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Analytic solutions of the Wheeler-DeWitt equation in spherically symmetric space-time

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The canonical quantum theory of Einstein gravity with a cosmological constant in spherically symmetric space-time is analyzed. A mass operator can be introduced as a dynamical variable due to spherically symmetry. The operator ordering in the Hamiltonian, the momentum, and the mass operator is properly fixed so that they form a closed algebra. In this scheme, we obtain the analytic solution which simultaneously satisfies the Wheeler-DeWitt equation, the momentum constraint, and the mass constraint. [S0556-2821(99)00312-4]

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I. INTRODUCTION

The canonical quantum theory of Einstein gravity was developed by Arnowitt, Deser and Misner (ADM), and Dirac [1]. According to the ADM decomposition the canonical gravity theory reduces to a totally constrained system: the Hamiltonian and the momentum (or the diffeomorphism) constraints [2]. With the Dirac approach these constraints are imposed as operator restrictions on the wave function. The quantum version of the Hamiltonian constraint is called the Wheeler-DeWitt equation [3]. There are various difficulties to solve the constraint equations. So some simplifications of the problem are adopted. In particular, the quantum cosmological black hole solutions in spherically symmetric space-time have been investigated extensively. Even in this case it is not easy to solve the constraints analytically and further simplifications have been made: to assume a special coordinate condition [4], to restrict the theory in the semiclassical approach [5], and to take the reduced Hamiltonian method in asymptotically flat geometry [6,7].

One of the serious problems of quantum gravity is operator ordering, which is related to the regularization in defining the product of operators at the same space-time point [8], and the question of whether the constraints form a closed algebra [9]. There is no resolution of the ordering problem, and the ordering is often neglected for simplicity [10]. Then, the problem is reduced to semiclassical one.

In this paper, we consider the canonical quantization of Einstein gravity with a cosmological constant. Particularly we seek spherically symmetric solutions. In the spherically symmetric geometry, there appears the mass constraint in addition to the Hamiltonian and the momentum constraints. In order to set a consistent theoretical framework, we consider the operator ordering in these constraints seriously. We succeeded in fixing the ordering such that the commutation relations between them form a closed algebra. Then, in this scheme, we explored solutions of three constraint equations. It is fortunate that we are able to find analytic solutions which satisfy three constraints simultaneously. To our knowledge, our solution is the first one which is consistent with all quantum requirements.

This paper is organized as follows. In Sec. II, we review the canonical formalism of the Einstein theory with a cosmological constant in spherically symmetric space-time. We introduce the mass function as a dynamical variable. In Sec. III, the canonical quantization is presented and the operator ordering in the Hamiltonian, the momentum and the mass operators is fixed. Then, it is shown that these constraints form a closed algebra. In Sec. IV, the analytic solutions of three constraints are presented. Summary and discussion are given in Sec. V.

II. CANONICAL FORMALISM OF THE EINSTEIN THEORY IN SPHERICALLY SYMMETRIC SPACE-TIME

In this section, we briefly review the canonical formalism of the Einstein theory of the spherically symmetric space-time with a cosmological constant in four dimensions. We use the natural units $c=\hbar=G=1$ and follow mainly the convention adopted in Kuchař's work [6].

We start to consider the general metric for spherically symmetric space-time

$$ds^2=-N^2dt^2+\Lambda^2(dr+N^r dt)^2+R^2d\Omega^2, \tag{2.1}$$

where $d\Omega^2$ is the line element on the unit sphere. The lapse function N, the shift vector N^r, and the metrics Λ and R are functions of the time coordinate t and the radial coordinate r.

The action of the Einstein theory is of the form

$$I=\frac{1}{16\pi}\int d^4x\sqrt{-{}^{(4)}g}({}^{(4)}R-2\lambda), \tag{2.2}$$

where λ denotes the cosmological constant, and ${}^{(4)}R$ and ${}^{(4)}g$ are the scalar curvature and the determinant of metrics in four dimensions, respectively.

Substituting the metrics (2.1) into the action (2.2), the action in the ADM decomposition is given as

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$$I=\int dtdr\bigg[-N^{-1}\bigg(R(-\dot{\Lambda}+(\Lambda N^r)')(-\dot{R}+R'N^r)+\frac{1}{2}\Lambda(-\dot{R}+R'N^r)^2\bigg)+N\bigg(-\Lambda^{-1}RR''-\frac{1}{2}\Lambda^{-1}R'^2+\Lambda^{-2}RR'\Lambda'\bigg)+\frac{1}{2}N\Lambda(1-\lambda R^2)\bigg], \tag{2.3}$$

where the superscripts, dot and prime, denote the derivative with respect to t and r, respectively. This action can describe the space-time structure of the black holes and the universe, but does not include the gravitational waves, as the gravitational field depends only on the time and radial coordinates.

The canonical momenta are

$$P_\Lambda=-N^{-1}R(\dot{R}-R'N^r),$$
$$P_R=-N^{-1}[R(\dot{\Lambda}-(\Lambda N^r)')+\Lambda(\dot{R}-R'N^r)]. \tag{2.4}$$

The action can be written in the canonical form

$$I=\int dtdr[P_R\dot{R}+P_\Lambda\dot{\Lambda}-(NH+N^rH_r)], \tag{2.5}$$

where

$$H=-R^{-1}P_\Lambda P_R+\frac{1}{2}R^{-2}\Lambda P_\Lambda^2+\Lambda^{-1}RR''-\Lambda^{-2}RR'\Lambda'+\frac{1}{2}\Lambda^{-1}R'^2-\frac{1}{2}\Lambda+\frac{\lambda}{2}\Lambda R^2,$$
$$H_r=R'P_R-\Lambda P'_\Lambda. \tag{2.6}$$

The quantities H and H_r are called the super-Hamiltonian and the super-momentum of gravity and should vanish by the variational principle with respect to N and N^r,

$$H\approx 0,\quad H_r\approx 0. \tag{2.7}$$

They reflect the general covariance of the spherically symmetric space-time and form the first class constraints.

In order to study the canonical theory of the black hole, it is very useful to introduce the mass function, which is defined by the integration with respect to the space coordinate r of a linear combination of the super-Hamiltonian and the supermomentum as

$$-\int^r dr(\Lambda^{-1}R'H+R^{-1}P_\Lambda\Lambda^{-1}H_r)=-\frac{1}{2}\bigg(-R^{-1}P_\Lambda^2+\Lambda^{-2}RR'^2-R+\frac{\lambda}{3}R^3\bigg)-m\equiv M-m\ , \tag{2.8}$$

where m is a integration constant, which is assigned to a mass of the black hole. The physical meaning of the mass function is studied by Nambu and Sasaki [11], who identified it with the energy of the inner part of the black hole. The canonical treatment of the mass function was considered by Fischler, Morgan, and Polchinski [5], and by Kuchař [6]. The mass function can also be considered as a constraint,

$$M-m\approx 0. \tag{2.9}$$

Thus, we have three constraints. The Poisson brackets at the same time t among these constraints form involution relations:

$$\{H_r(r),H_r(r')\}=H_r(r)\delta'(r-r')-(r\leftrightarrow r'),$$
$$\{H(r),H_r(r')\}=H'(r)\delta(r-r')+H(r)\delta'(r-r'),$$
$$\{H(r),H(r')\}=\Lambda^{-2}(r)H_r(r)\delta'(r-r')-(r\leftrightarrow r'),$$
$$\{M(r),H_r(r')\}=M'(r)\delta(r-r'),$$
$$\{M(r),H(r')\}=-\Lambda(r)^{-3}R'(r)H_r(r)\delta(r-r'),$$
$$\{M(r),M(r')\}=0, \tag{2.10}$$

where r and r' are different space coordinates, $\delta'(r-r')=\partial\delta(r-r')/\partial r$, and $(r\leftrightarrow r')$ indicates the interchange of the argument in the preceding term. The Poisson brackets, Eq. (2.10) show that the mass function is a constant of motion in a weak sense.

III. THE QUANTIZATION AND THE OPERATOR ORDERING

We quantize the spherically symmetric geometry constructed in Sec. II. In the Schrödinger picture, the momenta corresponding to the metrics Λ and R in Eq. (2.4) are represented by functional differential operators

$$\hat{P}_\Lambda(r)=-i\frac{\delta}{\delta\Lambda(r)},$$
$$\hat{P}_R(r)=-i\frac{\delta}{\delta R(r)}. \tag{3.1}$$

In the following we use the notation ''hat'' for the quantized operators and we do not express the time t explicitly, because we always treat the product of operators at the same time.

According to the Dirac approach, the constraints of the system, Eq. (2.7), are treated as operator restrictions on the wave function Ψ:

$$\hat{H}\Psi=0,$$
$$\hat{H}_r\Psi=0,$$
$$\hat{M}\Psi=m\Psi. \tag{3.2}$$

The first equation is the Wheeler-DeWitt (WD) equation of a spherically symmetric geometry, and the integration constant

m has the meaning of the eigenvalue of the mass operator which is interpreted as a mass of the quantum black hole.

In order to construct the quantum theory, we have to define the operator ordering in $\hat{H}$, $\hat{H}_r$ and $\hat{M}$. The ordering is fixed such that the commutation relations between these operators form a closed algebra. Then, we proceed to solve the constraint equations.

A. The operator ordering

We fix the ordering as

$$\hat{H}=\frac{1}{2}\Lambda R^{-2}\hat{P}_\Lambda^{(C)}\hat{P}_\Lambda-R^{-1}\hat{P}_R\Lambda\hat{P}_\Lambda^{(B)}\Lambda^{-1}+\Lambda R'^{-1}\left(\frac{R}{2}(\chi-F)\right)',$$
$$\hat{H}_r=R'\hat{P}_R-\Lambda(\hat{P}_\Lambda)',$$
$$\hat{M}-m=\frac{1}{2}R^{-1}\hat{P}_\Lambda^{(A)}\hat{P}_\Lambda-\frac{1}{2}R(\chi-F), \tag{3.3}$$

where

$$\chi\equiv\Lambda^{-2}R'^2,$$
$$F=1-2mR^{-1}-\frac{\lambda}{3}R^2. \tag{3.4}$$

Here the quantities $\hat{P}_\Lambda^{(A)}$, $\hat{P}_\Lambda^{(B)}$ and $\hat{P}_\Lambda^{(C)}$ are

$$\hat{P}_\Lambda^{(A)}=A\hat{P}_\Lambda A^{-1},$$
$$\hat{P}_\Lambda^{(B)}=A^{1/2}\hat{P}_\Lambda A^{-1/2}=\frac{1}{2}(\hat{P}_\Lambda+\hat{P}_\Lambda^{(A)}),$$
$$\hat{P}_\Lambda^{(C)}=C\hat{P}_\Lambda C^{-1}=\hat{P}_\Lambda^{(A)}-iRR'^{-1}\left(A^{-1}\frac{\delta A}{\delta\Lambda}\right)', \tag{3.5}$$

where

$$C=A\exp\left[-\int^r drRR'^{-1}\int^\Lambda d\Lambda\left(A^{-1}\frac{\delta A}{\delta\Lambda}\right)'\right]. \tag{3.6}$$

The above ordering is taken by the following reasons: (i) In the limit of neglecting operator ordering, all operators reduce to the classical ones defined in Eqs. (2.6) and (2.8); (ii) the Hamiltonian and the mass operators are defined such that they satisfy the relation

$$(\hat{M})'=-\Lambda^{-1}R'\hat{H}-R^{-1}\hat{P}_\Lambda^{(B)}\Lambda^{-1}\hat{H}_r, \tag{3.7}$$

which is a quantum version of Eq. (2.8).

It is worthwhile to note that reason (ii) in the above is important, because once we find the wave function that satisfies the momentum and also the mass constraint, then this wave function satisfies the Hamiltonian constraint automatically by virtue of the relation in Eq. (3.7). Note that another form of operator ordering, for example, the symmetric ordering for the momentum constraint $\hat{H}_r$, is related to that in Eq. (3.3) by a similarity transformation [12]. So our choice of operator ordering does not lose generality.

B. The choice of the ordering factor A

The ordering factor A is chosen such that the algebra among constraints closes and also the constraint equations can be solved analytically. We take A as a function of R and $\chi\equiv(R'/\Lambda)^2$. The reason is that these quantities have a good property in the sense that they satisfy the following commutation relation:

$$[\phi(r),\hat{H}_r(r')]=i\phi'(r)\delta(r-r'), \tag{3.8}$$

where $\phi=R$ or χ. We call the operator that satisfies the commutation relation in Eq. (3.8) a good operator. In addition to R and χ, $\hat{P}_\Lambda$ is a good operator. Furthermore, we can define Z as a functional of R and χ,

$$Z=\int dr\Lambda f(R,\chi)=\int dr\int^\Lambda d\Lambda\overline{f}(R,\chi), \tag{3.9}$$

where f and $\overline{f}$ are arbitrary functions and the integration of r extends up to boundaries. The functions f and $\overline{f}$ are related as

$$f(R,\chi)=-\frac{\chi^{1/2}}{2}\int^\chi\frac{dx}{x^{-3/2}}\overline{f}(R,x). \tag{3.10}$$

The quantity Z has a special property

$$[Z,\hat{H}_r(r)]=i\left(R'(r)\frac{\delta Z}{\delta R(r)}-\Lambda(r)\left(\frac{\delta Z}{\delta\Lambda(r)}\right)'\right)=0, \tag{3.11}$$

thus Z is also a good operator, since $Z'=0$ by the definition, Eq. (3.9).

Now we choose A as

$$A=A_Z(Z)\overline{A}(R,\chi), \tag{3.12}$$

then A and, resultantly, $\hat{P}_\Lambda^{(A)}$ are good operators. It is worthwhile to note that we chose the factorizable form for A which is to guarantee

$$[\Lambda(r),\hat{P}_\Lambda^{(A)}(r')]=i\delta(r-r'),\quad[\hat{P}_\Lambda^{(A)}(r),\hat{P}_\Lambda^{(A)}(r')]=0. \tag{3.13}$$

Now, we observe that the mass operator $\hat{M}$ in Eq. (3.3) contains good operators only, so that $\hat{M}$ also becomes a good operator.

C. Commutation relations

The straightforward calculation leads to

$$[\hat{H}_r(r),\hat{H}_r(r')]=i\hat{H}_r(r)\delta'(r-r')-(r\leftrightarrow r'), \tag{3.14}$$

$$[\hat{M}(r),\hat{M}(r')]=0, \tag{3.15}$$

$$[\hat{M}(r),\hat{H}_r(r')]=i\hat{M}'(r)\delta(r-r'). \tag{3.16}$$

The above expression corresponds to the Poisson brackets in Eq. (2.10). The commutation relations involving $\hat{H}$ are complicated. By differentiating the commutator between $\hat{M}$ and $\hat{H}_r$ in Eq. (3.16) with respect to r, we find the commutator between $\hat{H}(r)$ and $\hat{H}_r(r')$. Similarly, by differentiating the commutator between $\hat{M}$'s in Eq. (3.15) with respect to r and also r', we find the commutators between $\hat{H}$ and $\hat{M}$ or $\hat{H}$'s. The result is as follows:

$$[\hat{H}(r),\hat{H}_r(r')]=i\hat{H}'(r)\delta(r-r')+i\hat{H}(r)\delta'(r-r'), \tag{3.17}$$

$$\begin{aligned}[\hat{H}(r),\hat{M}(r')]=&\,i\Lambda^{-3}R'\hat{H}_r(r)\delta(r-r')\\&+\Lambda^{-2}R^{-1}\hat{H}(r)\delta(0)\delta(r-r')\\&-\frac{1}{4}\frac{\Lambda(r)}{R'(r)R(r)R(r')}\{g(r,r')\hat{P}_\Lambda(r')\\&-\hat{P}_\Lambda^{(A)}(r')g(r,r')\}\Lambda^{-1}(r)\hat{H}_r(r),\end{aligned} \tag{3.18}$$

$$\begin{aligned}[\hat{H}(r),\hat{H}(r')]=&\,i\Lambda^{-2}(r)\hat{H}_r(r)\delta'(r-r')\\&-h(r,r')H_r(r')-(r\leftrightarrow r').\end{aligned} \tag{3.19}$$

The functions g and h are defined by

$$\begin{aligned}g(r,r')=&\,\frac{\delta}{\delta\Lambda(r)}\left(A^{-1}(r)\frac{\delta A(r)}{\delta\Lambda(r')}\right)=(A_Z^{-1}A_Z')'\overline{f}(r)\overline{f}(r')\\&+\left\{A_Z^{-1}A_Z'\frac{\partial\overline{f}}{\partial\Lambda}+\frac{\partial}{\partial\Lambda}\left(\overline{A}^{-1}\frac{\partial\overline{A}}{\partial\Lambda}\right)\delta(0)\right\}\delta(r-r'),\\h(r,r')=&\left(\frac{\Lambda}{R'}\right)_{r'}\left[\left(\frac{\Lambda}{2R^2}\hat{P}_\Lambda^{(C)}\hat{P}_\Lambda-R^{-1}P_R\Lambda\hat{P}_\Lambda^{(B)}\Lambda^{-1}\right)_r,\right.\\&\left.(R^{-1}\hat{P}_\Lambda^{(B)}\Lambda^{-1})_{r'}\right],\end{aligned} \tag{3.20}$$

where the dash for A_Z denotes the derivative with respect to Z such as $A_Z'=dA_Z/dZ$, and the function $h(r,r')$ is determined once functional forms of A_Z and $\overline{A}$ are given.

In comparison with the Poisson brackets, there appear new terms in the commutators between $\hat{H}$ and $\hat{M}$, and between $\hat{H}$'s. In particular, these terms are nonlocal in the sense that they are not proportional to the delta function. All nonlocal terms appear as coefficients of $\hat{H}_r$. These are due to the introduction of a functional $A_Z(Z)$. Since the commutator between $\hat{H}$ and $\hat{M}$ contains only terms proportional to $\hat{H}$ and $\hat{H}_r$, the mass operator $\hat{M}$ is a conserved quantity.

We computed all commutators and observed that the algebra is closed. We can also check that these operators satisfy the Jacobi identities.

IV. SOLUTIONS OF THE WHEELER-DEWITT EQUATION AND OTHER CONSTRAINTS

In general, it is quite hard to obtain the solution which satisfies the Wheeler-DeWitt equation and the momentum equation simultaneously. However, by our construction of the Hamiltonian, the simultaneous solution of both the momentum and the mass constraints satisfies the Wheeler-DeWitt equation automatically, as we can see in Eq. (3.7). Therefore, we concentrate on obtaining solutions of the momentum and the mass constraints.

As for the momentum constraint, we can immediately obtain the solution by observing the commutation relation in Eq. (3.11). That is, the momentum constraint requires that the wave function is the function of Z only,

$$\Psi=\Psi(Z). \tag{4.1}$$

Next, the mass operator $\hat{M}-m$ is applied to the above solution of the momentum constraint, Eq. (4.1). It leads to

$$\left(\frac{\delta Z}{\delta\Lambda}\right)^2\frac{d^2\Psi(Z)}{dZ^2}+A\left[\frac{\delta}{\delta\Lambda}\left(A^{-1}\frac{\delta Z}{\delta\Lambda}\right)\right]\frac{d\Psi(Z)}{dZ}+(R\sqrt{\chi-F})^2\Psi(Z)=0. \tag{4.2}$$

The function $\overline{A}$ is an arbitrary function of R and χ. We fix $\overline{A}$ such that the above equation becomes an ordinary differential equation with respect to Z. If we take

$$\overline{A}(R,\chi)=\frac{\delta Z}{\delta\Lambda}=R\sqrt{\chi-F}, \tag{4.3}$$

then we find

$$\frac{d^2\Psi}{dZ^2}-A_Z^{-1}\frac{\delta A_Z}{\delta\Lambda}\frac{d\Psi}{dZ}+\Psi=0. \tag{4.4}$$

Only the unfixed part is A_Z, for which we take some special form such that solutions of this equation become some special functions.

(i) Hypergeometric type solutions. If we choose $A_Z=Z^\sigma(Z-1)^\delta$ and transform the wave function as

$$\Psi^{(\sigma,\delta)}(Z)=Z^{\sigma+1}(Z-1)^{\delta+1}\psi(Z), \tag{4.5}$$

we find

$$\psi''+\left(\frac{\sigma+2}{Z}+\frac{\delta+2}{Z-1}\right)\psi'+\frac{\sigma+\delta+2}{Z(Z-1)}\psi=0. \tag{4.6}$$

Solutions are given by hypergeometric function as

$$\begin{aligned}\Psi^{(\sigma,\delta)}(Z)=&\,Z^{\sigma+1}(Z-1)^{\delta+1}\{a_1F(\alpha,\beta,\gamma;Z)\\&+a_2Z^{1-\gamma}F(\alpha-\gamma+1,\beta-\gamma+1,2-\gamma;Z)\},\end{aligned} \tag{4.7}$$

where a_1 and a_2 are integration constants, and α, β and γ are given by

$$\alpha\beta=\sigma+\delta+2, \quad \alpha+\beta=\sigma+\delta+3, \quad \gamma=\sigma+2. \tag{4.8}$$

(ii) Bessel type solutions. As a special case of (i), we take $A_Z = Z^{2\nu-1}$. Then, the equation becomes

$$\frac{d^2\Psi(Z)}{dZ^2} - \frac{2\nu-1}{Z}\frac{d\Psi(Z)}{dZ} + \Psi(Z) = 0. \tag{4.9}$$

Solutions of Eq. (4.9) are given by the Hankel (or Bessel) functions as

$$\Psi^{(\nu)}(Z) = Z^{\nu}\{b_1 H_{\nu}^{(1)}(Z) + b_2 H_{\nu}^{(2)}(Z)\}, \tag{4.10}$$

where b_1, b_2 are integration constants. In a case of $\nu = 1/2$, the solution takes a simple form of an exponential type as

$$\Psi^{(1/2)}(Z) = \sqrt{\frac{2}{\pi}}\ \{b_1 e^{i(Z-\pi/2)} + b_2 e^{-i(Z-\pi/2)}\} \ . \tag{4.11}$$

V. SUMMARY AND DISCUSSION

Under the anzatz of spherical symmetry, we have studied the canonical quantum theory of the Einstein action with a cosmological constant. We gave the operator ordering for $\hat{H}$, $\hat{H}_r$ and $\hat{M}$ and showed that they form a closed algebra. The commutators reproduced all terms that appeared in the Poisson bracket, but we found some terms which are proportional to H_r. All these terms contain the nonlocal terms in a sense that they are not proportional to δ function and its derivatives. In this operator ordering, we derived analytic solutions which satisfy the Hamiltonian, the momentum and the mass constraints. Extension to include the electromagnetic field is also interesting in view of extremal black holes and cosmological black holes [13]. Its corresponding quantum theory can be done in a straightforward way in our formalism.

In our analysis, the constraint equations, Eq. (3.2), are imposed on an arbitrary Cauchy surface. On the surface, through Eq. (3.7), the Hamiltonian and the momentum constraints require that the dynamical operator M takes a same value. Using this property, we introduced the mass eigenequation as a constraint. As a result, the mass operator is reduced to constant value and substantially nondynamical, and the remaining quantum fluctuation is the ambiguity of the choice of the Cauchy surface. Such separation of the quantum fluctuation is justified by the closed algebra between the new constraint and the Hamiltonian and momentum constraints. Therefore, our procedure is allowed if and only if the operator ordering in the Wheeler-DeWitt equation is chosen as Eq. (3.3).

We now compare our solutions with the WKB (semiclassical) solution. The WKB wave function is given by [5,7,10]

$$\Psi_G \sim c_1 e^{iZ_{\mathrm{WKB}}} + c_2 e^{-iZ_{\mathrm{WKB}}} \ , \tag{5.1}$$

where c_1, c_2 are constants and

$$\begin{aligned} Z_{\mathrm{WKB}} &= \int dr \int^{\Lambda} d\Lambda R\sqrt{\chi - F} \ , \\ &= \int dr R\Lambda\left(\sqrt{\chi-F} + \sqrt{\chi}\log\left|\frac{\sqrt{-F}}{\sqrt{\chi}+\sqrt{\chi-F}}\right|\right) \\ &\qquad (\text{for } F<0), \end{aligned} \tag{5.2}$$

where χ and F have been defined in Eq. (3.4). The function Z_{WKB} turns out to be the same form as Z in Eq. (3.9) and, therefore, the WKB solution agrees with our Bessel type solution for $\nu = 1/2$. Our Bessel type solutions with arbitrary ν approach the WKB solution for large values of Z.

Our solutions are general since we did not impose any coordinate conditions. All known solutions are derived by imposing some coordinate conditions. In the following, we see whether our solutions reduce to some known solutions of the Schwarzschild or the de Sitter geometry by imposing some coordinate conditions. Our solutions are given as a function of Z which is determined by Eqs. (3.9) and (4.3). Thus, the limiting procedure is the process to find the restricted functional form of Z.

(1) The Schwarzschild black hole case ($\lambda=0$). As for the coordinate conditions, we impose $\dot{R}=1$ and $R'=0$, which leads to $\chi = \Lambda^{-2}R'^2 = 0$. Furthermore, we require $\Lambda = \sqrt{-F}$. Then, the function Z is obtained as

$$Z = \left[\int dr \int^{\Lambda} d\Lambda R\sqrt{\chi-F}\right]_{\chi=0,\Lambda=\sqrt{-F}} = \int dr(-R+2m).$$

The Bessel type solution with the above Z agrees with the ones given by Nakamura *et al.* [14] and by our previous work [15]. This case was extended to the case of $\lambda \neq 0$ and solutions were obtained [16].

(2) The de Sitter universe case ($m=0$). The coordinate conditions, $\chi=1$ and $\dot{R} = \sqrt{\lambda/3}R'$, are imposed. If, in addition, we require the relation $\Lambda = \sqrt{\lambda/3}R = a(t)e^r$, where $a(t) = \exp(\sqrt{\lambda/3}t)$, Z is determined as $Z \propto a(t)^3$ for large Z. If we choose the index of the Bessel type solution as $\nu = 1/3$ (Airy function), our solution agrees with that of Horiguchi [17] asymptotically.

In summary, our solutions will be used to investigate the early universe, the inflation epoch, and the quantum black hole. In order to study the quantum epoch beyond the semiclassical region, we have to adopt some interpretation of the quantum universe. The de Broglie–Bohm (pilot wave, quantum potential) interpretation [18] lies in a unique position in the sense that this provides the notion of trajectory in the full quantum region. The study of the cosmological quantum black hole is now under investigation by using the de Broglie–Bohm interpretation [19].

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$$\hat{H}_r^{(\mathrm{sym})}\Psi^{(\mathrm{sym})}=0, \quad \hat{H}_r^{(\mathrm{sym})}\equiv\frac{1}{2}(R'\hat{P}_R+\hat{P}_R R'-\Lambda\hat{P}'_\Lambda-\hat{P}'_\Lambda\Lambda),$$

then the operator and the wave function are related to those in Eqs. (3.3) and (3.2) by the similarity transformation as

$$D^{-1}\Psi^{(\mathrm{sym})}=\Psi, \quad D^{-1}\hat{H}_r^{(\mathrm{sym})}D=\hat{H}_r=R'\hat{P}_R-\Lambda\hat{P}'_\Lambda,$$

where $D=\exp[-\frac{1}{2}\int dr \ln R'(r)\Lambda(r)]$.

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