Perdurance of multiply connected de Sitter space

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This paper deals with a study of the effects that spherically symmetric first-order metric perturbations and vacuum quantum fluctuations have on the stability of the multiply connected de Sitter spacetime recently proposed by Gott and Li. It is the main conclusion of this study that although such a spacetime is stable to the classical metric perturbations for any size of the nonchronal region, it is only stable against the quantum fluctuations of vacuum if the size of the multiply connected region is of the order of the Planck scale. Therefore, boundary conditions for the state of the universe based on the notion that the universe created itself in a regime where closed timelike curves were active and stable still appear to be physically and philosophically well supported as are those boundary conditions relying on the notion that the universe was created out of nothing. [S0556-2821(99)01612-4]

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I. INTRODUCTION

When he was arguing in support of his favorite condition, Hawking considered [1] that the choice of the boundary conditions for the quantum state of the universe should be made on just compact metrics or noncompact metrics which are asymptotic to metrics of maximal symmetry. While completeness seems to be an appealing property, spaces which are maximally symmetric are desirable for their elegance and great simplicity. At the time, the quest for the boundary conditions of the universe has already collected a number of proposals with the above or similar properties, including Vilenkin's tunneling wave function [2], the vanishing Weyl-tensor condition of Penrose [3], and, perhaps most popularly, the no-boundary proposal of Hartle and Hawking [4]. They have all been thoroughly discussed from different perpectives [5], particularly by invoking their capability to generate a suitable inflationary mechanism able to solve the original problems of standard cosmology [5,6], giving at the same time rise to a scale-invariant spectrum of density fluctuations [7,8], compatible with the present observational status of the universe.

Recent observations of the density of gravitational lenses in the universe [9,10] and estimates of the present value of the Hubble constant [11,12] have nevertheless led to the conclusion that the critical density Ω has to be smaller than unity, thus favoring an open rather than closed model for the universal expansion. The problem then is that most of the inflationary models compatible with the proposed boundary conditions predict a value for Ω very close to unity. In order to solve this problem, Hawking and Turok have recently suggested [13] the existence of a singular instanton which is able to generate an open inflationay universe within the framework of the no-boundary proposal. Vilenkin has argued [14] that the singularity of this instanton may have catastrophic consequences and, therefore, some procedures have been advanced [15,16] to make the instanton nonsingular, while still inducing an open inflationary process. Nevertheless, recent measurements of the most distant supernovas [17] may question the notion that the universe should be open when one allows the presence of a nonzero effective cosmological constant. This would open the possibility for the universe to be flat and then remove the need for open inflation and, thus, for the Hawking-Turok or Gott proposal.

It is worth noting that as early as 1982 Gott already proposed [18] a procedure leading to an open inflationary model for the universe. At the time, of course, it was largely disregarded, since it was then generally believed that the universe is closed. Motivated by the present observational status, Gott and Li have now suggested [19] new boundary conditions based on Gott's original model of open inflation. These boundary conditions assume the existence of a nonchronal region with closed timelike curves (CTC's) in de Sitter space, separated from the observable universe by a chronology horizon [20]. The resulting model can be compared to, e.g., the most popular no boundary paradigm as follows. Consider first de Sitter space and visualize it by means of the Schrödinger five-hyperboloid with Minkowskian coordinates [21,22]. If we slice this space along surfaces of constant timelike coordinate, then the slices become three-spheres and represent a closed universe. On this slicing both the no-boundary and the Vilenkin conditions hold, so that in these models the beginning of the universe can be pictured (in the Euclidean framework) as being the south pole of the Earth [23], with time running along the Earth's meridians. Thus, asking what happened before the beginning of the universe is like asking what is south of the south pole. Therefore, these boundary conditions can be regarded to implement the idea that the universe was created out of nothing. Although slicings in standard de Sitter space are, of course, completely arbitrary and, therefore, the key point about, e.g., the Hartle-Hawking proposal is that there is no initial boundary, not whether you consider time to begin at the south pole or anywhere else in the Euclidean region, the picture that follows when one chooses the above particular slicing is highly illustrative [23], so that we shall now choose another particular

slicing of de Sitter space in order to visualize the Gott-Li boundary condition in Lorentzian quantum cosmology. In contraposition to the above approach, one can also slice Schrödinger's five-hyperboloid vertically, i.e., along the only spacelike direction defined in terms of proper time [19]. The resulting slices are negatively curved surfaces, describable in terms of open cosmological solutions. It is on this slicing that the Gott-Li boundary condition can be defined. Since time should now lie along the Earth's equator (or actually any of its parallels), the origin of time can by no means be visualized or fixed; so asking what was the earliest point is like asking what is the easternmost point on the Earth's equator: there will always be an eastern (that is, earlier) point. This was implied to mean [19] that on some region of the slices there must be CTC's. These curves will take the job of always shifting the origin of the cosmological time. Saying then that the universe was created from nothing would be meaningless; what one should instead say is that it created itself [19].

We have thus two different types of boundary conditions of the universe that may induce it to be open. Whereas the no boundary condition does it by rather an indirect way which involves some suitably modified version of the Hawking-Turok instanton, the Gott-Li proposal creates the open universe directly. Is the latter proposal therefore more fashionable than the former? A positive answer to this question could only be made once the Gott-Li model satisfactorily passes some important tests of its consistency. First of all, one has to check whether the multiply connected de Sitter space is classically and quantum-mechanically stable. Li and Gott claimed [24] that all multiply connected spacetimes with a chronology horizon (derived from Misner space) are stable to quantum fluctuations of vacuum, but previous work by Kay, Radzikowski, and Wald [25] and by Cassidy [26] has raised compelling doubts on this conclusion. The present paper aims at partly filling the above requirements by studying the classical and quantum stability properties of multiply connected de Sitter space. This will be done using both a first-order perturbation procedure paralleling the method devised by Regge and Wheeler [27] to investigate the stability of Schwarzschild spacetime and a time-quantization procedure [28] to analyze the regularity of the solution against vacuum quantum fluctuations. Our main conclusion is that multiply connected de Sitter space is stable both classically and quantum mechanically. Clearly, a particular slicing of the de Sitter space may be picked out by the surfaces of constant inflation field, but this is relevant only when inflation ends, because the topology of these slices will determine whether we live in an open or closed Friedman-Robertson-Walker universe. For the same reason, our stability analysis for the static slicing is only developed as a tool to be used on the nonchronal region of de Sitter space. Obviously, the property that ordinary de Sitter space is perturbatively stable must be independent of the used coordinate system. Quantum stability is, however, restricted to hold only in the very small regions where time shows its essential quantum character.

We outline the paper as follows. In Sec. II we briefly review how a multiply connected de Sitter space can be constructed, and show why its nonchronal region must be confined inside the cosmological horizon. The stability of the whole de Sitter space has already been investigated in terms of a global Friedmann-Robertson-Walker metric [29,30]. However, as far as I know, no corresponding research has been hitherto attempted for the de Sitter region covered by static coordinates, that is, the region where multiply connectedness and CTC's should appear. We have performed this study here, first for the simply connected static case in Sec. III and then for the multiply connected case in Sec. IV. This section also contains an analysis of the stability of multiply connected de Sitter space against quantum fluctuations of vacuum. Finally, we summarize and conclude in Sec. V.

II. MULTIPLY CONNECTED de SITTER SPACE

de Sitter space is usually identified [22] as a maximally symmetric space of constant negative curvature (positive Ricci scalar) which is a solution to the vacuum Einstein equations with a positive cosmological constant $\Lambda>0$. Following Schrödinger [21,30], it can be visualized as a five-hyperboloid defined by

$$w^2+x^2+y^2+z^2-v^2=\rho_0^2, \tag{2.1}$$

where $\rho_0=\sqrt{3/\Lambda}$. This hyperboloid is embedded in E^5 and the most general expression for the metric of the de Sitter space is then that which is induced in this embedding, i.e.,

$$ds^2=-dv^2+dw^2+dx^2+dy^2+dz^2, \tag{2.2}$$

which has topology $R\times S^4$, invariance group SO(4,1), and shows ten Killing vectors (four boosts and six rotations).

Metric (2.2) can be coveniently exhibited in either global or static coordinates. Global coordinates $t'\in(-\infty,\infty)$, $\psi_3,\psi_2\in(0,\pi)$, and $\psi_1\in(0,2\pi)$ can be defined by [30]

$$\begin{aligned} z&=\rho_0\cosh(t'/\rho_0)\sin\psi_3\sin\psi_2\cos\psi_1,\\ y&=\rho_0\cosh(t'/\rho_0)\sin\psi_3\sin\psi_2\sin\psi_1,\\ x&=\rho_0\cosh(t'/\rho_0)\sin\psi_3\cos\psi_2,\\ w&=\rho_0\cosh(t'/\rho_0)\cos\psi_3,\\ v&=\rho_0\sinh(t'/\rho_0). \end{aligned} \tag{2.3}$$

In terms of these coordinates metric (2.2) becomes

$$ds^2=-dt'^2+\rho_0^2\cosh^2(t'/\rho_0)d\Omega_3^2, \tag{2.4}$$

where $d\Omega_3^2$ is the metric on the unit three-sphere. Metric (2.4) is a $k=+1$ Friedmann-Robertson-Walker metric whose spatial sections are three-spheres of radius $\rho_0\cosh(t'/\rho_0)$. Coordinates (2.3) entirely cover the four-

dimensional de Sitter space which would first contract until $t'=0$ and expand thereafter to infinity.

In order to exhibit metric (2.2) in static coordinates $t\in(-\infty,\infty)$, $\psi_3,\psi_2\in(0,\pi)$, $\psi_1\in(0,2\pi)$, one can use the definitions [30]

$$\begin{aligned} z&=\rho_0\sin\psi_3\sin\psi_2\cos\psi_1,\\ y&=\rho_0\sin\psi_3\sin\psi_2\sin\psi_1,\\ x&=\rho_0\sin\psi_3\cos\psi_2,\\ w&=\rho_0\cos\psi_3\cosh(t/\rho_0),\\ v&=\rho_0\cos\psi_3\sinh(t/\rho_0). \end{aligned} \tag{2.5}$$

Setting $r=\rho_0\sin\psi_3$ [i.e., defining $r\in(0,\rho_0)$], we obtain the static metric in de Sitter space:

$$ds^2=-\left(1-\frac{r^2}{\rho_0^2}\right)dt^2+\left(1-\frac{r^2}{\rho_0^2}\right)^{-1}dr^2+r^2\,d\Omega_2^2, \tag{2.6}$$

where $d\Omega_2^2$ is the metric on the unit two-sphere. The coordinates defined by Eqs. (2.5) cover only the portion of de Sitter space with $w>0$ and $x^2+y^2+z^2<\rho_0^2$, i.e., the region inside the particle and event horizons for an observer moving along $r=0$.

In order to see whether the whole or some restricted region of the de Sitter space can be made to have multiply connected topology, with CTC's on it, we will follow the procedure described by Gott and Li [19], thus checking whether a symmetry like that is satisfied by the Minkowskian covering to Misner space [22] somewhere holding in de Sitter space. On the Minkowskian five-hyperboloid visualizing de Sitter space, such a symmetry would be expressible by means of the identification [28]

$$\begin{aligned}(v,w,x,y,z)\Leftrightarrow(&v\,\cosh(nb)+w\sinh(nb),\\ &w\,\cosh(nb)+v\sinh(nb),x,y,z),\end{aligned} \tag{2.7}$$

where b is a dimensionless arbitrary quantity and n is any integer number. The boost transformation in the (v,w) plane implied by this identification will induce a boost tranformation in de Sitter space. Hence, since the boost group in de Sitter space is a subgroup of the de Sitter group, either the static or the global metric of de Sitter space can also be invariant under symmetry (2.7).

It is easy to see that there cannot exist any symmetry associated with identification (2.7) on the (v,w) plane which leaves metric (2.4) invariant for coordinates (2.3). It follows that the whole of de Sitter spacetime can neither be multiply connected nor have CTC's. However, for coordinates defined by Eqs. (2.5) leading to the static metric with an apparent horizon (2.6), the above symmetry can be satisfied in the region covered by such a metric, defined by $w>|v|$, where there are CTC's, with the boundaries at $w=\pm v$ and $x^2+y^2+z^2=\rho_0^2$ being the Cauchy horizons that limit the onset of the nonchronal region from the causal exterior [19]. Such boundaries become then appropriate chronology horizons for de Sitter space.

III. STABILITY OF STATIC de SITTER SPACE

While our discussion of Sec. II made it clear that multiply connected de Sitter space is mathematically rich and interesting, we still need to know if such a space is indeed a physical object. Therefore, in what follows we shall use a general-relativity perturbation method to investigate the stability of the multiply connected de Sitter universe. Since multiply connectedness and CTC's only appear in the region covered by static coordinates, the extension of the analysis of the cosmologically perturbed global metric for simply connected de Sitter space [29,30] to a multiply connected topology would unavoidably lead to rather inconclusive results. We instead shall proceed as follows. We first extend the perturbative procedure originally devised by Regge and Wheeler for the Schwarzschild problem [27] to a cosmological de Sitter space in this section, and then in the next section, we conveniently include the effects derived from the identification (2.7) in the resulting formalism. In the present paper we confine ourselves to a linear analysis, investigating the stability of simply and multiply connected de Sitter space in first-order perturbation theory by means of a generalization from the refined method developed by Vishveshwara [31] and Zerilli [32].

We take as the general background metric $g_{\mu\nu}$ and the perturbation on it as $h_{\mu\nu}$. The quantity $g_{\mu\nu}$ will be later specialized to be the static de Sitter metric, i.e.,

$$ds^2=-(1-H^2r^2)dt^2+(1-H^2r^2)^{-1}dr^2+r^2\,d\Omega_2^2, \tag{3.1}$$

where we have now denoted $H=\rho_0^{-1}$ for the sake of simplicity, and $x^0=t$, $x^1=r$, $x^2=\theta$, $x^3=\phi$. Metric (3.1) corresponds to the initial time-independent equilibrium configuration; so the problem to be solved is, then, if metric (3.1) is somehow perturbed, whether the perturbations will undergo oscillations about the equilibrium state or will grow exponentially with time. The static de Sitter space will be stable in the first case and unstable in the second one.

Since the background is spherically symmetric, any arbitrary perturbation can be decomposed in normal modes given by [27]

$$\sum f_0(t)f_1(r)f_2(\theta)f_3(\phi).$$

Associated with these modes we have an angular moment l and its projection on the z axis, M. For any given value of l there will be two independent classes of perturbations which are respectively characterized by their parities $(-1)^l$ (even parity) and $(-1)^{l+1}$ (odd parity). Furthermore, since the background is time independent, all time dependence of the perturbations will be given by the simple factor $\exp(-ikt)$, where k is the frequency of the given mode.

In order to derive the equations governing the perturbations, we shall start with the Einstein equations

$$R_{\mu\nu}(g)=\Lambda g_{\mu\nu}, \tag{3.2}$$

with g denoting the de Sitter background metric. For the perturbed spacetime, these field equations transform into

$$R_{\mu\nu}(g)+\delta R_{\mu\nu}(h)=\Lambda(g_{\mu\nu}+h_{\mu\nu}), \tag{3.3}$$

for small perturbation, with $\delta R_{\mu\nu}$ assumed to contain first-order terms in $h_{\mu\nu}$ only. Now, since Einstein equations are still valid in the perturbation scheme, we obtain in this approximation that the differential equations that govern the perturbations should be derived from the equations

$$\delta R_{\mu\nu}(h)=\Lambda h_{\mu\nu}. \tag{3.4}$$

The $\delta R_{\mu\nu}$'s will be here computed using the same general formulas as those employed by Regge and Wheeler [27] and Eisenhart [33], i.e.,

$$\delta R_{\mu\nu}=-\delta\Gamma^{\beta}_{\mu\nu;\beta}+\delta\Gamma^{\beta}_{\mu\beta;\nu}, \tag{3.5}$$

where the semicolon denotes covariant differentiation, and the variation of the Christoffel symbols is given by

$$\delta\Gamma^{\beta}_{\mu\nu}=\frac{1}{2}g^{\beta\alpha}(h_{\mu\alpha;\nu}+h_{\nu\alpha;\mu}-h_{\mu\nu;\alpha}). \tag{3.6}$$

After introducing suitable gauge transformations [31,32], the most general perturbations in static de Sitter space can be written in forms which are similar to those obtained for Schwarzschild space; i.e., for odd parity,

$$h_{\mu\nu}=\begin{bmatrix} 0 & 0 & 0 & h_0(r) \\ 0 & 0 & 0 & h_1(r) \\ 0 & 0 & 0 & 0 \\ \text{sym} & \text{sym} & 0 & 0 \end{bmatrix} \times\left[\sin\theta\left(\frac{\partial}{\partial\theta}\right)\right]P_l(\cos\theta)\exp(-ikt) \tag{3.7}$$

and, for even parity,

$$h_{\mu\nu}=\begin{bmatrix} H_0(1-H^2r^2) & H_1 & 0 & 0 \\ H_1 & H_2(1-H^2r^2)^{-1} & 0 & 0 \\ 0 & 0 & Kr^2 & 0 \\ 0 & 0 & 0 & Kr^2\sin^2\theta \end{bmatrix} \times P_l(\cos\theta)\exp(-ikt). \tag{3.8}$$

In these expressions ''sym'' indicates that $h_{\mu\nu}=h_{\nu\mu}$, $P_l(\cos\theta)$ is the Legendre polynomial, and h_0, h_1, H_0, H_1, H_2, and K are given functions of the radial coordinate r which must be determined as solutions to the respective wave equations, subject to suitable boundary conditions.

In order to ensure holding of regularity on the cosmological horizon at $r=H^{-1}$, we should transform the components of the perturbations in Eqs. (3.7) and (3.8) to a representation where the static metric is maximally extended and regular also on the cosmological horizon. In Kruskal coordinates u,v, the static de Sitter metric can thus be written as

$$ds^2=F(r)^2(du^2-dv^2)+r^2\,d\Omega_2^2, \tag{3.9}$$

where

$$F(r)^2=-\left(\frac{1+Hr}{H}\right)^2, \tag{3.10}$$

$$v^2-u^2=\frac{1-Hr}{1+Hr}, \tag{3.11}$$

and

$$Ht=-\tanh^{-1}\left(\frac{v}{u}\right). \tag{3.12}$$

In terms of the Kruskal coordinates, the components of the metric perturbations take the form (the components that invove only angular coordinates are the same in the two coordinate systems)

$$h^K_{00}=\frac{F^2}{u^2-v^2}[u^2(1-H^2r^2)^{-1}h_{00}+v^2(1-H^2r^2)h_{11}-2uvh_{01}],$$

$$h^K_{11}=\frac{F^2}{u^2-v^2}[v^2(1-H^2r^2)^{-1}h_{00}+u^2(1-H^2r^2)h_{11}-2uvh_{01}],$$

$$h^K_{01}=\frac{F^2}{u^2-v^2}\{(u^2+v^2)h_{01}-uv[(1-H^2r^2)^{-1}h_{00}+(1-H^2r^2)h_{11}]\},$$

$$h^K_{03}\propto\frac{1}{u^2-v^2}[uh_{03}-v(1-H^2r^2)h_{13}],$$

$$h^K_{13}\propto-\frac{1}{u^2-v^2}[vh_{03}-u(1-H^2r^2)h_{13}], \tag{3.13}$$

where the superscript K refers to Kruskal coordinates. For future reference, we introduce here the relation

$$\exp(Hr^*)=u^2-v^2, \tag{3.14}$$

where the new variable r^* is defined by

$$r^*=H^{-1}\ln\left(\frac{Hr-1}{Hr+1}\right). \tag{3.15}$$

A. Wave equations for the perturbations

We derive in what follows the differential equations which should be satisfied by the perturbations on the maximally extended de Sitter metric. We shall start with the odd-parity solutions, and choose the perturbed Einstein equations

$$\frac{kh_0}{1-H^2r^2}+\frac{d}{dr}(1-H^2r^2)h_1=0, \tag{3.16}$$

$$\frac{k(h_0'-kh_1-2h_0/r)}{(1-H^2r^2)}+(l-1)(l+2)\frac{h_1}{r^2}=24\pi GH^2h_1, \tag{3.17}$$

where the prime denotes differentiation with respect to the radial coordinate, $'=d/dr$. Let us first introduce the definition

$$rQ=(1-H^2r^2)h_1, \tag{3.18}$$

with which Eqs. (3.16) and (3.17) becomes

$$\frac{kh_0}{(1-H^2r^2)}+\frac{d}{dr}(rQ)=0, \tag{3.19}$$

$$kh_0'-k^2h_1-\frac{2kh_0}{r}+(l-1)(l+2)\frac{Q}{r}=24\pi GH^2(rQ). \tag{3.20}$$

We eliminate then kh_0 from Eqs. (3.19) and (3.20) to obtain

$$-(1-H^2r^2)\frac{d^2}{dr^2}(rQ)-\frac{k^2rQ}{(1-H^2r^2)}+\frac{2}{r}\frac{d}{dr}(rQ)$$

$$+(l-1)(l+2)\frac{Q}{r}=24\pi GH^2(rQ). \tag{3.21}$$

From Eq. (3.15) we finally obtain the wave equation

$$\frac{d^2Q}{dr^{*2}}+\frac{1}{4}(k^2-V_{eff})Q=0, \tag{3.22}$$

where

$$V_{eff}=(1-H^2r^2)\left[\frac{l(l+1)}{r^2}-24\pi GH^2\right] \tag{3.23}$$

is an effective potential and

$$h_0=\left(\frac{i}{k}\right)\left(\frac{d}{dr^*}\right)(rQ). \tag{3.24}$$

The derivation of the wave equation for the case of even-parity perturbations has more algebraic complications. From $\delta R_{\mu\nu}=\Lambda h_{\mu\nu}$ and the general expression (3.8), we obtain first the independent first-order perturbations of the Ricci tensor:

$$\frac{dK}{dr}+\frac{K-h}{r}+\frac{H^2rK}{(1-H^2r^2)}-\frac{il(l+1)h_1}{2kr^2}=0, \tag{3.25}$$

$$ikh_1+(1-H^2r^2)\left(\frac{dh}{dr}-\frac{dK}{dr}\right)-2H^2rh=0, \tag{3.26}$$

$$\frac{d}{dr}[(1-H^2r^2)h_1]+ik(h+K)=0, \tag{3.27}$$

$$(1-H^2r^2)^2\frac{d^2h}{dr^2}+\frac{2}{r}(1-H^2r^2)(1-3H^2r^2)\frac{dh}{dr}-k^2h$$
$$+\frac{2ik(1-H^2r^2)}{r^2}\frac{d}{dr}(r^2h_1)-(1-H^2r^2)\frac{l(l+1)h}{r^2}$$
$$-4ikH^2rh_1+2(1-H^2r^2)H^2r\frac{dK}{dr}-2k^2K$$
$$=-6H^2(1-H^2r^2)h, \tag{3.28}$$

$$2ik(1-H^2r^2)\frac{dh_1}{dr}-2ikH^2rh_1-k^2h$$
$$+\frac{(1-H^2r^2)^2}{r^2}\left[\frac{d}{dr}\left(r^2\frac{dh}{dr}\right)-2\frac{d}{dr}\left(r^2\frac{dK}{dr}\right)\right]$$
$$+(1-H^2r^2)\left[\frac{l(l+1)h}{r^2}-4H^2r\frac{dh}{dr}+2H^2r\frac{dK}{dr}\right]$$
$$=6H^2(1-H^2r^2)^2h, \tag{3.29}$$

$$\frac{d}{dr}\left\{(1-H^2r^2)\left[\frac{d}{dr}(r^2K)-2rh\right]\right\}+\frac{r^2k^2K}{(1-H^2r^2)}$$
$$-l(l+1)K-2ikrh_1=-6H^2r^2K, \tag{3.30}$$

where we have used $H_2=H_0\equiv h$, which is allowed by the high symmetry of the de Sitter space, and $H_1\equiv h_1$. The three first-order equations (3.25)–(3.27) can be used to derive any of the subsequent second-order equations (3.28)–(3.30), provided the following algebraic relationship is satisfied:

$$[(l-1)(l+2)-4H^2r^2(1-3H^2r^2)]h$$
$$+\left[-l(l+1)+\frac{2(k^2r^2+1+2H^2r^2)}{(1-H^2r^2)}\right]K$$
$$-\left[2ikr+\frac{3iH^2rl(l+1)}{k}\right]h_1=0. \tag{3.31}$$

The wave equation in a single unknown can now be derived from Eqs. (3.25), (3.26), (3.27), and (3.31). After a lot of algebraic manipulations we finally obtain

$$\frac{d^2\overline{S}}{dr^{*2}}-\frac{1}{2}\left\{4H^2r+\frac{4(1-H^2r^2)}{rD(r)}[4H^2r^2(1-3H^2r^2)-(l-1)(l+2)]\right\}\frac{d\overline{S}}{dr^*}$$
$$+\frac{1}{4}\left\{k^2+4H^2(2+H^2r^2)-\frac{(1-H^2r^2)}{r^2}(l-1)(l+2)-\frac{4}{r^2}+\frac{4(1-H^2r^2)}{D(r)}\right.$$
$$\left.\times\left[4H^2(1-H^2r^2)(1-3H^2r^2)+2k^2-\frac{(1-H^2r^2)}{r^2}(l-1)(l+2)+\frac{3H^2l(l+1)}{kr}\right]\right\}\overline{S}=0, \tag{3.32}$$

in which we have introduced the definition

$$\overline{S}=(1-H^2r^2)\frac{h_1}{r} \tag{3.33}$$

and the function $D(r)$ has the form

$$D(r)=\frac{2k^2r^2}{(1-H^2r^2)}-2(l-1)(l+2)+\frac{6H^2r^2}{(1-H^2r^2)}+4H^2r^2(1-3H^2r^2). \tag{3.34}$$

We note that $D(r)$ approaches infinity as r tends to the cosmological horizon. The wave equation (3.32) can also be expressed in terms of another function $S=h_1/r$:

$$\frac{d^2S}{dr^{*2}}-\frac{2(1-H^2r^2)}{rD(r)}[4H^2r^2(1-3H^2r^2)-(l-1)(l+2)]\frac{dS}{dr^*}+\frac{1}{4}\left\{3H^2(3+H^2r^2)-\frac{(1-H^2r^2)}{r^2}(l-1)(l+2)\right.$$
$$\left.+k^2-\frac{4}{r^2}+\frac{4(1-H^2r^2)}{D(r)}\left[4H^2(1-3H^2r^2)^2+2k^2-\frac{(1-3H^2r^2)}{r^2}(l-1)(l+2)+\frac{3H^2}{kr}l(l+1)\right]\right\}S=0, \tag{3.35}$$

with $D(r)$ also as given by Eq. (3.34), which will be most useful in what follows.

B. Odd-parity perturbations

The stability of the static region of de Sitter space to the odd-parity perturbations will be now examined. As in the Schwarzschild case [31,32], we shall distinguish two different cases: we first consider the situation when the frequency k is pure imaginary, and then we analyze the wave problem that results when that frequency is kept real.

Set $k=i\alpha$. Then, from Eqs. (3.22) and (3.24), we have

$$\frac{d^2Q}{dr^{*2}}=\frac{1}{4}(\alpha^2+V_{eff})Q=0, \tag{3.36}$$

$$h_0=\frac{1}{\alpha}\left(\frac{d}{dr^*}\right)(rQ). \tag{3.37}$$

The coordinate r^* ranges from $-\infty$ to $i\pi/H$. The upper limit can of course be made zero by a redefinition of r^*, such that now

$$r^*=\frac{1}{H}\ln\left[\frac{1-Hr}{1+Hr}\right].$$

The effective potential V_{eff} is real and positive everywhere along this range, and vanishes at $r^*=-\infty$, i.e., on the boundary at $r=H^{-1}$. The asymptotic solution to Eq. (3.36) as r approaches the cosmological horizon is

$$Q_H\sim\exp\left(\pm\frac{1}{2}\alpha r^*\right). \tag{3.38}$$

As $r\to0$, i.e., as one approaches the other boundary where we should require the perturbations to fall off to zero, one can write Eq. (3.36) in the form

$$\frac{d^2Q_0}{dr^2}+\frac{1}{4}\left(24\pi GH^2-\frac{l(l+1)}{r^2}-\alpha^2\right)Q_0\simeq0, \tag{3.39}$$

whose general solution can be given in terms of the Bessel functions [34]

$$Q_0\sim\sqrt{r}\mathcal{C}_{\sqrt{l(l+1)+1}/2}\left(\frac{1}{2}\sqrt{H^2-\alpha^2}r\right). \tag{3.40}$$

Since we should require the pertutbations to vanish as $r\to0$, we have to choose for the Bessel function $\mathcal{C}=\mathcal{J}$ [34]. However, if we take the function Q to be positive, one can see from Eq. (3.36) that d^2Q/dr^{*2} can never become negative within the entire range of r, from 0 to H^{-1}, and the solution that goes to zero at the origin $r=0$ [that is, Eq.

(3.40) with $\mathcal{C}=\mathcal{J}$] cannot be matched to the solution that goes to zero on the horizon at $r=H^{-1}$ [that is, Eq. (3.38) with the + sign in the argument of the exponential]. It follows that the asymptotic solution near $r=H^{-1}$ has to be

$$Q_H=A\exp\left(-\frac{1}{2}\alpha r^*\right), \tag{3.41}$$

where A is an arbitrary constant; so the radial solution near $r=H^{-1}$, Eq. (3.37), becomes

$$h_0=-\frac{Q_H}{2H}. \tag{3.42}$$

We can now compute the Kruskal perturbations given by Eqs. (3.13). Let us concentrate on the expression that results for h_{03}^K near the surface $r=H^{-1}$ (i.e., on $u=v$). Suppressing all angular dependence, we have

$$h_{03}^K\propto\frac{[uh_0-v(1-H^2r^2)h_1]e^{-\alpha t}}{u^2-v^2}=-\frac{(u/2H+rv)Q_H}{u^2-v^2}$$
$$\simeq-\frac{(v+u/2)}{H(u^2-v^2)}A\exp[-\alpha(r^*+t)].$$

Note that the sign of time has been reversed in the above equation with respect to those perturbations that appear in Schwarzschild space. This reversal expresses the fact that the expansion of the universe is in many ways similar to the collapse of a star, except that the sense of time is reversed [22].

Using then $\exp(-r^*)=1/(u^2-v^2)^{1/H}$ and $e^t=[(v-u)/(u+v)]^{1/2H}$, and taking into account that $u=v$ on the surface $r=H^{-1}$, we finally get

$$h_{03}^K\propto\frac{3}{4}(-1)^{1+\alpha/2H}A(u-v)^{-(1+3\alpha/2H)}(u+v)^{-\alpha/2H}. \tag{3.43}$$

Now, since at $t=0$ (i.e., on $u=0$) this perturbation becomes

$$h_{03}^K\propto\frac{3}{4H}(-1)^{-2/H}Av^{-(1+2\alpha/H)}, \tag{3.44}$$

which is clearly divergent at $t=0$ on the horizon, and the physically allowable perturbations should be regular everywhere in space at $t=0$, we see that this perturbation is physically unacceptable, and hence cannot exist. It follows that the odd-parity perturbations with purely imaginary frequency ought to be ruled out.

Let us consider now solutions that correspond to real frequencies k. We shall look first at the case of ingoing waves for which the asymptotic solutions near the horizon at $r=H^{-1}$ are ($V_{eff}=0$)

$$Q_H=A\exp\left(-\frac{ik}{2r^*}\right),\quad r^*\to-\infty. \tag{3.45}$$

For these solutions it holds that

$$h_1=\frac{A\exp(-ik/2r^*)}{H(1-H^2r^2)}, \tag{3.46}$$

$$h_0=\frac{A}{2H}\exp(-ik/2r^*). \tag{3.47}$$

For the cosmological perturbations in Kruskal coordinates we have then

$$h_{03}^K\propto\frac{A(u/2-v)}{H(u^2-v^2)}\exp(-ikr^*/2+ikt).$$

We note that for ingoing waves the horizon $r=H^{-1}$ should be taken at $u=-v$. Therefore,

$$h_{03}^K\propto\frac{3A}{4H(u+v)}\exp[ik(t-r^*/2)].$$

Using again $\exp(-r^*)=1/(u^2-v^2)^{1/H}$ and $e^t=[(v-u)/(u+v)]^{1/2H}$, we finally obtain, for this perturbation,

$$h_{03}^K\propto\frac{3A}{4H(u+v)}[i(u+v)]^{-ik/H}. \tag{3.48}$$

Since $u+v\to0$ on the horizon, at first glance this perturbation appears to be seriously divergent. However, as also happens in the Schwarzschild case [31], one can build wave packets which are convergent everywhere in space out of the monochromatic waves. If we form the purely ingoing waves into a wave packet by taking A to be a function of k given by the Fourier transform of a function $f(q)=\int A(k)\exp(-ikq)dk$, which vanishes for $q<1$, by integrating over k, Eq. (3.48) transforms into

$$h_{03}^K\propto\frac{3}{4H(u+v)}f\left\{\frac{1}{H}\ln[i(u+v)]\right\}. \tag{3.49}$$

There cannot be any singularity from the $(u+v)^{-1}$ factor in Eq. (3.49) because f is nonzero only when $i(u+v)>1$. Thus, h_{03}^K does not diverge anywhere in space, but it is always purely imaginary.

For outgoing waves, the asymptotic solutions near $r=H^{-1}$ are given by ($u=v$)

$$Q_H=A\exp(+ik/2r^*),\quad r^*\to-\infty, \tag{3.50}$$

and, for wave packets formed as before, we finally obtain

$$h_{03}^K\propto-\frac{3}{4H(u-v)}f\left\{\frac{1}{H}\ln[i(u-v)]\right\}, \tag{3.51}$$

which, although coverging everywhere in space, is always purely imaginary, such as happened for perturbation (3.49). The reason for these perturbations to be imaginary resides in the fact that the argument of the logarithm in the above expressions should be larger than unity, which in turn requires that both u and v be imaginary simultaneously. Note that this

does not imply rotation of the time t into the imaginary axis because Kruskal coordinates u and v appear in the form v/u in Eq. (3.12).

Thus, the main conclusion for odd-parity perturbations is that either they cannot exist whenever their frequency is purely imaginary or they are stable and physically acceptable if their frequency is real. Since in the latter case the solutions are purely imaginary, one can also conclude that such perturbations would not be observable.

C. Even-parity perturbations

In terms of the most covenient function $S=\overline{S}/(1-H^2r^2)$, the second-order wave equation for the even-parity perturbations given by Eq. (3.35), near the horizon at $r=H^{-1}$, becomes

$$\frac{d^2S}{dr^{*2}}+\frac{1}{4}(k^2+8H^2)S=0. \tag{3.52}$$

Thus, the asymptotic form of the general solution at $r=H^{-1}$, S_H, would read

$$S_H\sim\exp\left(\pm i\sqrt{\frac{1}{4}k^2+2H^2}r^*\right). \tag{3.53}$$

As for the boundary at $r=0$, one attains that no perturbation can consistently be expected, since from Eq. (3.35) we obtain $S_0=0$ (unless for $l=0$ for which case S_0 is an arbitrary constant), and therefore $h=h_1=0$. Hence, we can readily get an expression for h^K_{00}, independently of the value of the frequency. It follows that the two signs involved in the exponent of Eq. (3.53) are allowed and should therefore be taken into account in our analysis.

Again we first consider the case where the frequency k is purely imaginary, $k=i\alpha$. Then

$$S_H\sim\exp\left(\pm i\sqrt{2H^2-\frac{1}{4}\alpha^2}r^*\right). \tag{3.54}$$

We have two distinct cases: case I, for which $|\alpha|>\alpha_c\equiv 2\sqrt{2}H$, and case II, for which $|\alpha|<\alpha_c$. In case I the asymptotic solution reduces to

$$S_H\sim\exp(\pm\xi r^*), \tag{3.55}$$

where

$$\xi=\sqrt{\frac{1}{4}\alpha^2-2H^2} \tag{3.56}$$

is real. Because case II is qualitatively the same as that for real frequencies (to be dealt with later on), we shall concentrate now on case I only. Since $S=h_1/r$, near $r=H^{-1}$, one can assume the asymptotic forms

$$h_1=A\exp(\pm\xi r^*),\quad h=B\exp(\pm\xi r^*).$$

Choosing first the minus sign in the exponent of these two functions, from the equation relating radial functions h and h_1 to one another [which can be obtained by suitably combining Eqs. (3.26) and (3.27)], after specializing to $r=H^{-1}$ and $k=i\alpha$, i.e.,

$$4H\frac{dh_1}{dr^*}+4\frac{d^2h_1}{dr^{*2}}+\alpha^2h_1+2\alpha\left(Hh+2\frac{dh}{dr^*}\right)=0,$$

we get

$$B=\frac{2H\xi+4H^2-\alpha^2}{\alpha(2\xi-H)}A. \tag{3.57}$$

Let us now denote $\alpha=\pm\epsilon H$, with $\epsilon\geqslant 2\sqrt{2}$. Then the analysis of all possible resulting cases [including the use of both signs in the exponent of Eq. (3.54)] will require considering

$$\xi=\pm\sqrt{\frac{\epsilon^2}{4}-2}H.$$

We shall look at three significant values of ϵ, namely, $2\sqrt{2}$, 3, and ∞, in the following cases: (i) $\alpha>0$, $\xi>0$ ($B=\sqrt{2}A$ for $\epsilon=2\sqrt{2}$, $A=0$ for $\epsilon=3$, and $B=-A$ for $\epsilon\to\infty$), (ii) $\alpha>0$, $\xi<0$ ($B=\sqrt{2}A$ for $\epsilon=2\sqrt{2}$, and $A=B$ for $\epsilon=3$ and $\xi\to\infty$), (iii) $\alpha<0$, $\xi<0$ ($B=-\sqrt{2}A$ for $\epsilon=2\sqrt{2}$, and $B=-A$ for $\epsilon=3$ and $\epsilon\to\infty$), and (iv) $\alpha<0$, $\xi>0$ ($B=-\sqrt{2}A$ for $\epsilon=2\sqrt{2}$, $A=0$ for $\epsilon=3$, and $A=B$ for $\epsilon\to\infty$). Clearly, if all of these particular cases led to stability of de Sitter space against the considered perturbations, one could conclude that de Sitter space is stable to such perturbations in all cases. Thus, for case I, the relation between the coefficients A and B for the asymptotic solutions h and h_1 must run between the extreme values for A (B fixed), $A=B$ and $A=-B$, passing on $A=0$.

For $A=B$, the perturbation in the Kruskal coordinates, e.g., h^K_{00}, are given by (angular dependence suppressed)

$$h^K_{00}=\frac{F^2(u-v)^2}{u^2-v^2}he^{-\alpha t}$$
$$=(-1)^{-\alpha/2H}F^2A\frac{(u-v)^{1-\sqrt{\alpha^2/4H^2-2}-\alpha/2H}}{(u+v)^{1+\sqrt{\alpha^2/4H^2-2}+\alpha/2H}}. \tag{3.58}$$

At $t=0$ $(u=0)$,

$$h^K_{00}=(-1)^{1-\sqrt{\alpha^2/4H^2-2}-\alpha/H}v^{-(2\sqrt{\alpha^2/4H^2-2}+\alpha/H)}. \tag{3.59}$$

Equation (3.59) is divergent as $v\to 0$, except for the case $\alpha=-\epsilon H$, $\xi>0$ or ξ very large, for which case $h^K_{00}=-F^2A$. Hence, except for this case, all perturbations are physically unacceptable, as they all diverge at the initial time $t=0$.

For $B=-A$, we have

$$h_{00}^{K}=(-1)^{-\alpha/2H}F^{2}A\frac{(u+v)^{1-\sqrt{\alpha^{2}/4H^{2}-2}+\alpha/2H}}{(u-v)^{1+\sqrt{\alpha^{2}/4H^{2}-2}+\alpha/2H}}, \tag{3.60}$$

which at $t=0$ $(u=0)$ reduces to

$$h_{00}^{K}=(-1)^{-(1+\sqrt{\alpha^{2}/4H^{2}-2}+\alpha/H)}F^{2}Av^{-2\sqrt{\alpha^{2}/4H^{2}-2}}. \tag{3.61}$$

We note that all of these perturbations are physically unacceptable, except for the case $\alpha=-\epsilon H$, $\xi<0$, with $\epsilon=3$, where $h_{00}^{K}=iF^{2}Av$.

Finally, when $A=0$, we obtain

$$h_{00}^{K}=(-1)^{-\alpha/2H}F^{2}B(u^{2}+v^{2})\frac{(u+v)^{\alpha/2H-1-\sqrt{\alpha^{2}/4H^{2}-2}}}{(u-v)^{1+\sqrt{\alpha^{2}/4H^{2}-2}+\alpha/2H}}. \tag{3.62}$$

Again at $t=0$ $(u=0)$, this perturbation reduces to

$$h_{00}^{K}=(-1)^{-(1+\sqrt{\alpha^{2}/4H^{2}-2}+\alpha/H)}F^{2}Bv^{-2\sqrt{\alpha^{2}/4H^{2}-2}}. \tag{3.63}$$

It can easily be checked that in all cases, without any exception, Eq. (3.63) diverges as $v\to 0$, and therefore this perturbation is physically unacceptable and should be ruled out.

We are in this way left with two physically acceptable even-parity perturbations for purely imaginary frequencies in case I: that given by Eq. (3.58) for negative α, positive ξ, and very large ϵ and that given by Eq. (3.60) for negative α, negative ξ, and ϵ about 3. Since negative values of α correspond to the case of outgoing perturbations for which $u=v$ on the horizon $r=H^{-1}$, these perturbations will be stable as the resulting powers to the factor $(u-v)$ are positive definite in both cases. Note, furthermore, that at least the perturbation given by Eq. (3.60) is always purely imaginary. Thus, even-parity perturbations with purely imaginary frequency in case I either are physically unacceptable or, unlike in Schwarzschild space [31], are stable and most of them imaginary.

In the case that k is kept real the asymptotic general solution on the cosmological horizon has already been given by Eq. (3.53). The analysis to follow will also be valid for purely imaginary frequencies satisfying the condition implied by case II, with the asymptotic solution being given by Eq. (3.54) in this case. For real frequencies, the relation between h and h_1 is

$$i\left[k^{2}h_{1}-4\left(H\frac{dh_{1}}{dr^{*}}+\frac{d^{2}h_{1}}{dr^{*2}}\right)\right]=2k\left(Hh+2\frac{dh}{dr^{*}}\right), \tag{3.64}$$

where we have specialized to the region $r=H^{-1}$. Let us now assume, as was made for purely imaginary frequencies, that the asymptotic forms for h and h_1 near the cosmological horizon are given by

$$h_{1}=A\exp(-i\nu r^{*}),\quad h=B\exp(-i\nu r^{*}), \tag{3.65}$$

in which we have introduced the shorthand notation

$$\nu=\sqrt{2H^{2}+\frac{1}{4}k^{2}}. \tag{3.66}$$

From Eqs. (3.64) and (3.65) we get

$$B=-\left[\frac{2\nu(k^{2}+5H^{2})}{k(k^{2}+9H^{2})}+\frac{4iH^{3}}{k(k^{2}+9H^{2})}\right]A. \tag{3.67}$$

Restricting ourselves to the case $k^{2}\gg H^{2}$, so that $\nu\simeq k/2$, we see that there are two solutions: when $k\simeq 2\nu$ (ingoing waves), $A=-B$, and when $k\simeq -2\nu$ (outgoing waves), $A=B$. Suppressing again the angular dependence of the perturbations, we can then compute such perturbations in Kruskal coordinates. In the case of ingoing waves, $u=-v$ on the horizon, and we have

$$h_{00}^{K}=F^{2}A(-1)^{ik/2H}\frac{(u+v)^{1-ik/H}}{u-v} \tag{3.68}$$

and, for outgoing waves $(u=v)$,

$$h_{00}^{K}=F^{2}A(-1)^{ik/2H}\frac{(u-v)^{1+ik/H}}{u+v}. \tag{3.69}$$

Clearly, the perturbations (3.68) and (3.69) are stable near the cosmological horizon and everywhere inside it, even in the forms given by these equations, without building suitable wave packets by superposing them. This analysis can readily be generalized to any values of k and H, obtaining the same conclusion.

IV. STABILITY OF MULTIPLY CONNECTED de SITTER SPACE

In Sec. III we have investigated the stability properties of the simply connected de Sitter region covered by static coordinates. We had to do so because, as far as I know, such an analysis had not been carried out so far, and we needed it to prepare our system to study the perturbations when the coordinates involved are identified in such a way that this region of de Sitter space becomes multiply connected topologically, such as was discussed in Sec. II. We have obtained that on the static region, the de Sitter space is also stable to the first-order perturbations that satisfy its symmetries.

A. Classical perturbations

We shall now study the effect that topological multiple connectedness has on the stability of de Sitter space. Because of the high symmetry of this space, the time parameter t always appears in the form of a factorized exponential factor in all perturbations, either as $\exp(\pm\alpha t)$, if the frequency of the perturbative modes is purely imaginary, or as $\exp(\pm ikt)$, if that frequency is kept real. Thus, we can generically write the time factor as $\exp[g(k)t]$, with $g(k)$ a given function of the mode frequency. From our discussion in Sec. II, it fol-

lows that de Sitter space can be made multiply connected by simply including the time identification $t \leftrightarrow t + nb/H$, with b a dimensionless arbitrary period and n any integer number. This will amount to the insertion of an additional factor

$$\exp[nbg(k)/H]$$

for each value of the integer n in the distinct expressions for the first-order perturbations obtained in Sec. III. In order to take into account all possible values of n, along its infinite range, one then should coveniently sum over all n, from 0 to ∞, introducing the statistical factor $1/n!$ to account for the equivalent statistical weight one must attribute to all of these contributions. Thus, the general expression for the perturbations in multiply connected de Sitter space would be

$$h^K_{ij}(b) = \sum_{n=0}^{\infty} \frac{h^K_{ij} \exp[nbg(k)/H]}{n!}, \tag{4.1}$$

where h^K_{ij} generically denotes the first-order perturbations in Kruskal coordinates for simply connected de Sitter space which were computed in Sec. III.

In what follows we shall perform the calculation of the relevant $h^K_{ij}(b)$ for all physically acceptable perturbations. We shall first restrict ourselves to the regime where both kb/H and $\alpha b/H$ are much smaller than unity; i.e., we will work in the regime characterized by nonchronal regions and CTC's whose size is very small. As will be seen below, this is the regime of most physical interest where vacuum quantum fluctuations can be kept convergent everywhere. Let us start with odd-parity perturbations with real frequency. For the case of ingoing waves, we have, for the asymptotic solution at $r = H^{-1}$ [35],

$$h^K_{03}(b) = \frac{3A}{4H(u+v)} [i(u+v)]^{-ik/H} \sum_{n=0}^{\infty} \frac{e^{inkb/H}}{n!}$$

$$= \frac{3A}{4H(u+v)} \exp(e^{ikb/H}) e^{-ik \ln[i(u+v)]/H}. \tag{4.2}$$

For small values of kb/H, Eq. (4.2) can be approximated to

$$h^K_{03}(b) \simeq \frac{3Ae}{4H(u+v)} \exp\left\{ -\frac{ik}{H} \{\ln[i(u+v)] - b\} \right\}. \tag{4.3}$$

Forming again a wave packet out of monochromatic perturbations (4.3), we finally obtain, for this type of wave,

$$h^K_{03}(b) \simeq \frac{3e}{4H(u+v)} f\left[\frac{1}{H} \left(\ln(u+v) + \frac{i\pi}{2} - b \right) \right], \tag{4.4}$$

which still is a convergent expression for all times. If we let b be complex, so that $b = b + i\pi/2$, then Eq. (4.4) becomes not only convergent but purely real as well.

For outgoing waves, an analogous calculation leads finally to the perturbation

$$h^K_{03}(b) \simeq -\frac{3e}{4H(u-v)} f\left[\frac{1}{H} \left(\ln(u-v) + \frac{i\pi}{2} + b \right) \right], \tag{4.5}$$

which is also always covergent and purely real if we similarly let b be complex and given by $b = b - i\pi/2$. We can then conclude that in the considered regime, multiply connected de Sitter space is stable to all odd-parity perturbations that are physically acceptable.

For even-parity perturbations which also are physically acceptable we have [35]

$$h^K_{00}(b) = h^K_{00} \sum_{n=0}^{\infty} \frac{e^{-nb\alpha/H}}{n!} = h^K_{00} \exp[e^{-b\alpha/H}] \tag{4.6}$$

for any value of kb/H. Thus, making the de Sitter space multiply connected preserves the stability of these perturbations and increases their amplitude, especially for small values of $b\alpha/H$.

Next we consider even-parity perturbations with real frequency. We first note that in this case the perturbations corresponding to the asymptotic solutions near $r = H^{-1}$ can also be expressed in terms of wave packets in the simply connected case. They are

$$h^K_{00} = F^2\left(\frac{u+v}{u-v} \right) f\left[\frac{1}{H} \left(\ln(u+v) + \frac{i\pi}{2} \right) \right], \tag{4.7}$$

for ingoing waves, and

$$h^K_{00} = F^2\left(\frac{u-v}{u+v} \right) f\left[\frac{1}{H} \left(\ln(u-v) + \frac{i\pi}{2} \right) \right], \tag{4.8}$$

for outgoing waves. Because of the form of the Kruskal-coordinate-dependent prefactor, these expressions are real in any case.

When we multiply connect the de Sitter space in the regime of small values of kb/H, Eqs. (4.7) and (4.8) transform into

$$h^K_{00}(b) = eF^2\left(\frac{u+v}{u-v} \right) f\left[\frac{1}{H} \left(\ln(u+v) + \frac{i\pi}{2} - b \right) \right], \tag{4.9}$$

for ingoing waves, and

$$h^K_{00}(b) = eF^2\left(\frac{u-v}{u+v} \right) f\left[\frac{1}{H} \left(\ln(u-v) + \frac{i\pi}{2} + b \right) \right], \tag{4.10}$$

for outgoing waves. Note that the argument of the function f becomes real when we allow b to be complex and given by $b = b \pm i\pi/2$, with the $+$ sign for ingoing waves and the $-$ sign for outgoing waves. Anyway, the perturbations given by Eqs. (4.9) and (4.10) keep being convergent and, therefore, one can conclude that in the regime of very small values of

b/H, multiply connected de Sitter space is also stable to first-order perturbations which respect the symmetry of this space.

As to the perturbations for larger values of b/H and real frequencies, we first note that the expressions for the components $h^K_{03}(b)$ and $h^K_{00}(b)$, before forming any wave packets, are given by expressions which are the same as those obtained above in the regime of very small values of b/H, but with the parameter b replaced by the k-dependent function $(-iH/k)\exp(ikb/H)$. The wave packets formed using the same procedure as for all the above cases will therefore involve characteristic functions of the form

$$f\left\{\frac{H}{b}\int_0^1 \frac{dxbe^x}{b+\ln[i(u\pm v)]}\right\},$$

where $x=\exp(ikb/H)$, and the sign $+/-$ in the argument of the logarithm stands for ingoing/outgoing waves. Now, for $b\gg\ln[i(u\pm v)]$ we can obtain [34,35]

$$f\left[\frac{H}{b}\mathrm{Ei}(x)\Big|_0^1\right],$$

with $\mathrm{Ei}(x)$ the exponential integral function. Since the argument of f is then always smaller than unity, we have $f=0$ on this regime.

As $(u\pm v)$ becomes very small, so that $b\ll y=|\mathrm{Re}\ln[i(u\pm v)]|$, the wave packet function approaches the form [34,35]

$$f\left\{\frac{H}{b}\lim_{y\to\infty}\left[\frac{\Phi(y,y+1;1)}{y}\right]\right\}=0,$$

with Φ the degenerate (confluent) hypergeometric function. Thus, also for arbitrarily large nonchronal regions, the multiply connected de Sitter space is stable to all physically allowable classical perturbations.

B. Quantum fluctuations

In what follows we shall briefly discuss the possible influence that multiple connectedness may have on the quantum stability of de Sitter space. Because of the presence of a chronology horizon on the surface $r=H^{-1}$, it could at first sight be thought that the quantum renormalized stress-energy tensor $\langle T_{\mu\nu}\rangle_{ren}$ for vacuum quantum fluctuations generated in multiply connected de Sitter space ought to diverge [36]. However, it has recently been stressed that this could not be actually the case if either we consistently redefine the quantum vacuum [24] or we introduce a suitable quantization of the relevant time parameter, beyond semiclassical approximation [28]. To see how these ideas apply to the case under study, let us work in the Euclidean framework where the Kruskal metric is obtained by rotating coordinates u and t to the imaginary axis, starting with Eq. (3.9), i.e., $u=i\eta$ and $t=i\tau$. We get then

$$ds^2=\frac{4}{H^2(1+\eta^2+v^2)}(d\eta^2+dv^2)+r^2\,d\Omega_2^2, \quad (4.11)$$

which in fact is definite positive. Corresponding to this metric, Euclidean time τ will be defined by the relation

$$\exp(2iH\tau)=\frac{v-i\eta}{v+i\eta}. \quad (4.12)$$

Wick rotating also in the identification $t\leftrightarrow n\beta$ (where the nonperiodic time term is disregarded and $\beta=b/H$ is the period), which makes de Sitter space multiply connected, Eq. (4.12) transforms into

$$\exp(2inH\beta)=\frac{v-i\eta}{v+i\eta}, \quad (4.13)$$

from which one can get the complex relation

$$v-i\eta=\sqrt{v^2+\eta^2}\exp(inH\beta). \quad (4.14)$$

It follows from Eq. (4.14) that Euclidean time preserves a periodic character also on the Euclidean sector $\beta=2\pi/H$, that is to say, $b=2\pi$. This result can be interpreted along the following lines. First of all, one can readily see that multiple connectedness in de Sitter space is nothing but the Lorentzian counterpart of the existing thermal states that are uncovered in the Euclidean description [37]. This relation might be reflecting the origin of the excess of some perturbative waves which we have found above for multiply connected de Sitter space with respect to its simply connected space.

Quite more importantly, the value $b=2\pi$ can be used (as Li and Gott did [24]) to redefine a conformal vacuum in Euclidean space for which $\langle T_{\mu\nu}\rangle_{ren}$ does not diverge even on the chronology horizon. However, the meaning of this horizon in such a vacuum has been discussed by Kay, Radzikowski, and Wald [25] and Cassidy [26], so that some quite well-founded doubts can be cast on its real existence and capability to restore quantum stability this way. But if we adhere to the also recently suggested [28] kind of time quantization by which $t=(n+\gamma)t_0$ (with t_0 a constant time whose value is of the order of Planck time and γ the automorphic parameter [38,39], $0\leqslant\gamma\leqslant 1/2$), and note the formal analogy of this expression with that which is implied by the identification $t\leftrightarrow t+nb/H$ when we take $b=2\pi$ and $t=2\pi\gamma/H$, we see that quantum stability could be unambiguously restored in multiply connected de Sitter space, provided we accept restricting the nonchronal region and the CTC's on it to be essentially at the Planck scale, $1/H\sim l_p$ [28].

V. SUMMARY AND CONCLUSIONS

The main aim of this paper is to study the classical and quantum stability of multiply connected de Sitter space. This space arises when we introduce some periodicity conditions on the coordinates describing the five-dimensional Minkowski hyperboloid and can only be exhibited on the region covered by static de Sitter coordinates. Using a first-

order perturbation formalism analogous to that which was originally developed by Regge and Wheeler for the Schwarzschild metric [27], we have shown that multiply connected de Sitter space is classically stable to these perturbations, no matter the size of the static region. Although stability against higher order perturbations has not been checked in this paper, one would expect these perturbations not to introduce any instabilities, such as occurs in Schwarzschild spacetime [27].

By continuing the Kruskal extension of the multiply connected, static de Sitter metric into its Euclidean section, we have also argued that quantum vacuum fluctuations should not induce any divergence on this space, provided the nonchronal region and the CTC's on it are all sufficiently small, probably of the order of the Planck size [28]. We therefore consider multiply connected de Sitter universes to be genuine components of any future description of a well-defined theory of quantum gravity. In particular, the considered stable little multiply connected universes should be included, together with Euclidean and multiply connected wormholes [20], ringholes [40], Klein bottleholes [41], and virtual black holes [42], as components of the vacuum quantum spacetime foam, where their CTC's would contribute the required violation of causality that governs the foam. Thus, Planck-sized de Sitter universes containing CTC's can help to define the boundary conditions of the universe we live in, probably along the lines recently suggested by Gott and Li [19] and discussed in the Introduction of this paper.

To close up, I would like to refer to the interesting possibility that the Hartle-Hawking and the Gott-Li conditions might both be seen to imply initial physical pictures which, at least in a way, appear to be complementary. In classical relativity time and spatial coordinates can still be distinguished by the fact that, whereas spatial coordinates do not single out a particular sign to run over, time can only run forwards. This does not help, nonetheless, to understand the Lorentzian signature of the classical metrics which are not positive definite. However, as we approach the regime of the quantum spacetime foam, the entirety of this distinction can be thought to vanish, since in such a regime there will be CTC's everywhere and hence the two time directions would become equally allowable and, at the same time, metrics can be taken to be positive definite. In order to describe the quantum origin of the universe, one then can either keep CTC's and Lorentzian signatures simultaneously, as assumed by Gott and Li, or disregard CTC's while using the Euclidean signature where time becomes spacelike, as suggested by Hartle and Hawking. As seen in this way, the two pictures would actually describe rather complementary aspects of the initial physical situation.

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