, for example, Figs. 6, 7, and 8).

V. STABILITY

A. Local stability

As mentioned before there is a second condition of local stability that has not yet been investigated. We will follow the same procedure as given in Ref. [11]. This is the condition that the Hessian matrix of the reduced action be positive definite. For convenience in this analysis we will use the variable $S=\pi r_+^2$ instead r_+. The Hessian matrix of $I^*(S,e)$ is

$$I^*_{,ij}=\begin{pmatrix} I^*_{,ee} & I^*_{,eS} \\ I^*_{,eS} & I^*_{,SS}\end{pmatrix}. \tag{43}$$

The matrix is positive definite if its pivots are all positive. The pivots of Eq. (43) are

$$I^*_{,ee} \tag{44}$$

and

$$\frac{\det(I^*_{,ij})}{I^*_{,ee}}. \tag{45}$$

The first condition of local stability, $\nabla I^*=0$ yields

$$\left(\frac{\partial I^*}{\partial e}\right)_S=\beta\left(\frac{\partial E^*}{\partial e}\right)_S-\beta\phi=0\Rightarrow\phi=\left(\frac{\partial E}{\partial e}\right)_S,$$
$$\left(\frac{\partial I^*}{\partial S}\right)_e=\beta\left(\frac{\partial E^*}{\partial S}\right)_e-1=0\Rightarrow\beta^{-1}=T=\left(\frac{\partial E}{\partial S}\right)_e. \tag{46}$$

These are well-known thermodynamical relations [16].

From Eq. (46), we compute the second derivatives of I^* in the stationary points of I^*

$$\left.\frac{\partial^2 I^*}{\partial e^2}\right|_{\rm eq}=\beta\left.\frac{\partial^2 E^*}{\partial e^2}\right|_{\rm eq}=\beta\left(\frac{\partial\phi}{\partial e}\right)_S,$$
$$\left.\frac{\partial^2 I^*}{\partial e\partial S}\right|_{\rm eq}=\beta\left.\frac{\partial^2 E^*}{\partial e\partial S}\right|_{\rm eq}=\beta\left(\frac{\partial\phi}{\partial S}\right)_e=-\frac{1}{\beta}\left(\frac{\partial\beta}{\partial e}\right)_S,$$
$$\left.\frac{\partial^2 I^*}{\partial S^2}\right|_{\rm eq}=\beta\left.\frac{\partial^2 E^*}{\partial S^2}\right|_{\rm eq}=-\frac{1}{\beta}\left(\frac{\partial\beta}{\partial S}\right)_e, \tag{47}$$

where ''eq'' means quantities evaluated at equilibrium, i.e., at the stationary points of the reduced action I^*.

The first pivot (44) is simply the first of these equations. The second pivot (45) is easily obtained from Eq. (47)

$$\frac{\det I^*_{,ij}}{I^*_{,ee}}=-\frac{1}{\beta}\left(\frac{\partial\beta}{\partial S}\right)_e+\frac{1}{\beta}\left(\frac{\partial\beta}{\partial e}\right)_S\left(\frac{\partial\phi}{\partial S}\right)_e\Big/\left(\frac{\partial\phi}{\partial e}\right)_S$$
$$=-\frac{1}{\beta}\left[\left(\frac{\partial\beta}{\partial S}\right)_e+\left(\frac{\partial\beta}{\partial e}\right)_S\left(\frac{\partial e}{\partial S}\right)_\phi\right]$$
$$=-\frac{1}{\beta}\left(\frac{\partial\beta}{\partial S}\right)_\phi=\frac{1}{C_{\phi,r_B}}, \tag{48}$$

where C_{ϕ,r_B} is the heat capacity at constant ϕ and r_B. Notice that in all this calculation we have implicitly held r_B constant, since in the grand canonical ensemble the dimension of the system is held constant.

Imposing positive pivots for local stability yields

$$\left(\frac{\partial\phi}{\partial e}\right)_{S,r_B}\geqslant 0,$$
$$C_{\phi,r_B}\geqslant 0. \tag{49}$$

These conditions are identical to the classical thermodynamical stability conditions [16]. Therefore one can conclude that the dynamical stability conditions given in Eqs. (46) and (49) are identical to the thermodynamical stability conditions [11].

For the Reissner–Nordström–anti-de Sitter black hole, the pivots obtained from Eqs. (47) and (48) are

$$\left(\frac{\partial\phi}{\partial e}\right)_{S,r_B}=\left(\frac{1}{r_+}-\frac{1}{r_B}\right)\times\left(1-\frac{r_+}{r_B}-\frac{e^2}{r_+r_B}-\frac{\alpha^2 r_+^3}{r_B}+\frac{e^2}{r_B^2}+\alpha^2 r_B^2\right)^{1/2}$$
$$+\frac{e^2}{r_B}\left(\frac{1}{r_+}-\frac{1}{r_B}\right)^2\left(1-\frac{r_+}{r_B}-\frac{e^2}{r_+r_B}-\frac{\alpha^2 r_+^3}{r_B}+\frac{e^2}{r_B^2}+\alpha^2 r_B^2\right)^{3/2}, \tag{50}$$

which is positive, and

$$C_{\phi,r_B}=4\pi r_+^3(r-r_+)(r_+^2-e^2+3\alpha^2 r_+^4)\times[1+\alpha^2(r_B^2+r_+r_B+r_+^2)]\times\{e^4+r_+^3[(6\alpha^2r_+^2-2)(r_B+\alpha^2r_B^3)+3r_++2\alpha^2r_+^3+3\alpha^4r_+^5]+2e^2r_+\times[-2r_++r_B(1+\alpha^2(r_B^2-2r_+r_B-2r_+^2))]\}^{-1}. \tag{51}$$

The numerator of C_{ϕ,r_B} is positive, therefore the condition $C_{\phi,r_B}>0$ is verified if the denominator in Eq. (51) is positive. Using Eqs. (27) and (30), we obtain the following condition of stability for the solutions of Eq. (31):

$$[-2(1+\bar{\alpha}^2)+3x+6\bar{\alpha}^2(1+\bar{\alpha}^2)x^2+2\bar{\alpha}^2x^3+3\bar{\alpha}^4x^5]\times(1-x+x\phi^2)^2-\{2\phi^2[1+\bar{\alpha}^2(1+x+x^2)]\times[-1+2x+\bar{\alpha}^2(-1+2x+2x^2)]\}\times(1-x+x\phi^2)+\phi^4x[1+\bar{\alpha}^2(1+x+x^2)]^2>0. \tag{52}$$

By numerical computation, we can verify the solutions that obey this condition. We have found that the lower branches of the curves presented in Figs. 1 to 4 correspond to unstable solutions, while the upper branches correspond to stable solutions. Therefore we can say that in general only the solutions with the higher value of x, that is with higher event horizon radius, are stable.

In more detail we can distinguish three cases. (i) for low $\bar{\alpha}$ (see Figs. 1 and 2), and for values of β and ϕ for which there is one solution, it corresponds to a lower branch and therefore this solution is unstable; (ii) for high values of $\bar{\alpha}$ (see Fig. 4), whenever there is only one solution, it corresponds to an upper branch and therefore this solution is stable; (iii) whenever there are two solutions, for given values of $\bar{\beta},\phi,\bar{\alpha}$, the smaller one is an instanton (i.e., it is an unstable solution and dominates the semi-classical evaluation of the rate of nucleation of black holes [21]) and the one with larger event horizon radius is a stable black hole.

These three cases can easily be distinguished in Figs. 5–9. In these figures there are in general two separated regions with one solution. The region with lower values of a, i.e., lower values of $\bar{\alpha}$, corresponds to case (i) and these are unstable solutions. The region with higher values of a corresponds to case (ii) and these are stable solutions. Obviously, the region with two solutions in each Fig. 5–9, corresponds to case (iii), i.e., one of those solutions is stable, the one with higher value of x, and the other unstable.

B. Global stability

The stable solutions computed in the previous subsection are not necessarily global minimum of the action. In this case they will not dominate the partition function and the zero-loop approximation cannot be considered accurate [6,11].

As the reduced action given in Eq. (21) grows without bound in the directions where r_+ or e tend to infinity, the global minimum must be found either at the local minimum or at $r_+=e=0$. The action tends to zero as r_+ and e tend to zero, therefore the condition of global stability of the stable solutions is that the classical action I must be negative.

If the classical action is positive, the partition function is dominated by points near the origin. But these points do not correspond to a black hole in thermal equilibrium. In this case, the black hole, that corresponds to the stable solution of the reduced action, is metastable.

We verified numerically which boundary conditions (given by the values of β, ϕ, and α) correspond to globally stable black holes, i.e., to solutions that dominate the partition function. We can also see, using a simple argument, that for $\bar{\beta}<\frac{8}{27}$ all locally stable solutions are also globally stable. Indeed, York [6] has shown that for the Schwarzschild black hole the condition of global stability is $\bar{\beta}<\frac{8}{27}$ and since the classical action I is a decreasing function of ϕ and α, this condition is still a bound for global stability for all ϕ and α. However, for $\bar{\beta}>\frac{8}{27}$, there is always a certain region of $\phi\times\alpha$, for which the locally stable solutions do not correspond to global minima of the action. We show these regions in Figs. 5–9, where the regions for which the solutions do not correspond to a global minimum of the action, i.e., the metastable solutions, are shaded. In particular Figs. 5 and 9 do not present a shaded region because all the stable solutions are dominant in the limit $\bar{\beta}\to 0$ as said above and for $\bar{\beta}\to\infty$ the region with metastable solutions is too thin to be presented in graphic.

VI. THE $r_B\to\infty$ LIMIT AND THE HAWKING-PAGE SOLUTIONS

One can study the case where the boundary goes to infinity. There are two different ways for taking this limit: (i) fixing the horizon radius r_+ and the charge e of the black hole; (ii) fixing the boundary conditions, i.e., fixing β and ϕ. In this section we will study only the pure Reissner–Nordström–anti-de Sitter cases, i.e., the cases with $\alpha,\phi\neq 0$. Other cases, with either α or ϕ equal to zero are considered in the next section.

We will start by studying the first case: (i) Fixing the black hole solution, i.e., fixing r_+ and e and taking the limit $r_B\to\infty$, we obtain from Eqs. (25) and (26) that the temperature $T=\beta^{-1}$ and electrostatic potential ϕ go to zero as $T\sim[(r_+^2-e^2+3\alpha^2r_+^4)/4\pi\alpha r_+^3]r_B^{-1}$ and $\phi\sim(e/\alpha r_+)r_B^{-1}$, respectively. In this case the thermal energy goes to zero as $E\sim M(1+\alpha^2r_B^2)^{-1/2}$, where M is the mass of the black hole given in Eq. (13). As also previously found in Ref. [12], the thermal energy at infinity is not equal to the ADM mass of the black hole, due to the the fact that the spacetime is not asymptotically flat. Note that in Refs. [13,17] it is found that $E=M$ due to a different definition of the temperature of the ensemble.

To determine the stability of the black hole solutions in this limit, we compute the heat capacity, from Eq. (51) and obtain

$$C_\phi = \frac{2\pi r_+^2 (r_+^2 - e^2 + 3\alpha^2 r_+^4)}{e^2 - r_+^2 + 3\alpha^2 r_+^4}. \tag{53}$$

The numerator in Eq. (53) is necessarily positive due to condition (32), that r_+ must obey to represent the event horizon radius. Therefore, the stability condition for these solutions is simply $e^2 - r_+^2 + 3\alpha^2 r_+^4 > 0$. Notice that Eq. (53) is a generalization to $e \neq 0$ of the heat capacity found by Hawking and Page in Ref. [17]. Computing the action in this limit yields

$$I = \frac{\pi r_+^2 (r_+^2 - e^2 - \alpha^2 r_+^4)}{r_+^2 - e^2 + 3\alpha^2 r_+^4}. \tag{54}$$

This is precisely Hawking-Page action [17,13], which here we have recovered from York's formalism taking the appropriate limit.

From Eq. (54), we can verify the global stability of the black hole by imposing that $I < 0$. Therefore we conclude that the solution given by (r_+, e) corresponds to a globally stable black hole if conditions $r_+^2 - e^2 - \alpha^2 r_+^4 < 0$ and Eq. (32) are both verified. These conditions are similar to those found in Ref. [13].

We conclude that taking different boundary conditions, i.e., choosing to fix the boundary conditions in the boundary at infinity (as done here) or in the region where spacetime is flat like Hawking and Page [17], yields a different value for the energy only. Furthermore this difference is so that all the other physical quantities (like the action, entropy, mean-value of the charge and heat capacity) remain the same.

On the other hand, we can take a different limit: (ii) we can fix the boundary conditions, i.e., β and ϕ, when taking the limit. This is done by recovering the variables β and α in Eq. (31), using Eq. (27). Taking the limit $r_B \to \infty$, we obtain the equation $\alpha^2 \beta^2 [\phi^2(1 + 2x + 3x^2) - 3x^2]^2 - x^2(1 + x + x^2)(1 - x + \phi^2 x) = 0$. This is a fifth degree equation in x. For fixed β, ϕ, and α and taking the limit $r_B \to \infty$, we obtain a solution r_+ that tends to infinity as $r_+ \sim c_+ r_B$, where c_+ is a constant that depends only on β, ϕ, and α. Considering now Eq. (30), we can see that the charge e goes to infinity as $e \sim c_e r_B^2$, where again c_e is a constant that depends only on β, ϕ, and α. Therefore the entropy (24) and the mean-value of the charge (23) both go to infinity as r_B^2. In this limit the action, the thermal energy and the heat capacity, given in Eqs. (21), (22), and (51), respectively, also diverge as r_B^2. The heat capacity is always positive, which means these solutions are stable. The action is always positive, therefore the solutions are not globally stable and represent metastable black holes.

VII. COMMENTS ON SPECIAL CASES

Several black holes may be considered as special cases of the Reissner–Nordström–anti-de Sitter black hole.

(I) Putting $\phi = 0$ and $\Lambda = 0$, we obtain the Schwarzschild black hole studied in Ref. [5]. There are two solutions for $\overline{\beta} < \overline{\beta}_{\max} = 2/\sqrt{27}$. These solutions can be computed analytically, since Eq. (31) becomes a third degree equation for $\Lambda = 0$. Only the solution with higher event horizon radius, i.e., higher mass, is stable. In the limit $r_B \to \infty$ the unstable solution, i.e., the solution with lower value of the horizon radius, goes to $r_+ = 1/(4\pi r_+)$, while the stable solution goes to infinity as $r_+ \sim r_B$ [5].

(II) Putting $\Lambda = 0$ we obtain the Reissner-Nordström black hole. This has been studied in Ref. [11]. There are one or two solutions for $\overline{\beta} < \overline{\beta}_{\max}$. These as stated above can be computed analytically. Once again only the solution with higher event horizon radius, when it exists, is stable. In the limit $r_B \to \infty$, the black hole horizon radius is $r_+ = (\beta/4\pi)(1 - \phi^2)$ and the charge is given by $e = \phi r_+$. The thermal energy of these solutions is equal to their ADM mass $E = M$. The heat capacity is negative $C_\phi = -2\pi r_+^2$. Therefore the solutions are unstable.

(III) Putting $\phi = 0$ we obtain the Schwarzschild–anti-de Sitter. This black hole has been studied before in Refs. [17,12]. This black hole has two solutions for $\beta < \beta_{\max}$, and again only the one with higher event horizon radius is stable. The limit $r_B \to \infty$ can be taken in two ways: (i) fixing the temperature and the cosmological constant, there is one unstable solution that tends to zero as $r_+ \sim (\beta/4\pi\alpha) r_B^{-1}$ and one stable solution that tends to infinity as $r_+ \sim c r_B$, where c is the solution of equation $c^3 + (3\alpha\beta/4\pi)^2 c^2 - 1 = 0$; (ii) fixing the horizon radius r_+, the temperature goes to zero as $T \sim (1 + 3\alpha^2 r_+^2)/4\pi\alpha r_+ r_B^{-1}$, these solutions are stable if $3\alpha^2 r_+^2 > 1$, see Ref. [12].

(IV) The extremal cases require special care [22–24] and were not studied in any detail in this paper.

VIII. CONCLUSIONS

We have studied the thermodynamics of the Reissner–Nordström–anti-de Sitter black hole in York's formalism. In the grand canonical ensemble where the temperature and the electrostatic potential are fixed at a boundary with finite radius, we have found one or two black hole solutions depending on the boundary conditions and the value of the cosmological constant. In general when there are two solutions, one is stable, the one with larger event horizon radius, and the other is an instanton. On the other hand, the cases with a single solution can correspond either to a stable or unstable solution. We have found that both high values of the cosmological constant and low temperatures favor stable solutions.

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[1] J. B. Hartle and S. W. Hawking, Phys. Rev. D **13**, 2188 (1976).
[2] G. W. Gibbons and S. W. Hawking, Phys. Rev. D **15**, 2752 (1977).
[3] S. W. Hawking, in *General Relativity: An Einstein Centenary Survey*, edited by S. W. Hawking and W. Israel (Cambridge University Press, Cambridge, England, 1979).
[4] S. W. Hawking, Phys. Rev. D **13**, 191 (1976).
[5] J. W. York, Phys. Rev. D **33**, 2092 (1986).
[6] B. F. Whiting and J. W. York, Phys. Rev. Lett. **61**, 1336 (1988).
[7] B. F. Whiting, Class. Quantum Grav. **7**, 15 (1990).
[8] J. D. Brown and J. W. York, in *The Black Hole 25 Years After*, edited by C. Teitelboim and J. Zanelli (Plenum, New York, in press), gr-qc/9405024.
[9] J. D. E. Creighton, Ph.D. thesis, Waterloo University, 1996, gr-qc/9610038.
[10] J. D. Brown *et al.*, Class. Quantum Grav. **7**, 1433 (1990).
[11] H. W. Braden, J. D. Brown, B. F. Whiting, and J. W. York, Phys. Rev. D **42**, 3376 (1990).
[12] J. D. Brown, J. Creighton, and R. B. Mann, Phys. Rev. D **50**, 6394 (1994).
[13] J. Louko and S. N. Winters-Hilt, Phys. Rev. D **54**, 2647 (1996).
[14] J. P. S. Lemos, Phys. Rev. D **54**, 6206 (1996).
[15] J. W. York, Physica A **158**, 425 (1989).
[16] H. B. Callen, *Thermodynamics and an Introduction to Thermostatistics* (Wiley, New York, 1985).
[17] S. W. Hawking and D. N. Page, Commun. Math. Phys. **87**, 577 (1983).
[18] D. N. Page, in *Black Hole Physics*, edited by V. D. Sabbata and Z. Zhang (Kluwer Academic, Dordrecht, 1992).
[19] S. W. Hawking, Commun. Math. Phys. **43**, 109 (1975).
[20] J.D. Bekenstein, Phys. Rev. D **7**, 2333 (1973).
[21] D. J. Gross, M. J. Perry, and L. G. Yaffe, Phys. Rev. D **25**, 330 (1982).
[22] S. W. Hawking, G. T. Horowitz, and S. F. Ross, Phys. Rev. D **51**, 4302 (1995).
[23] O. B. Zaslavskii, Phys. Rev. Lett. **76**, 2211 (1996).
[24] A. Ghosh and P. Mitra, Phys. Rev. Lett. **78**, 1858 (1997).