# Cosmological scaling solutions of nonminimally coupled scalar fields

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We study the existence and stability of cosmological scaling solutions of a nonminimally coupled scalar field evolving in either an exponential or inverse power law potential. We show that, for inverse power law potentials, there exist scaling solutions the stability of which does not depend on the coupling constant $\xi$. We then study the more involved case of exponential potentials and show that the scalar field will asymptotically behave as a baryotropic fluid when $\xi \ll 1$. The general case $\xi \not\ll 1$ is then discussed and we illustrate these results by some numerical examples. [S0556-2821(99)06510-8]

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## I. INTRODUCTION

On the one hand, recent cosmological observations, and particuarly the Hubble diagram for type Ia supernovae [1], have led to the idea that the universe may be dominated by a component with negative pressure [2] and thus that today the universe is accelerating. Such a component can also, if one sticks to the prediction of inflation that $\Omega = 1$, account for the ''missing energy.'' Yet many candidates have been proposed, such as a cosmological constant, a ''dynamical'' cosmological constant [3], cosmic strings [4], or a spatially homogeneous scalar field rolling down a potential [5].

On the other hand, potentials decreasing to zero for an infinite value of the field have been shown to appear in particle physics models (see, e.g., [6,7]). For instance, exponential potentials arise in high order gravity [8], in Kaluza-Klein theories which are compactified to produce the four-dimensional observed universe [9], or due to nonperturbative effects such as gaugino condensation [10]. Inverse power law potentials can be obtained in models where supersymmetry is broken through fermion condensates [7]. This gives one more theoretical motivation to study the cosmological implications of a field with such potentials.

The cosmological solutions with such a field were first studied by Ratra and Peebles [11] (see also [12,13]) who showed the existence and stability of scaling solutions in, respectively, a field, radiation, or matter dominated universe for a field evolving in an exponential and inverse power law potential. A complete study in the framework of baryotropic cosmologies in the case of the exponential potential [14,15] shows that the solutions were stable to shear perturbations and to curvature perturbations when $P/\rho < -1/3$, but that for realistic matter (such as dust) these solutions were unstable, essentialy to curvature perturbations. Liddle and Scherrer [16] made a complete classification of the field potentials and show that power law potentials also lead to scaling solutions (i.e., to solutions such that the field energy density $\rho_\phi$ behaves as the scale factor at a given power) and the coupling of the field to ordinary matter has been considered in [17]. Such solutions are indeed of interest in cosmology since they provide a candidate for a component with negative pressure.

Cosmological models with a scalar field have started to be investigated [18,19] for different kind of potentials such as the cosine potential [3], exponential potential [20,21], and inverse power law potentials [22]. It has also been shown that the luminosity distance as a function of redshift [23,24] or the behavior of density perturbations in the dust era as a function of redshift [24] can be used to reconstruct the field potential.

However, all these studies have been done under the hypothesis that the field is minimally coupled to the metric. It is known that terms with such a nonminimal coupling $\bar{\mathcal{R}} f(\phi)$ between the scalar curvature $\bar{\mathcal{R}}$ and the field $\phi$ can appear when quantizing fields in curved spacetime [25,26] and in multidimensional theories [27] such as superstring and induced gravity theories [28]. Since these theories predict both the existence of scalar fields with potential or power law potential and nonminimal coupling, it is of interest to study the influence of this coupling and, for instance, the robustness of the existence and stability of scaling solutions. The influence of such a coupling during an inflationary period and the existence of inflationary attractors have yet been examined (see, e.g., [29]).

In this article, we study the stability of scaling solutions of a nonminimally coupled scalar field. We first present (Sec. II) the main notations and equations. After having, in Sec. III, briefly recalled the standard approach for determining the potentials that can give rise to such behavior for a minimally coupled scalar field, we investigate the inverse power law potentials (Sec. IV) and the exponential potential (Sec. V). In the latter case, we study the two limiting situations $\xi \ll 1$ and $\xi \gg 1$ and then have a heuristic discussion for the general case.

## II. DESCRIPTION OF THE MODEL

We assume that the universe is described by a Friedmann-Lemaître model with Euclidean spatial sections so that the line element reads

$$ds^2 = -dt^2 + a^2(t)[\delta_{ij} dx^i dx^j] \equiv \bar{g}_{\mu\nu} dx^\mu dx^\nu, \qquad (1)$$

where $a(t)$ is the scale factor and $t$ the cosmic time. Greek indices run from 0 to 3 and latin indices from 1 to 3. The Hubble parameter $H$ is defined as $H \equiv \dot{a}/a$, where an overdot denotes a derivative with respect to $t$. We also introduce $\bar{\nabla}_\mu$, the covariant derivative associated with $\bar{g}_{\mu\nu}$.

We assume that the matter content of this universe is composed of a perfect fluid and a homogeneous scalar field $\phi$ coupled to gravity and coupled to matter only through gravity. The fluid energy density $\rho_B$ and pressure $P_B$ are related through the equation of state

$$P_B = \omega_B \rho_B, \tag{2}$$

where $B$ refers to ''background.'' The conservation of the energy momentum of the fluid reduces to

$$\dot{\rho}_B + 3H(\rho_B + P_B) = 0. \tag{3}$$

The scalar field $\phi$ evolves in a potential $V(\phi)$ and its dynamics is given by the Lagrangian

$$S_\phi = -\frac{1}{2}\int [\partial_\mu \phi \partial^\mu \phi + 2\xi \bar{\mathcal{R}} f(\phi) + 2V(\phi)]\sqrt{-\bar{g}}d^4x, \tag{4}$$

where $\xi$ is the field-metric coupling constant ($\xi=0$ corresponds to a minimally coupled field and $\xi=1/6$ to a conformally coupled field), and $\bar{\mathcal{R}}$ is the scalar curvature of the spacetime. No known fundamental principle predicts the functional form $f(\phi)$ and we will assume that $f(\phi)=\phi^2/2$. This is, however, the only choice that allows for a dimensionless $\xi$. The equation of evolution is then obtained by varying the action (4) with respect to the field which leads to the Klein-Gordon equation

$$\frac{\delta S_\phi}{\delta \phi} = 0 \Leftrightarrow \Box \phi - \xi \bar{\mathcal{R}} \phi - a^2 \frac{dV}{d\phi} = 0, \tag{5}$$

where $\Box \equiv \bar{\nabla}_\mu \bar{\nabla}^\mu$. For an homogeneous field in the spacetime (1), it reduces to

$$\ddot{\phi} + 3H\dot{\phi} + \frac{dV}{d\phi} + 6\xi(2H^2 + \dot{H})\phi = 0. \tag{6}$$

The energy density and the pressure of this field are defined by

$$\rho_\phi \equiv a^{-2} T_{00}, \quad P_\phi \equiv \frac{1}{3} a^{-2} T_{ij}\delta^{ij}, \tag{7}$$

and we define $\omega_\phi \equiv P_\phi/\rho_\phi$. The energy-momentum tensor $T_{\mu\nu}$ is obtained by varying the action (4) with respect to the metric $\bar{g}_{\mu\nu}$ and reads

$$\begin{aligned} T_{\mu\nu} = {} & (1-2\xi)\bar{\nabla}_\mu \phi \bar{\nabla}_\nu \phi + \left(2\xi - \frac{1}{2}\right)\bar{g}_{\mu\nu}\bar{\nabla}_\lambda \phi \bar{\nabla}^\lambda \phi \\ & - 2\xi \phi \bar{\nabla}_\mu \bar{\nabla}_\nu \phi + 2\xi \phi \Box \phi \bar{g}_{\mu\nu} + \xi \bar{G}_{\mu\nu}\phi^2 - V(\phi)\bar{g}_{\mu\nu}, \end{aligned} \tag{8}$$

$\bar{G}_{\mu\nu}$ being the Einstein tensor of the background metric. This expression can be compared to standard results (see, e.g., [25]). Note, however, that when the signature of the metric is $(+,-,-,-)$, one has to change the sign of $\bar{g}_{\mu\nu}$, $\Box$, $\bar{R}_{\mu\nu}$, and $\bar{G}_{\mu\nu}$ while $T_{\mu\nu}$ and $\bar{\mathcal{R}}$ remain unaffected.

It is then straigthforward to check that the density and the pressure (7) of the scalar field are given by

$$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi) + 3H\xi\phi(2\dot{\phi} + H\phi), \tag{9}$$

$$\begin{aligned} P_\phi = {} & \frac{1}{2}\dot{\phi}^2 - V(\phi) - \xi[(2\dot{H} + 3H^2)\phi^2 + 4H\phi\dot{\phi} \\ & + 2\phi\ddot{\phi} + 2\dot{\phi}^2], \end{aligned} \tag{10}$$

and that the conservation of the energy momentum of the field ($\bar{\nabla}_\mu T^{\mu\nu}=0$) reduces to the Klein-Gordon equation (6):

$$\begin{aligned} & \dot{\rho}_\phi + 3H(\rho_\phi + P_\phi) = 0 \\ & \qquad \Leftrightarrow \ddot{\phi} + 3H\dot{\phi} + \frac{dV}{d\phi} + 6\xi(2H^2 + \dot{H})\phi \\ & \qquad = 0. \end{aligned} \tag{11}$$

[We have used that the scalar curvature and the Einstein tensor are, respectively, given by $\bar{\mathcal{R}} = 6(2H^2 + \dot{H})$ and $\bar{G}_{00} = 3a^2H^2$, $\bar{G}_{ij} = -(2\dot{H} + 3H^2)a^2\delta_{ij}$.] The equation of state of the field is defined by

$$\omega_\phi = \frac{P_\phi}{\rho_\phi}. \tag{12}$$

The matter content being described, we can write the Einstein equations which dictate the dynamics of the spacetime and, in our case, reduce to the Friedmann equations

$$H^2 = \frac{\kappa}{3}(\rho_B + \rho_\phi), \tag{13}$$

$$\dot{H} = -\frac{\kappa}{2}[(\omega_B + 1)\rho_B + \rho_\phi + P_\phi] \tag{14}$$

with $\kappa = 8\pi G$. One equation of the set (3), (11), and (13), (14) is redundant due to the Bianchi identities. It will also be useful to introduce the density parameter of a component $X$ as $\Omega_X \equiv \kappa\rho_X/3H^2$. They must satisfy [from Eq. (13)] the constraint

$$\Omega_B + \Omega_\phi = 1. \tag{15}$$

## III. SCALING SOLUTIONS FOR MINIMALLY COUPLED SCALAR FIELDS ($\xi = 0$)

In this section, we briefly recall the ''standard'' procedure to show that there exist scaling solutions and to determine the potentials that give rise to such solutions. This presentation will also enable us to understand the differences with the more general case of nonminimally coupled scalar fields. We follow the approach initiated by Ratra and Peebles [11] and others [13,16] where one, assuming a scaling form for $\rho_\phi$, derives an equation for $\phi(a)$ and then uses Eqs. (9), (10) to deduce the associated potential.

The equation of evolution (3) of the background fluid density can be integrated to give

$$\rho_B = \rho_{B0} x^{-m} \quad \text{with} \quad x \equiv \frac{a}{a_0}, \tag{16}$$

where a subscript 0 refers to quantities evaluated at a given initial time. We now look for scaling solutions, i.e., solutions such that

$$\rho_\phi = \rho_{\phi 0} x^{-n} \Leftrightarrow P_\phi = \left(\frac{n}{3} - 1\right) \rho_\phi . \tag{17}$$

Since $n/3 - 1 = 1 - 2V/\rho_\phi \in [-1;1]$, we deduce that $n \in [0,6]$ (let us emphasize that this is *a priori* no longer true when $\xi \neq 0$). Using Eqs. (9), (10), such a solution must satisfy

$$\dot{\phi}^2 = \frac{n}{3} \rho_\phi \quad \text{and} \quad V(\phi) = \left(1 - \frac{n}{6}\right) \rho_\phi . \tag{18}$$

Now, the Friedmann equation (13) implies that the field should satisfy

$$\frac{d\phi}{dx} = \frac{A}{x\sqrt{1 + B^2 x^{n-m}}} \quad \text{with} \quad B \equiv \sqrt{\frac{\rho_{B0}}{\rho_{\phi 0}}} \quad \text{and}$$

$$A \equiv \sqrt{\frac{n}{\kappa}} \sqrt{\frac{\rho_{\phi 0}}{\rho_{\phi 0} + \rho_0}} = \sqrt{\frac{n}{\kappa} \Omega_{\phi 0}}. \tag{19}$$

This can be integrated for different relative values of $m$ and $n$.

### A. $m = n$ case

In that situation, Eq. (19) leads to

$$\phi - \phi_0 = \frac{n}{\lambda} \ln x \quad \text{with} \quad \lambda^{-1} \equiv \frac{\Omega_{\phi 0}}{\sqrt{n\kappa}} \tag{20}$$

and, then using Eq. (18), to the potential

$$V(\phi) = \left(1 - \frac{n}{6}\right) \rho_{\phi 0} e^{-\lambda(\phi - \phi_0)}. \tag{21}$$

This solution corresponds to the scalar field dominated universe of Ratra and Peebles [11] and to their scaling solution in the case $m = 3$ and $m = 4$ (i.e., radiation or matter dominated universe). Note that with such a potential, $\rho_\phi$ will, by construction, mimic the evolution of the background fluid and that we do not have to assume that $\Omega_\phi \ll 1$.

Note, however, that, if the scalar field has reached the attractor from very early time, $\rho_\phi$ behaves like radiation and thus contributes to a non-negligible part of the radiation content during the nucleosynthesis and it has been shown that it implies the constraint $\Omega_{\phi 0} < 0.15$ [20,12]. Moreover, since $\omega_\phi = 0$ in the matter era, such a field will not explain the supernovae measurements (which seem to favor $\omega_\phi = -0.6$ [30]) even if it can account for a substantial part of the dark matter.

### B. $m \neq n$ case

In this case we have

$$\frac{d\phi}{dx} = \frac{A}{x\sqrt{1 + B^2 x^{n-m}}}, \tag{22}$$

which can be integrated ($B > 0$) to give [31]

$$\phi - \phi_0 = \frac{2A}{m-n} \ln\left[ \sqrt{1 + (B^{-1} x^{(m-n)/2})^2} + B^{-1} x^{(m-n)/2} \right]. \tag{23}$$

Again, using Eq. (18), we can deduce the potential

$$V(\phi) = \left(1 - \frac{n}{6}\right) \rho_{\phi 0} x^{-n}, \tag{24}$$

$x$ being given by Eq. (23). Indeed, we only get the potential in a parametric form, but when $B \gg 1$ (i.e., when the perfect fluid drives the evolution of the universe) $x$ can be eliminated from Eqs. (23), (24) to give

$$V(\phi) = \left(1 - \frac{n}{6}\right) \rho_{\phi 0} \left(\frac{m-n}{2A} B\right)^{-2n/(m-n)} (\phi - \phi_0)^{-2n/(m-n)}. \tag{25}$$

When $m = 3$ and $m = 4$, we recover the Ratra-Peebles result [11] as well as the Liddle-Scherrer result [16] for all $m$. This parametric general form of the potential seems, however, not to have been exhibited before.

## IV. NONMINIMALLY COUPLED SCALAR FIELDS WITH A POWER LAW POTENTIAL

### A. Existence of a scaling solution

The former procedure cannot be applied when the field is nonminimally coupled since it is impossible, for instance, to write a closed equation for $d\phi/dx$ as in Eq. (19). Moreover, we are interested in a field evolving in a given potential. We assume that the field evolves in an inverse power law potential and show that there exist scaling solutions, the stability of which is then studied.

We assume that the potential takes the form

$$V(\phi) = V_0 M_p^4 \left(\frac{\phi}{M_p}\right)^{-\alpha} \quad \text{with} \quad \alpha > 0, \tag{26}$$

with $M_p$ being the Planck mass. The universe is dominated by the perfect fluid so that (we assume $m \neq 0$)

$$H = \frac{2}{m} \frac{1}{t - t_0}, \quad a = a_0 (t - t_0)^{2/m}, \quad \omega_B = \frac{m}{3} - 1. \tag{27}$$

Redefining $M_p (t - t_0) \sqrt{V_0}$ as $t$ and $\phi/M_p$ as $\phi$, the Klein-Gordon equation (6) becomes

$$\ddot{\phi}+\frac{6}{m}\frac{1}{t}\dot{\phi}+\frac{12}{m}\left(\frac{4}{m}-1\right)\xi\frac{1}{t^2}\phi-\alpha\phi^{-(\alpha+1)}=0. \tag{28}$$

Looking for a solution of this equations of the form $\phi\propto t^{\beta}$, one obtains

$$\phi=\phi_0 t^{\beta},\quad \phi_0^{\alpha+2}=\frac{\alpha}{\beta\left(\beta+\frac{6}{m}-1\right)+\frac{12}{m}\left(\frac{4}{m}-1\right)\xi},$$

$$\beta=\frac{2}{\alpha+2}, \tag{29}$$

so that $\rho_{\phi}\propto a^{-n}$ with $n/m=\alpha/(\alpha+2)$. This solution is only well defined if

$$\frac{6}{m}-\frac{\alpha}{\alpha+2}+\frac{6}{m}\left(\frac{4}{m}-1\right)(\alpha+2)\xi>0. \tag{30}$$

One can then compute the energy and the pressure of this field by inserting this solution into Eqs. (9), (10) and verify, after some algebra, that

$$\omega_{\phi}=\frac{\omega_{B}\alpha-2}{\alpha+2}, \tag{31}$$

whatever the value of $\xi$. This shows that the scaling solution does not depend on the coupling in the sense that $\omega_{\phi}$ is independent of $\xi$. This relation generalizes the one found for minimally coupled scalar fields [2,11,16].

### B. Stability

As emphasized in the previous section, the scaling solution does not depend on the coupling $\xi$. The stability of such a solution is known when $\xi=0$ [11,16]; we now have to study it when $\xi\neq 0$. Following [11], we define the new set of variables

$$t=\mathrm{e}^{\tau}, \tag{32}$$

$$u(\tau)=\frac{\phi(\tau)}{\phi_s(\tau)}, \tag{33}$$

where $\phi_s$ is the scaling solution (29). We set $\phi'\equiv d\phi/d\tau$. Using $\dot{\phi}=\mathrm{e}^{-\tau}\phi'$, $\ddot{\phi}=\mathrm{e}^{-2\tau}(\phi''-\phi')$, and $\phi_s'/\phi_s=2/(\alpha+2)$, Eq. (28) reduces to

$$u''+\left(\frac{6}{m}+\frac{4}{\alpha+2}-1\right)u'+\left(\frac{2}{\alpha+2}\left[\frac{6}{m}-\frac{\alpha}{\alpha+2}\right]+\frac{12}{m}\left[\frac{4}{m}-1\right]\xi\right)(u-u^{-(\alpha+1)})=0. \tag{34}$$

The scaling solution corresponds to the critical point $u=1$. Introducing $v=u'$ and linearizing around this critical point $[u=1+\epsilon]$, we obtain

$$\begin{pmatrix}\epsilon\\ v\end{pmatrix}'=\begin{pmatrix}0 & 1\\ -2\left(\left[\frac{6}{m}-\frac{\alpha}{\alpha+2}\right]+\frac{6(\alpha+2)}{m}\left[\frac{4}{m}-1\right]\xi\right) & \left(1-\frac{6}{m}-\frac{4}{\alpha+2}\right)\end{pmatrix}\begin{pmatrix}\epsilon\\ v\end{pmatrix}. \tag{35}$$

The eigenvalues $\lambda_{\pm}$ of this system are

$$2\lambda_{\pm}=\left(1-\frac{6}{m}-\frac{4}{\alpha+2}\right)\pm\sqrt{\left(1-\frac{6}{m}-\frac{4}{\alpha+2}\right)^2-8\left[\left(\frac{6}{m}-\frac{\alpha}{\alpha+2}\right)+\frac{6}{m}(\alpha+2)\left(\frac{4}{m}-1\right)\xi\right]}. \tag{36}$$

This expression reduces to the Liddle-Scherrer result [16] when $\xi=0$ [with $\alpha/(\alpha+2)=n/m$] and to the Ratra-Peebles result [11] when either $m=3$ or $m=4$.

A necessary and sufficient condition for the critical point to be stable is the negativity of the real part of the two eigenvalues. Defining $\bar{\xi}$ as

$$\bar{\xi}=\frac{\left(m-6-\frac{4m}{\alpha+2}\right)^2-8m\left(6-\frac{\alpha m}{\alpha+2}\right)}{48(\alpha+2)(4-m)}, \tag{37}$$

the two solutions of Eq. (36) are real when $(\xi-\bar{\xi})(m-4)\geqslant 0$. Thus, (i) when the two eigenvalues are complex, $\lambda_{-}=\lambda_{+}^{*}$ and their real part is given by $\mathrm{Re}(\lambda_{\pm})=[1-6/m-4/(\alpha+2)]/2$ and the scaling solution is stable if

$$1-\frac{6}{m}-\frac{4}{\alpha+2}<0. \tag{38}$$

(ii) When the two eigenvalues $\lambda_{\pm}$ are real, their product is [from Eq. (34)]

$$\lambda_{-}\lambda_{+}=2\left(\left[\frac{6}{m}-\frac{\alpha}{\alpha+2}\right]+\frac{6(\alpha+2)}{m}\left[\frac{4}{m}-1\right]\xi\right)>0, \tag{39}$$

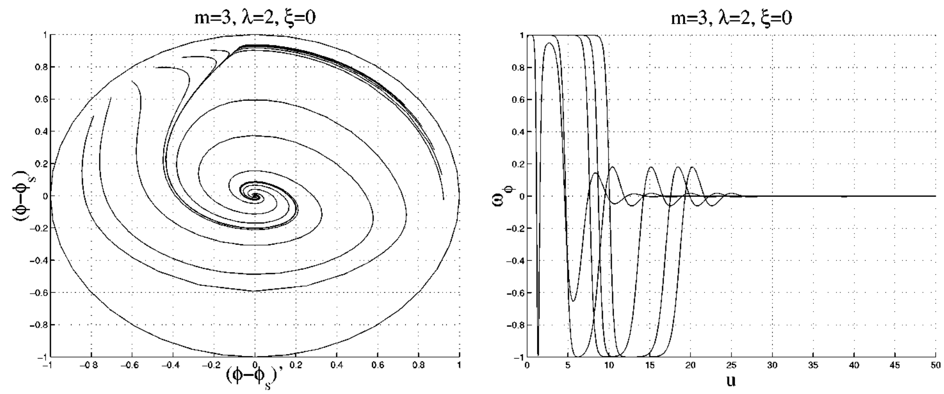


FIG. 1. (left) The convergence towards the scaling solution when $\xi=0$. (right) The convergence towards the equation of state $\omega_\phi=0$ for different initial conditions.

because of the condition (30). They are thus of the same sign, their sum being

$$\lambda_+ + \lambda_- = 1 - \frac{6}{m} - \frac{4}{\alpha+2}; \tag{40}$$

they will both be negative if $\lambda_+ + \lambda_- < 0$, and thus the solution will be stable only if

$$1 - \frac{6}{m} - \frac{4}{\alpha+2} < 0, \tag{41}$$

as in the case $\xi \geqslant \overline{\xi}$.

In conclusion, we find that *whatever* the coupling constant $\xi$, the scaling solution (29) will be stable if and only if

$$1 - \frac{6}{m} - \frac{4}{\alpha+2} < 0. \tag{42}$$

The value of $\xi$ only determines the nature of the stable point, i.e., wether it is a stable spiral or a stable node. This generalizes the study by Liddle and Scherrer [16] to a nonminimally coupled scalar field.

## V. NONMINIMALLY COUPLED SCALAR FIELD IN AN EXPONENTIAL POTENTIAL

### A. Scaling solutions

We now focus on potentials of the form

$$V(\phi) = V_0 M_p^4 \mathrm{e}^{-\lambda\phi/M_p} \quad \text{with} \quad \lambda > 0 \tag{43}$$

and work under the same assumptions as in Sec. IV. Redefining $t$ and $\phi$ as in Sec. IV, the Klein-Gordon equation now reads

$$\ddot{\phi} + \frac{6}{m}\frac{1}{t}\dot{\phi} + \frac{12}{m}\left(\frac{4}{m}-1\right)\xi\frac{1}{t^2}\phi - \lambda\,\mathrm{e}^{-\lambda\phi} = 0. \tag{44}$$

When the coeficient of $\phi$ vanishes, i.e., in a radiation dominated universe ($m=4$) or when $\xi=0$, one can find a special solution of the form $\phi=\ln(At^\beta)$. We get that (see Sec. III A with $\kappa = M_p^{-2}$)

$$\phi_s = \ln(At^\beta), \quad A^{-\lambda} = \frac{2}{\lambda^2}\left(\frac{6}{m}-1\right), \quad \beta = \frac{2}{\lambda} \Rightarrow \omega_{\phi_s} = \frac{m}{3} - 1, \tag{45}$$

where the subscript $s$ refers to the scaling solution. In the radiation era, this implies, for instance, that $\omega_{\phi_s}=1/3$ and the scalar field behaves like radiation. Indeed, this is a very special case since in that period $\overline{\mathcal{R}}=0$ and the field does not ''feel'' the nonminimal coupling and it evolves as if it were minimally coupled. The complete study of these solution in function of the two parameters $(\lambda, m)$ [14,15] shows that when $\lambda^2 > m$, the scaling solution $\phi_s$ is a stable node or spiral whereas, when $\lambda^2 < m$, the late time attractor is the field dominated solution, which we do not consider here. The convergence towards the solution $\phi_s$ is illustrated in Fig. 1.

Now, in the most general case where $m \neq 4$, it is easy to realize that a solution of the form $\phi = \ln(At^\beta)$ cannot be a solution of Eq. (44).

#### 1. $|\xi| \ll 1$ case

Let us first look at the effect of a small perturbation in $\xi$ in the sense that the potential term dominates over the coupling term in the Klein-Gordon equation. For that purpose, we set

$$u \equiv \ln t,$$
$$\phi = \phi_s + \xi\psi + \mathcal{O}(\xi^2). \tag{46}$$

The equation of evolution for $\psi$ can be deduced from the Klein-Gordon equation (44):

$$\psi'' + \left(\frac{6}{m}-1\right)\psi' + \frac{12}{m}\left(\frac{4}{m}-1\right)\xi\psi = -\frac{\lambda}{\xi}[1-\mathrm{e}^{-\lambda\xi\psi}]\mathrm{e}^{2u-\lambda\phi_s} - \frac{12}{m}\left(\frac{4}{m}-1\right)\phi_s, \tag{47}$$

where a prime denotes a derivative with respect to $u$. Now, if we restrict ourselves to $\xi \ll 1$ and linearize this equation using expression (45) for $\phi_s$, we obtain, at zeroth order in $\xi$,

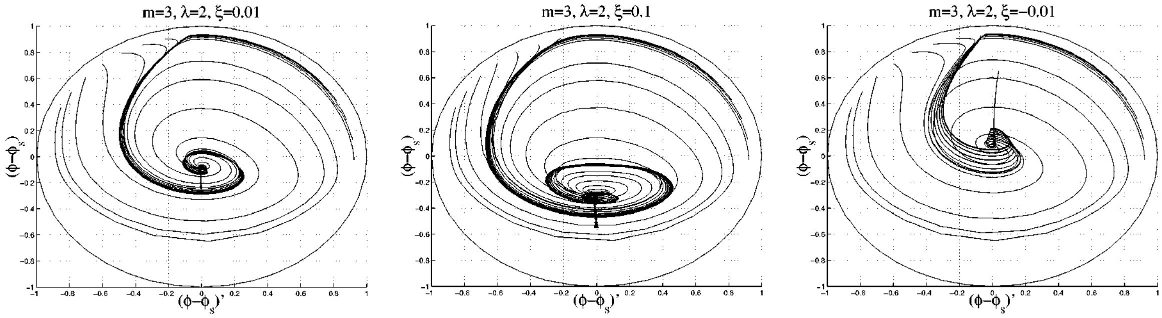


FIG. 2. The deformation of the phase space from Fig. 1 due to the coupling for $\xi=10^{-2},10^{-1},-10^{-2}$. The solution converges towards the attractor which drifts from $\phi_s$, since $\phi-\phi_s\propto\xi u+\mathcal{O}(\xi^2)$.

$$\psi''+\left(\frac{6}{m}-1\right)\psi'+2\left(\frac{6}{m}-1\right)\psi=\frac{12}{m\lambda}\left(\frac{4}{m}-1\right)[\ln A^{-\lambda}-2u], \tag{48}$$

the solution of which has the general form

$$\psi=B_+e^{\alpha_+u}+B_-e^{\alpha_-u}-\frac{12}{m\lambda}\frac{4-m}{6-m}\left[u-\frac{1+\ln A^{-\lambda}}{2}\right]. \tag{49}$$

$B_+e^{\alpha_+u}$ and $B_-e^{\alpha_-u}$ are two independent solutions of the homogeneous equation. Since $\alpha_+\alpha_-=2(6/m-1)=-2(\alpha_++\alpha_-)$, we deduce that if $m<6$, the real parts of both $\alpha_+$ and $\alpha_-$ are negative, so that the two homogeneous solutions correspond to decaying modes and the particular solution is then an attractor (if $m<2/3$, then $\alpha_+$ and $\alpha_-$ are real; otherwise, they are complex and the homogeneous part will decay while oscillating).

Indeed, this analysis is valid only as long as the potential term dominates over the coupling term in Eq. (44), that is, as long as

$$\left|\frac{dV}{d\phi}\right|\gg\left|\frac{12}{m}\left(\frac{4}{m}-1\right)\xi\phi e^{-2u}\right|\Leftrightarrow u\ll u_{eq},$$

$$u_{eq}\equiv\frac{1}{12|\xi|}\frac{m(6-m)}{|4-m|}\ln\frac{1}{\lambda}\sqrt{2\left(\frac{6}{m}-1\right)}. \tag{50}$$

We recover that when $\xi\to0$ or $m\to4$, $u_{eq}\to\infty$ and we are back to the minimally coupled case (Sec. III A). The behavior of $|dV/d\phi|$ and $(12/m)(4/m-1)\xi\phi$ as a function of $u$ is shown in Fig. 4, below. When $\xi\ll1$, we see that, as expected, the solution is first dominated by the potential term but that as time elapses the coupling term tends to become more dominant. In Fig. 2, we illustrate how the phase space trajectories are deformed due to the existence of this small coupling.

Now, as long as $u\ll u_{eq}$, we can compute the equation of state of the field by inserting the particular part of the solution (49) in Eqs. (9), (10) which will take the form

$$\begin{aligned}\omega_\phi(u)&=\omega_{\phi_s}[1+\mathcal{A}(m,\lambda)u^{-1}+\mathcal{B}(m,\lambda,\xi)u^{-2}\\&\quad+\mathcal{O}(u^{-3})]\quad(\text{if}\ \ m\neq3)\\&=-\frac{1}{u}[1+\mathcal{C}(m,\lambda)u^{-1}+\mathcal{O}(u^{-2})]\quad(\text{if}\ \ m=3).\end{aligned} \tag{51}$$

The difference in these two behaviors comes from the fact that $P_\phi\propto u^2$ if $m\neq3$ and $P_\phi\propto u$ when $m=3$. The exact forms of the functions $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ can be obtained by doing an expansion of $P_\phi/\rho_\phi$ in $u^{-1}$. In Fig. 3, we show the deviation of the equation of state from pure scaling and compare the former expansion to the numerical integration. In conclusion we have that when $\xi\ll1$ and $u\ll u_{eq}$, the field converges towards a barotropic fluid.

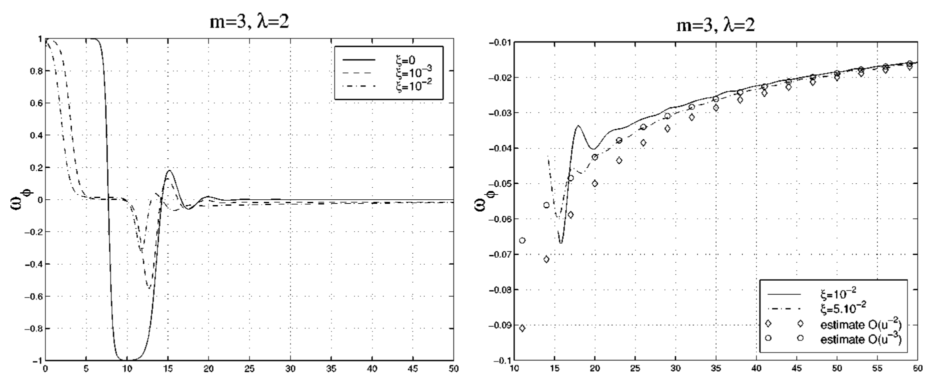


FIG. 3. The time varying equation of state and its deviation from the minimally coupling case for $\xi=0$, $10^{-3}$, $10^{-2}$ and the comparaison of this equation of state with the estimate (51) at first order and second order in $u^{-1}$.

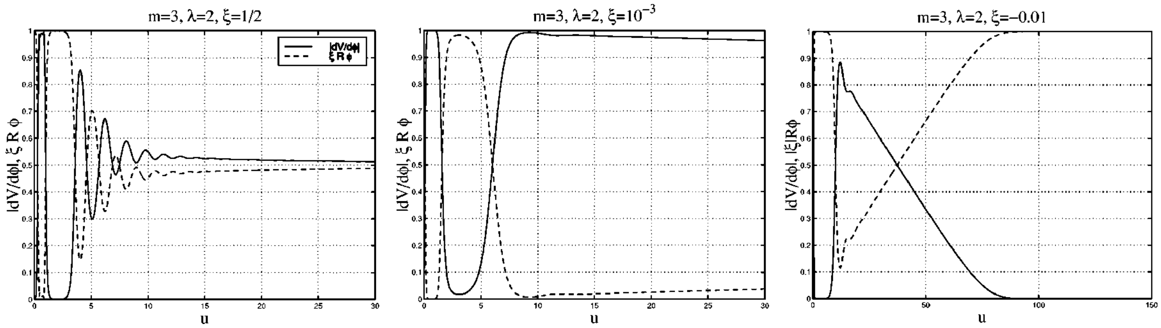


FIG. 4. The respective influence of the coupling term and the potential term in the Klein-Gordon equation. We have plotted $|dV/d\phi|$ and $|\bar{\mathcal{R}}\xi\phi|$ normalized to their sum for $\xi=1/2$ (left). The two terms alternatively dominate and then converge. When $\xi\ll 1$, after some oscillations corresponding to the convergence towards $\phi_s$, the potential term dominates over the coupling as long as $u\ll u_{eq}$. When $\xi<0$, the coupling term will dominate forever.

### 2. $|\xi|\gg 1$ *case*

Let us now consider the case where initially the coupling term dominates over the potential term in the Klein-Gordon equation. At lowest order, one has

$$\phi''+\left(\frac{6}{m}-1\right)\phi'+\frac{12}{m}\left(\frac{4}{m}-1\right)\xi\phi=0, \tag{52}$$

the solution of which is of the form

$$\phi=A_+e^{\alpha_+u}+A_-e^{\alpha_-u}. \tag{53}$$

When $\xi>0$, since $\alpha_+\alpha_-\propto 4/m-1$ and $\alpha_++\alpha_-=-(6/m-1)$, we deduce that if $0<m<4$, the real part of both $\alpha_+$ and $\alpha_-$ is negative so that Eq. (53) corresponds to two decaying modes (it can be seen as a proof that the critical point $\phi=0$ is an attractor) so that

$$\bar{\mathcal{R}}\xi\phi\rightarrow 0 \quad\text{and}\quad \lambda e^{-\lambda\phi}\rightarrow\lambda \quad\text{when}\quad 0<m<4 \quad\text{and}\quad \xi>0, \tag{54}$$

and the potential term will rapidly catch up the coupling term.

When $\xi<0$, $\alpha_+\alpha_-\propto -4/m+1$, and the real part of one of the two quantities, $\alpha_+$, say, is positive when $0<m<4$ so that

$$\bar{\mathcal{R}}\xi\phi\rightarrow\infty \quad\text{and}\quad \lambda e^{-\lambda\phi}\rightarrow 0 \quad\text{when}\quad 0<m<4 \quad\text{and}\quad \xi<0, \tag{55}$$

and the coupling term dominates forever (see Fig. 4) and $\phi$ behaves as $A_+e^{\alpha_+u}$ with

$$2\alpha_+=\left(1-\frac{6}{m}\right)+\sqrt{\left(\frac{6}{m}-1\right)^2-\frac{48}{m}\left(\frac{4}{m}-1\right)\xi}, \tag{56}$$

for which, since the potential term is negligible,

$$\omega_\phi(m,\xi)\simeq\frac{\alpha_+^2-4\xi\left[\frac{2}{m^2}(3-m)+\left(\frac{4}{m}-1\right)\alpha_++2\alpha_+^2\right]}{\alpha_+^2+\frac{24}{m}\xi\left[\alpha_++\frac{1}{m}\right]}. \tag{57}$$

This solution has, however, to be excluded since one can check that it leads to $\rho_\phi<0$.

### 3. *General case*

The general case is more involved since we cannot find any analytic solution to Eq. (44). First, when $\xi>0$, we have seen that when either the potential or the coupling term dominates, the other slowly catches up. We thus expect a late time solution which satisfies

$$\bar{\mathcal{R}}\xi\phi\simeq\lambda e^{-\lambda\phi}. \tag{58}$$

In Fig. 4, we plot the evolution of these two terms in the case where $\xi\ll 1$ and in a more general case. The two terms alternatively dominate and then converge to the same value. Indeed, this is no proof.

We can, however, look for a general solution of the form

$$\phi=\sum_{n=0}^{n=\infty}\xi^n\psi_n, \tag{59}$$

with $\psi_0$ given by Eq. (45) and $\psi_1$ given by Eq. (49). Inserting this expansion into the Klein-Gordon equation (44), we obtain the hierarchy of equations

$$\psi_n''+\left(\frac{6}{m}-1\right)\psi_n'+2\left(\frac{6}{m}-1\right)\psi_n =-\frac{12}{m}\left(\frac{4}{m}-1\right)\psi_{n-1}+\lambda A^{-\lambda}f_n(\psi_0,\ldots,\psi_{n-1}) \quad \text{if}\quad n\geq 1. \tag{60}$$

The function $f_n$ depends on all the solutions $\psi_i$ for $i<n$. All these equations have a solution of the form

$$\psi=B_+e^{\alpha_+u}+B_-e^{\alpha_-u}+\hat{\psi}_n. \tag{61}$$

As in Sec. V A 2, the two homogeneous modes decay if $m<6$ and $\forall n,\psi_n\rightarrow\hat{\psi}_n$. It can also be seen that $\hat{\psi}_n$ will be a polynomial in $u$ of degree $n$. If the series $\Sigma\xi^n\hat{\psi}_n$ converges, then

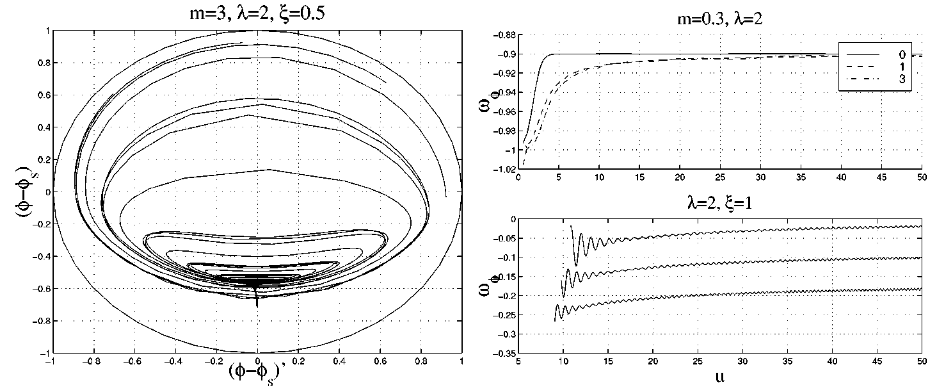


FIG. 5. The phase space analysis and the evolution of the density, pressure, and equation of state in a case where $\xi \not\ll 1$. $\omega_\phi$ converges towards $\omega_{\phi_s}$ whatever the value of $\xi$ ($\xi$=0,1,3 in the upper right plot) with or without oscillations according to the value of $m$ ($m=2$, 2.5, 3 from bottom to top in the lower right plot).

$$\phi \rightarrow \sum_{n=0}^{n=\infty} \xi^n \hat{\psi}_n \quad \text{when} \quad u \rightarrow \infty, \tag{62}$$

from which we can conclude that if $\Sigma \xi^n \hat{\psi}_n$ converges, there exists an attractor to Eq. (44) given by Eq. (62). Indeed, we cannot demonstrate the convergence of this series in the general case. In Fig. 5, we show the phase space trajectories showing the convergence towards this attractor.

When $m > 2/3$, the solution converges towards the attractor while oscillating so that the equation of state has wiggles and converges towards $\omega_{\phi_s}$ (see Fig. 5). When $m < 2/3$, this oscillatory behavior does not appear whatever $\xi$ (see Fig. 5).

#### *4. Case of a radiation dominated universe*

As explained above, one has the solution (45) to the Klein-Gordon equation and this solution is an attractor (see the phase analysis of Fig. 6) and the field behaves as radiation. The solution (49) reduces to the two decaying modes. Indeed the derivation of Eq. (51) is no longer valid but, with the same method, one can show that, to first order in $u^{-1}$,

$$\omega_\phi = \frac{1}{3}[1 + \mathcal{A}(\lambda) u^{-1} + \mathcal{B}(\lambda, \xi) u^{-2}], \tag{63}$$

where the two functions $\mathcal{A}$ and $\mathcal{B}$ are obained as in Sec. V A 2. Note that for $u \gg 1$ the $\xi$ terms of Eqs. (9), (10) dominate the density and the pressure. In Fig. 6, we have plotted the phase space showing the convergence towards the attractor $\phi_s$ in the case of a conformally coupled scalar field and the evolution of its equation of state.

### B. Numerical results

We integrate numerically Eq. (44) and use Poincaré projection [32] to represent the result in the plane $(\phi, \phi')$:

$$\phi = \frac{r}{1-r} \sin \theta,$$

$$\phi' = \frac{r}{1-r} \cos \theta. \tag{64}$$

This projection shrinks all the trajectory in the phase space to the unit disk. The points $N \equiv (0,1)$, $E \equiv (-1,0)$, and $W \equiv (1,0)$, respectively, correspond to $(\phi = \infty, \phi' = 0)$, $(\phi = 0, \phi' = -\infty)$, and $(\phi = 0, \phi' = \infty)$. Note that the system is nonautonomous since Eq. (47) depends on $u$. Thus, curves can cross in the Poincaré representation but they will not cross in a 3D representation $(\phi, \phi', u)$. The 1 to 6 plots

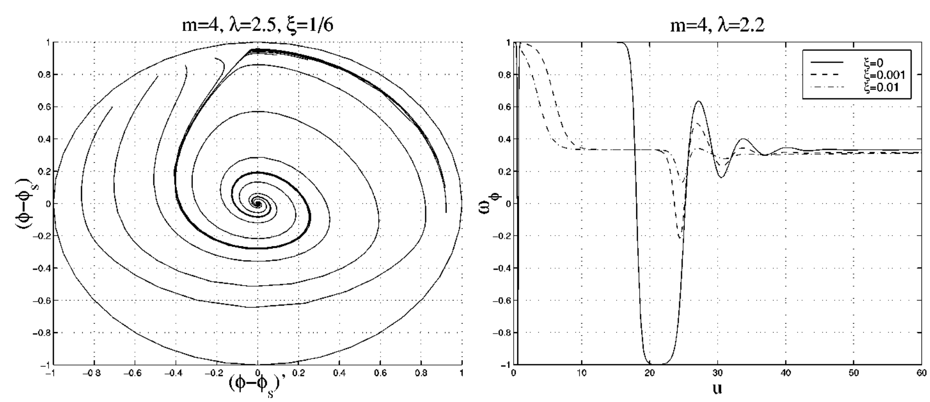


FIG. 6. The phase space analysis for a field in a radiation dominated universe. The scaling solution is the late time attractor whatever $\xi$ [$\xi = 0$, $10^{-2}$, and $10^{-3}$].

correspond to the situations studied in the previous paragraphs. We plot the evolution of the solution in the phase space $(\phi,\phi')$ and the evolution of the equation of state when $\xi \ll 1$. To finish, we give some examples of evolution in the case $\xi \not\ll 1$.

## VI. CONCLUSION

In this article, we studied the influence of the coupling between the scalar curvature and the scalar field on the existence and stability of scaling solution of this field evolving in either an exponential or an inverse power law potential. The motivation for considering such solutions is first that they can be a candidate for a matter component with negative pressure and second that they appear for a large class of potentials predicted by some theories of high energy physics.

We first found a new parametric form of the potential that reduces to the inverse power law potential when the energy density of the fluid drives the evolution of the spacetime.

Concerning the inverse power law potentials, we showed analytically that the existence and stability of a scaling solution does not depend on the coupling $\xi$ and that the equation of state of the field was always given by $\omega_\phi = (\omega_B \alpha - 2)/(\alpha + 2)$. This generalizes the work by Ratra and Peebles [11] and Liddle and Scherrer [16].

The situation is more involved with exponential potentials since one cannot find an analytic form for a scaling solution (apart for a radiation dominated universe). We then studied the effect of a small perturbation coupled to the scalar curvature and computed $\omega_\phi$ to first order in $\xi \ll 1$ and showed that there always exists a time after which one cannot neglect the effect of the coupling. In that limit, we show that the equation of state was converging towards a baryotropic equation of state and some numerical examples tend to show us that it should be the case whatever $\xi > 0$ (but this has not been demonstrated). Indeed, such potentials are not the most favored since they are constrained by nucleosynthesis to $\Omega_{\phi 0} < 0.15$ [20] and cannot explain (when $\xi = 0$) the supernova measurements since $\omega_\phi = 0$. Note, however, that the convergence toward the baryotropic equation $\omega_{\phi_s}$ is much slower when $\xi \neq 0$. When $\xi < 0$, we have shown that there always exists a time after which the coupling term will dominate and thus that there exists a scaling solution but with a different equation of state (as long as $0 < m < 4$ and, thus, in a matter dominated era). Unfortunately, such a solution has to be rejected since it has negative energy.

As pointed out by Caldwell *et al.* [22], a smooth time-dependent field is unphysical since ''one has to take into account the back-reaction of the fluctuations in the matter components.'' The cosmological consequences of such an inhomogeneous coupled scalar field in both exponential and inverse power law potentials [such as the computation of the cosmic microwave background anisotropies and the matter power spectrum] will be presented later [33] and our present study is only related to the implication of the homogeneous part of such a field.

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[1] S. Perlmutter *et al.*, Nature (London) **391**, 51 (1998); Astrophys. J. **493**, 565 (1997); A. G. Riess *et al.*, Astrophys. J. (to be published), astro-ph/9805201.
[2] I. Zlatev, L. Wang, and P. J. Steinhardt, Phys. Rev. Lett. **82**, 896 (1999).
[3] K. Coble, S. Dodelson, and J. A. Frieman, Phys. Rev. D **55**, 1851 (1997).
[4] D. Spergel and U.-L. Pen, Astrophys. J. Lett. **491**, L67 (1997).
[5] G. Huey, L. Wang, R. Dave, R. R. Caldwell, and P. J. Steinhardt, Phys. Rev. D **59**, 063008 (1999).
[6] P. Binétruy, M. K. Gaillard, and Y.-Y. Wu, Phys. Lett. B **412**, 288 (1997); Nucl. Phys. **B493**, 27 (1997).
[7] P. Binétruy, Phys. Rev. D (to be published), hep-ph/9810553.
[8] J. D. Barrow and S. Cotsakis, Phys. Lett. B **214**, 515 (1988); D. Wands, Class. Quantum Grav. **11**, 269 (1988).
[9] C. Wetterich, Nucl. Phys. **B252**, 309 (1985).
[10] B. de Carlos, J. A. Casas, and C. Muñoz, Nucl. Phys. **B399**, 623 (1993).
[11] B. Ratra and P. J. E. Peebles, Phys. Rev. D **37**, 3406 (1988).
[12] D. Wands, E. J. Copeland, and A. R. Liddle, Ann. (N.Y.) Acad. Sci. **688**, 647 (1993).
[13] C. Wetterich, Astron. Astrophys. **301**, 321 (1995).
[14] E. J. Copeland, A. R. Liddle, and D. Wands, Phys. Rev. D **57**, 4686 (1997).
[15] A. P. Billyard, A. A. Coley, and R. J. Van den Hoogen, Phys. Rev. D **58**, 123501 (1998).
[16] A. R. Liddle and R. J. Scherrer, Phys. Rev. D **59**, 023509 (1999).
[17] S. Carroll, Phys. Rev. Lett. **81**, 3067 (1998).
[18] F. Perrotta and C. Baccigalupi, Phys. Rev. D (to be published), astro-ph/9811156.
[19] P. J. E. Peebles and A. Vilenkin, Phys. Rev. D **59**, 063507 (1999).
[20] P. G. Ferreira and M. Joyce, Phys. Rev. D **58**, 023503 (1997).
[21] P. T. Viana and A. R. Liddle, Phys. Rev. D **57**, 674 (1998).
[22] R. R. Caldwell, R. Dave, and P. J. Steinhardt, Phys. Rev. Lett. **80**, 1582 (1998).
[23] D. Huterer and M. Turner, astro-ph/9808133, 1998; T. Nakamura and T. Chiba, Mon. Not. R. Astron. Soc. (to be published), gr-qc/9810447.
[24] A. A. Starobinsky, JETP Lett. **68**, 757 (1998).
[25] N. D. Birrell and P. C. W. Davis, *Quantum Fields in Curved Spacetime* (Cambridge University Press, Cambridge, England,

1982).
[26] L. H. Ford, Phys. Rev. D **35**, 2955 (1987).
[27] K. Maeda, Class. Quantum Grav. **3**, 233 (1986).
[28] F. S. Accetta, D. J. Zoller, and M. S. Turner, Phys. Rev. D **31**, 3046 (1985).
[29] L. Amendola, M. Litterio, and F. Occhionero, Int. J. Mod. Phys. A **5**, 3861 (1990); L. Amendola, D. Bellisai, and F. Occhionero, Phys. Rev. D **47**, 4267 (1993); T. Futumase and K. Maeda, *ibid.* **39**, 399 (1989).
[30] M. S. Turner and M. White, Phys. Rev. D **56**, R4439 (1997).
[31] I. S. Gradshteyn and I. M. Rizhik, *Table of Integrals, Series and Products* (Academic, New York, 1980).
[32] J. Wainwright and G. F. R. Ellis, *Dynamical Systems in Cosmology* (Cambridge University Press, Cambridge, England, 1997).
[33] A. Melchiorri and J-P. Uzan (work in progress).