Early time perturbations behavior in scalar field cosmologies

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We consider the problem of the initial conditions and behavior of the perturbations in scalar field cosmology with a general potential. We use the general definition of adiabatic and isocurvature conditions to set the appropriate initial values for the perturbation in the scalar field and in the ordinary matter and radiation components. In both the cases of initial adiabaticity and isocurvature, we solve the Einstein and fluid equations at early times and on superhorizon scales to find the initial behavior of the relevant quantities. In particular, in the isocurvature case, we consider models in which the initial perturbation arises from the matter as well as from the scalar field itself, provided that the initial value of the gauge-invariant curvature is zero. We extend the standard code to include all these cases, and we show some results concerning the power spectrum of the cosmic microwave background temperature and polarization anisotropies. In particular, it turns out that the acoustic peaks follow opposite behavior in the adiabatic and isocurvature regimes: in the first case their amplitude is higher than in the corresponding pure cold dark matter model, while it is the opposite for pure isocurvature initial perturbations. [S0556-2821(99)03110-0]

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I. INTRODUCTION

In recent times, the need for a ''quintessence'' component has come out due to the several difficulties of the standard $\Omega_m = 1$ cold dark matter (CDM) model which was unable to explain the observed features of large scale structure. In the context of inflationary cosmologies, we expect that the present spatial curvature of the Universe is negligible and the total energy density equals the critical energy density; on the other hand, there is growing observational evidence that the matter energy density is remarkably below the critical value, even taking into account the exotic and so far undetectable particles known as cold dark matter. Thus, we are faced with figuring out how to explain the missing-energy values of as much as 70% or 80% of the critical density.

Further, there is need to have the age of the Universe, t_0, exceed the age of globular clusters in our galaxy; the limits on t_0 are holding at about 13 Gyr or more [1], and when combined with current estimates of the Hubble expansion parameter, converging from different methods to $H_0 \approx 60 \pm 10$ km/(sec Mpc) [2], give rise to an observed value of the ''expansion-age'' parameter $H_0 t_0 \simeq 0.8$, sensibly higher than 2/3 as predicted by the standard Einstein–de Sitter model.

Preserving the flatness of the Universe, its age could be enhanced by lowering the matter content in models involving a component whose equation of state is different from matter and radiation, for example, in models including a cosmological constant. One more problem with the CDM model arises from the mismatching of the galaxy clustering power spectrum shape, when only Cosmic Background Explorer (COBE) normalized spectra are considered. All these motivations have ratified the demise of the standard CDM model, leaving cosmologists with the open question of what the missing-energy candidate could be.

Apart from the cosmological constant, introduced in some models in order to maintaining spatial flatness, but still retaining serious unsolved theoretical issues, several ''quintessence'' models were proposed as candidates for the missing energy, often modeled as scalar fields rolling down their potentials [3–11] or, more generally, described in terms of an unspecified equation of state different from that of matter and radiation [12–14]. We refer to any component whose properties are well described in terms of a scalar field evolving in a potential which couples to ordinary matter only through gravity. In some sense, it can behave like a cosmological constant when its kinetic energy is negligible with respect to the potential energy, so that the scalar field equation of state approaches -1; because of the strongly relativistic nature of such a component, the characteristic scale of clustering processes for a scalar field is just the horizon [11], giving a similarity with a cosmological constant in the undetectability of quintessence energy concentrations on scales smaller than the horizon.

The interesting feature of the ''quintessence'' component is just that, contrary to the cosmological constant, it is time varying and spatially inhomogeneous, so that it can develop fluctuations which can be relevant in perturbation growth and can leave a characteristic signature in the cosmic microwave background (CMB) and in the large scale structure. Even though many of these imprints have been studied in previous works, the issue of initial conditions and scalar field perturbations has often been underestimated; in particular, we found a gap regarding the opportunity to impose isocurvature initial conditions in a several-component system including a minimally coupled scalar field. Our aim is to give a complete prescription for describing adiabatic and isocurvature initial conditions if an additional component is present in the form of such a scalar field; this can be acquired by giving the set of equations relating all the fluid components needed in the two cases.

In order to do that, we need background and perturbation equations which we briefly review on Secs. II and III. The original results of our work are presented in Secs. IV and V, where we work in the formalism of the synchronous gauge, by generalizing the work of Ma and Bertschinger [15] and finding the super-horizon-scale behavior of perturbations at early times, starting from initial zero-entropy perturbations (adiabatic case) or initial zero-curvature perturbations (isocurvature case). We express the needed gauge-invariant quantities, namely, entropy and curvature perturbations, in terms of synchronous perturbations of baryons, photons, massless neutrinos, cold dark matter, and a minimally coupled scalar field, as well as metric perturbations. In Sec. VI the results are translated in conformal Newtonian gauge, and in Sec. VII we numerically investigate the growth of entropy and curvature perturbations starting from different initial conditions, and we compare them with the corresponding behavior in the standard CDM model. In that computation, the adopted scalar field is associated with an ultralight pseudo Nambu-Goldstone boson [16], with global spontaneous symmetry breaking scale $f \simeq 10^{18}$ GeV and explicit breaking scale $M \sim 10^{-3}$ eV; such a field should be acting at present like an effective cosmological constant and dominating the energy density of the Universe. Also we plot, and discuss, the pure adiabatic and pure isocurvature CMB anisotropies spectra, again making a comparison with the standard CDM.

II. EINSTEIN AND CONSERVATION EQUATIONS

We begin by a brief review of a homogeneous Friedmann-Robertson-Walker (FRW) cosmology in which there is an additional contribution coming from a minimally coupled real scalar field ϕ evolving in a potential $V(\phi)$.

We consider only models with total $\Omega = 1$ in this paper, and we work in conformal coordinates, so that the line element is $ds^2 = a^2[-d\tau^2 + \delta_{ij}\, dx^i\, dx^j]$ where a is the cosmic scale factor and τ is the conformal time.

The scalar field energy density and pressure, associated with the Lagrangian describing the classical behavior of ϕ,

$$\mathcal{L} = -\frac{1}{2}\sqrt{-g}[g^{\mu\nu}\,\partial_\mu\phi\,\partial_\nu\phi + 2V(\phi)], \tag{1}$$

follow from the expression of the stress-energy tensor

$$T^\mu_\nu = \phi^{;\mu}\phi_{;\nu} - \frac{1}{2}(\phi^{;\alpha}\phi_{;\alpha} + 2V)\,\delta^\mu_\nu \tag{2}$$

and are given by

$$\rho_\phi = \frac{1}{2a^2}\dot{\phi}^2 + V(\phi), \quad p_\phi = \frac{1}{2a^2}\dot{\phi}^2 - V(\phi), \tag{3}$$

where the overdot denotes a derivative with respect to conformal time τ. The above quantities evolve according to the Friedmann equation, in which we separate the contributes of matter, radiation, and scalar field to the total energy density,

$$\mathcal{H}^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}a^2[\rho_m + \rho_r + \rho_\phi], \tag{4}$$

together with the conservation equations

$$\ddot{\phi} + 2\mathcal{H}\dot{\phi} + a^2\frac{dV}{d\phi} = \frac{1}{a^2}\frac{d}{d\tau}(a^2\dot{\phi}) + a^2 V'(\phi) = 0, \tag{5}$$

$$\dot{\rho}_n + n\mathcal{H}\rho_n = 0, \tag{6}$$

where $\mathcal{H}$ is the conformal expansion rate of the Universe, ρ_n is the energy density contributed by radiation ($n=4$) or nonrelativistic matter ($n=3$), and $V' = dV/d\phi$. Note that, from Eqs. (3), the second-order Klein-Gordon equation (5) is equivalent to the conservation law

$$\dot{\rho}_\phi = -3\mathcal{H}(\rho_\phi + p_\phi). \tag{7}$$

Including all the modifications due to the additional scalar field component, we shall carry out a fully relativistic treatment of the perturbations of this background, based on the notation of Ma and Bertschinger [15]. We work in Fourier space and we perform the parametrization of the perturbed quantities in the formalism of the synchronous gauge, in which the perturbed line element is $ds^2 = a^2[-d\tau^2 + (\delta_{ij} + h_{ij})dx^i\, dx^j]$. Since we are interested here only in scalar-type perturbations, the metric perturbations can be parametrized as

$$h_{ij}(\mathbf{x},\tau) = \int d^3k\, e^{i\mathbf{k}\cdot\mathbf{x}}\left[\hat{\mathbf{k}}_\mathbf{i}\hat{\mathbf{k}}_\mathbf{j} h(\mathbf{k},\tau) + \left(\hat{\mathbf{k}}_\mathbf{i}\hat{\mathbf{k}}_\mathbf{j} - \frac{1}{3}\delta_{ij}\right)6\,\eta(\mathbf{k},\tau)\right], \tag{8}$$

with $\mathbf{k} = k\hat{k}$ and h denoting the trace of h_{ij}.

Note that the synchronous potentials h and η in k space are related to the gauge-invariant variables $\mathbf{\Phi}_A$ of Bardeen [17] and $\mathbf{\Psi}$ of Kodama and Sasaky [18] by the relation

$$\mathbf{\Phi}_A = \mathbf{\Psi} = \frac{1}{2k^2}\left[\ddot{h} + 6\,\ddot{\eta} + \frac{\dot{a}}{a}(\dot{h} + 6\,\dot{\eta})\right], \tag{9}$$

which allows us to relate h and η to the gauge-invariant curvature perturbation ζ (see [19]),

$$\zeta = \frac{2}{3}(\mathcal{H}^{-1}\dot{\mathbf{\Psi}} + \mathbf{\Psi})/(1+w) + \mathbf{\Psi}, \tag{10}$$

where $w = p/\rho$; the above expression will be useful in the following. Now, focus on the equations describing the evolution of perturbations involving the various components.

As is well known, the scalar field can mimic a cosmological constant if its kinetic energy is negligible with respect to the potential one. However, a substantial difference is that it admits perturbations around the homogeneous solution of Eq. (5); in linear theory, they are described by small fluctuations $\delta\phi$ and $\dot{\delta\phi}$ around the background values, driven by the equation of motion

$$\ddot{\delta\phi}+2\frac{\dot{a}}{a}\dot{\delta\phi}-\nabla^2\delta\phi+a^2\frac{d^2V}{d\phi^2}\delta\phi+\frac{1}{2}\dot{\phi}\dot{h}=0. \quad (11)$$

The density, pressure, and velocity perturbations for the scalar field are described as usual by the following quantities:

$$\delta\rho_\phi=-\delta T_0^0=\frac{\dot{\phi}\dot{\delta\phi}}{a^2}+V'\,\delta\phi, \quad (12)$$

$$\delta p_\phi=\frac{1}{3}\delta T_i^i=\frac{\dot{\phi}\dot{\delta\phi}}{a^2}-V'\,\delta\phi, \quad (13)$$

$$(\rho_\phi+p_\phi)\theta_\phi=\delta T_i^0=ka^{-2}\dot{\phi}\;\delta\phi, \quad (14)$$

$$p_\phi\pi_\phi=0, \quad (15)$$

$$\delta_\phi=\frac{\delta\rho_\phi}{\rho_\phi},\quad \theta_\phi=\frac{k}{a^2}\frac{\dot{\phi}\delta\dot{\phi}}{\rho_\phi+p_\phi}=k\frac{\delta\dot{\phi}}{\dot{\phi}}, \quad (16)$$

therefore, we define the differential ratio

$$\frac{\delta p_\phi}{\delta\rho_\phi}=\frac{\dot{\phi}\delta\dot{\phi}-a^2V'\,\delta\phi}{\dot{\phi}\delta\dot{\phi}+a^2V'\,\delta\phi}=c_\phi^2+\frac{p_\phi}{\delta\rho_\phi}\Gamma_\phi, \quad (17)$$

which differs from the scalar field sound velocity

$$c_\phi^2=\frac{\dot{p}_\phi}{\dot{\rho}_\phi} \quad (18)$$

by the term Γ_ϕ describing the entropy contribution [18].

It is useful to describe radiation in terms of the coefficients characterizing the Legendre expansion of the temperature and polarization brightness functions, $\Delta_T(\mathbf{k},\hat{\mathbf{n}},\tau)$ and $\Delta_P(\mathbf{k},\hat{\mathbf{n}},\tau)$:

$$\Delta_T(\mathbf{k},\hat{\mathbf{n}},\tau)=\sum_{l=0}^{\infty}(-i^l)(2l+1)\Delta_{Tl}(k,\tau)P_l(\hat{\mathbf{k}}\cdot\hat{\mathbf{n}}), \quad (19)$$

$$\Delta_P(\mathbf{k},\hat{\mathbf{n}},\tau)=\sum_{l=0}^{\infty}(-i^l)(2l+1)\Delta_{Pl}(k,\tau)P_l(\hat{\mathbf{k}}\cdot\hat{\mathbf{n}}). \quad (20)$$

Their evolution is completely determined by the Boltzmann equations; denoting by σ_T the Thomson scattering cross section and by n_e the electron density, we have, for photons,

$$\dot{\delta}_\gamma=-\frac{4}{3}\theta_\gamma-\frac{2}{3}\dot{h}, \quad (21)$$

$$\dot{\theta}_\gamma=k^2\left(\frac{1}{4}\delta_\gamma-\sigma_\gamma\right)+an_e\sigma_T(\theta_b-\theta_\gamma), \quad (22)$$

$$2\dot{\sigma}_\gamma=\frac{8}{15}\theta_\gamma-\frac{3}{5}k\Delta_{\gamma 3}+\frac{4}{15}\dot{h}+\frac{8}{5}\dot{\eta}-\frac{9}{5}an_e\sigma_T\sigma_\gamma+\frac{1}{10}an_e\sigma_T(\Delta_{P0(\gamma)}+\Delta_{P2(\gamma)}), \quad (23)$$

$$\dot{\Delta}_{Tl(\gamma)}=\frac{k}{2l+1}[l\Delta_{T(l-1)(\gamma)}-(l+1)\Delta_{T(l+1)(\gamma)}]-an_e\sigma_T\Delta_{Tl(\gamma)}\quad (l\geqslant 3), \quad (24)$$

$$\dot{\Delta}_{Pl(\gamma)}=\frac{k}{2l+1}[l\Delta_{P(l-1)(\gamma)}-(l+1)\Delta_{P(l+1)(\gamma)}]+an_e\sigma_T\left[\frac{1}{2}(\Delta_{T2(\gamma)}+\Delta_{P0(\gamma)}+\Delta_{P2(\gamma)})\times\left(\delta_{l0}+\frac{\delta_{l2}}{5}\right)-\Delta_{Pl(\gamma)}\right], \quad (25)$$

where

$$\delta_\gamma=\Delta_{T0},\quad \theta_\gamma=\frac{3}{4}k\Delta_{T1},\quad \sigma_\gamma=\frac{1}{2}\Delta_{T2}. \quad (26)$$

The perturbed stress-energy tensor for radiation contributes to the following nonzero quantities:

$$\delta T_0^0=-\rho_\gamma\delta_\gamma, \quad (27)$$

$$ik^i\delta T_i^0=\frac{4}{3}\rho_\gamma\theta_\gamma, \quad (28)$$

$$\delta T_j^i=\frac{1}{3}\rho_\gamma\delta_\gamma+\Sigma_j^i, \quad (29)$$

$$\left(\hat{\mathbf{k}}_i\hat{\mathbf{k}}_j-\frac{1}{3}\delta_{ij}\right)\Sigma_j^i=-\frac{4}{3}\rho_\gamma\sigma_\gamma. \quad (30)$$

The expansion (19) also applies for massless neutrinos; their evolution equations in the synchronous gauge are given by the following system:

$$\dot{\delta}_\nu=-\frac{4}{3}\theta_\nu-\frac{2}{3}\dot{h}, \quad (31)$$

$$\dot{\theta}_\nu=k^2\left(\frac{1}{4}\delta_\nu-\sigma_\nu\right), \quad (32)$$

$$\dot{\Delta}_{T2(\nu)}=2\dot{\sigma}_\nu=\frac{8}{15}\theta_\nu-\frac{3}{5}k\Delta_{T3(\nu)}+\frac{4}{15}\dot{h}+\frac{8}{5}\dot{\eta}, \quad (33)$$

$$\dot{\Delta}_{Tl(\nu)}=\frac{k}{2l+1}[l\Delta_{T(l-1)(\nu)}-(l+1)\Delta_{T(l+1)(\nu)}]\quad (l\geqslant 3). \quad (34)$$

Pressureless cold dark matter interacts only gravitationally with other particles and in the synchronous gauge its peculiar velocity is zero; setting $\theta_c=0$, the evolution of CDM density perturbations is given by

$$\dot{\delta}_c=-\frac{1}{2}\dot{h}, \tag{35}$$

and the nonzero component of its perturbed stress-energy tensor is

$$\delta T_0^0=-\rho_c\delta_c. \tag{36}$$

Taking into account the coupling between photons and baryons by Thomson scattering,

$$\dot{\delta}_b=-\theta_b-\frac{1}{2}\dot{h}, \tag{37}$$

$$\dot{\theta}_b=\frac{\dot{a}}{a}\theta_b+c_s^2k^2\delta_b+\frac{4\rho_\gamma}{3\rho_b}an_e\sigma_T(\theta_\gamma-\theta_b), \tag{38}$$

the perturbed stress-energy tensor for baryons contributes by

$$\delta T_0^0=-\rho_b\delta_b, \tag{39}$$

$$ik^iT_i^0=\frac{4}{3}\rho_\gamma\theta_b. \tag{40}$$

All these ingredients are to be implemented in the perturbed Einstein equations

$$k^2\eta-\frac{1}{2}\mathcal{H}\dot{h}=4\pi Ga^2\delta T_0^0, \tag{41}$$

$$k^2\dot{\eta}=4\pi Ga^2ik^i\delta T_i^0, \tag{42}$$

$$\ddot{h}+2\mathcal{H}\dot{h}-2k^2\eta=-8\pi Ga^2\delta T_i^i, \tag{43}$$

$$\ddot{h}+6\ddot{\eta}+2\mathcal{H}(\dot{h}+6\dot{\eta})-2k^2\eta=24\pi Ga^2\left(\hat{\mathbf{k}}_i\hat{\mathbf{k}}_j-\frac{1}{3}\delta_{ij}\right)\Sigma_j^i. \tag{44}$$

This system of differential equations can be integrated once the appropriate initial conditions are fixed, which will be the content of the following sections.

III. INITIAL CONDITIONS AND SUPERHORIZON EVOLUTION

In order to start the numerical integration of the evolution equations given in the previous section, one has to impose appropriate initial conditions to the fluid and metric perturbations. Although a general perturbation need not be either isothermal (entropic) or adiabatic (isoentropic), it can always be expressed as a linear superposition of adiabatic and isothermal components [20]. Also, it is useful to recall that isocurvature perturbations may be present in this kind of model [21]. We explore both these conditions in scalar field cosmology. In particular we search for the initial values of the field perturbations $\delta\phi_0$ and $\delta\phi_{t0}$ (initial real time derivative) that realize initial adiabaticity and isocurvature, together with appropriate initial conditions on the other perturbed quantities. For this purpose, we will focus on the initial fluctuation of the real time derivative of the scalar field perturbation, since the conformal time derivative is always zero at $a=0$ by definition ($\delta\dot{\phi}=a\,\delta\phi_t$). The same happens of course to the initial conformal time derivative of the background scalar field ϕ; if in general the latter has an initial nonvanishing kinetic energy, so that $\phi_{t0}=(d\phi/dt)_0\neq 0$, its conformal time velocity $\dot{\phi}_0$ is zero since $\dot{\phi}=a\phi_t$.

In the following, we will need to use the scale factor behavior at early times, when $a\ll 1$. We will often use the expansion of the scale factor in powers of the dimensionless parameter ϵ:

$$\epsilon=\sqrt{\frac{8\pi G\rho_c\Omega_r}{3}}\tau=H_0\sqrt{\Omega_r}\tau=\mathcal{C}\tau, \tag{45}$$

where ρ_c and H_0 are the present critical density and Hubble parameter, respectively, and $\Omega_r=\Omega_\gamma+\Omega_\nu$ is the total radiation density contribution at the present; indeed, as can be easily verified, the scale factor being $a\ll 1$ at early times, we can neglect the scalar field contribution in Eq. (4), which admits the simple solution

$$a(\epsilon)=\epsilon+\frac{1}{4}\frac{\Omega_m}{\Omega_r}\epsilon^2+O(\epsilon^3); \tag{46}$$

besides, the expansion rate behaves as

$$\mathcal{H}=\mathcal{C}\left[\frac{1}{\epsilon}+\frac{1}{4}\frac{\Omega_m}{\Omega_r}-\frac{1}{16}\left(\frac{\Omega_m}{\Omega_r}\right)^2\epsilon+O(\epsilon^2)\right]. \tag{47}$$

Before going on, it is worth recalling some general results concerning the synchronous gauge behavior of metric and density perturbations on superhorizon scales (we refer to the work of Ma and Bertschinger [15], although they did not include the scalar field component). We impose initial conditions at an early time, deep in the radiation era, when photons and baryons are tightly coupled and can be considered as a single coupled fluid; as a result of the large Thomson scattering opacity, the $l\geq 2$ moments of the photon temperature brightness function (24) (in particular, the shear σ_γ) and the polarization brightness function (25) are driven to zero; similarly, to the lowest order in $k\tau$, one can ignore the $l\geq 3$ moments of the neutrino temperature brightness function. Thus, Eqs. (21), (22), (31), and (32) become

$$\dot{\delta}_\gamma+\frac{4}{3}\theta_\gamma+\frac{2}{3}\dot{h}=0,\quad \dot{\theta}_\gamma-\frac{1}{4}k^2\delta_\gamma=0, \tag{48}$$

$$\dot{\delta}_\nu+\frac{4}{3}\theta_\nu+\frac{2}{3}\dot{h}=0,\quad \dot{\theta}_\nu-\frac{1}{4}k^2(\delta_\nu-4\sigma_\nu)=0, \tag{49}$$

$$\dot{\sigma}_\gamma-\frac{2}{15}(2\theta_\gamma+\dot{h}+6\dot{\eta})=0,$$

$$\dot{\sigma}_\nu - \frac{2}{15}(2\theta_\nu + \dot{h} + 6\dot{\eta}) = 0. \qquad (50)$$

When we impose initial conditions, at $\epsilon \ll 1$, to get starting values for numerical integration, all the k modes are still outside the horizon, i.e., $k \ll aH = 1/\tau$ (the last equality holds in a radiation-dominated universe). Our aim is to extract the analytical time dependence of superhorizon-sized perturbations at early times, once the initial conditions are realized: thus we find the early time form of Eqs. (41)–(44), (48)–(50), and (11) and we find their solutions in successive powers of $k\tau$. To set up the growth of perturbations, we must assume that at least a single perturbation is nonzero at initial time, in order to generate all the others.

IV. ADIABATIC INITIAL CONDITIONS

The first necessary step to impose adiabatic conditions is setting to zero the initial entropy perturbation; ultimately, the origin of this result is that there is initially a single curvature perturbation (generated we suppose by inflation) and all later perturbations are inherited from it. The entropy exchange between any two fluid species a and b is ruled by the gauge-invariant quantity

$$S_{ab} = \frac{\delta_a}{1+w_a} - \frac{\delta_b}{1+w_b} \ , \qquad (51)$$

which must be set to zero initially [18]. The second request comes from setting to zero the first time derivative of S_{ab}; actually, S_{ab} obeys the following differential equation:

$$\dot{S}_{ab} = -kV_{ab} - 3\mathcal{H}\Gamma_{ab}, \qquad (52)$$

where Γ_{ab} is defined as

$$\Gamma_{ab} = \frac{w_a}{1+w_a}\Gamma_a - \frac{w_b}{1+w_b}\Gamma_b, \qquad (53)$$

Γ_a being the gauge-invariant amplitude of the entropy perturbation of the fluid species a. The quantity $V_{ab} = v_a - v_b$ is the gauge-invariant difference between the gauge-dependent velocity perturbations of the species a and b. In order to have adiabatic initial conditions, both these terms on the right hand side of Eq. (52) are initially set to zero. Thus, for each pair of fluid components, we impose

$$S_{ab} = \dot{S}_{ab} = 0. \qquad (54)$$

In particular, applying Eq. (54) to the scalar field and another component (which we leave unspecified and label with x) will relate the initial values of $\delta\phi$ and $\delta\dot{\phi}$ to the other energy component and metric perturbations. The first condition in Eq. (54) gives

$$\phi_t \delta\phi_t + V' \delta\phi = \phi_t^2 \frac{\delta_x}{1+w_x}. \qquad (55)$$

Posing $\dot{S}_{\phi x} = aS_{\phi x t} = 0$ and using the Klein-Gordon equations (5),(11), we obtain

$$\delta\phi_t \left(1 - \frac{k^2\phi_t}{6a^2HV'}\right) = \frac{1}{6H}\left[-\frac{1}{2}\phi_t h_t - \left(\frac{\delta_x}{1+w_x}\right)_t \phi_t + \frac{\delta_x}{1+w_x}\left(6H\phi_t + 2V' - \frac{k^2\phi_t^2}{a^2V'}\right)\right]. \qquad (56)$$

Combining them together we find

$$\delta\phi = \frac{1}{V'}\left(\phi_t^2 \frac{\delta_x}{1+w_x} - \phi_t \delta\phi_t\right), \qquad (57)$$

$$\delta\phi_t = \frac{1}{6H - k^2\phi_t/(a^2V')}\left[-\frac{1}{2}\phi_t h_t - \left(\frac{\delta_x}{1+w_x}\right)_t \phi_t + \frac{\delta_x}{1+w_x}\left(6H\phi_t + 2V' - \frac{k^2\phi_t^2}{a^2V'}\right)\right]. \qquad (58)$$

The above expressions specify the general adiabatic conditions for the scalar field. Now, let us make a link to previous works; in [15] the adiabatic initial values and early time behavior of the matter and the radiation components were found in the synchronous gauge; these results apply here, too. Indeed, as can be easily seen from the Einstein equations, the contribution of the scalar field fluctuations is negligible at early times $a \ll 1$ with respect to the matter and radiation ones, by a factor of a^3 and a^4, respectively. Thus the approximations and treatment developed in [15] are valid also here for what concerns the ordinary fluid components, i.e., photons, massless neutrinos, baryons, and dark matter; the time dependence of the resulting superhorizon-sized perturbations ($k\tau \ll 1$) is found by expanding the Einstein equations into powers of $k\tau$ and resolving the system of coupled differential equations to obtain the leading-order terms

$$\delta_\gamma = \delta_\nu = \frac{4}{3}\delta_b = \frac{4}{3}\delta_c = -\frac{2}{3}\mathcal{N}(k\tau)^2, \qquad (59)$$

$$\theta_\gamma = \frac{15+4R_\nu}{23+4R_\nu}\theta_\nu = \theta_b = -\frac{1}{18}\mathcal{N}k^4\tau^3, \quad \theta_c = 0, \qquad (60)$$

$$\sigma_\gamma = 0, \quad \sigma_\nu = \frac{4\mathcal{N}}{45+12R_\nu}(k\tau)^2, \qquad (61)$$

$$h = \mathcal{N}(k\tau)^2, \quad \eta = 2\mathcal{N} - \frac{5+4R_\nu}{90+24R_\nu}\mathcal{N}(k\tau)^2, \qquad (62)$$

where $R_\nu = \rho_\nu/(\rho_\nu + \rho_\gamma)$ and $\mathcal{N}$ is a normalization constant. Using these results, it is immediate to see from Eqs. (57), (58) that, imposing adiabatic initial conditions, the initial values of $\delta\phi$ and $\delta\phi_t$ must be set to zero. Adiabatic conditions can be strictly verified *only* at this initial time, due to the effect of the mutual coupling between total density perturbations and entropy perturbations which appear in a generic multicomponent fluid. Starting from initial zero values and using Eq. (62), $\delta\phi$ and $\delta\dot{\phi}$ will evolve according to Eq.

(11) which can be easily integrated once terms of the highest order in τ are dropped; this gives the following behavior at early times ($a\ll 1$):

$$\delta\phi=-\frac{1}{20}\phi_{t0}\mathcal{N}\mathcal{C}k^2\tau^4, \tag{63}$$

$$\delta\dot{\phi}=-\frac{1}{5}\phi_{t0}\mathcal{N}\mathcal{C}k^2\tau^3, \tag{64}$$

having considered the lowest terms in τ, thereby approximating the time derivative of ϕ with its value at the initial time ϕ_{t0}. We have inserted these inputs into the standard CMB code and in Sec. VII we shall give some numerical results. Now, let us turn to the second class of initial conditions.

V. ISOCURVATURE INITIAL CONDITIONS

The isocurvature initial conditions are obtained by setting to zero the gauge-invariant curvature perturbation. Its expression is given in terms of the gauge-invariant perturbation potential Ψ [19]:

$$\zeta=\frac{2}{3}\left(\frac{\mathcal{H}^{-1}\dot{\Psi}+\Psi}{1+w}\right)+\Psi; \tag{65}$$

we point out that here, as in [15,18], Ψ indicates the gauge-invariant Φ_A of the original work of Bardeen [17], while in [19] the same quantity is indicated as Φ. Its expression in terms of the metric perturbations h and η in the synchronous gauge is

$$\Psi=\frac{1}{2k^2}\left[\ddot{h}+6\ddot{\eta}+\frac{\dot{a}}{a}(\dot{h}+6\dot{\eta})\right]. \tag{66}$$

Therefore, the appropriate isocurvature initial conditions are realized by the time growing solutions of the system (41)–(44) in which Ψ and $\mathcal{H}^{-1}\dot{\Psi}$ are zero initially. First, let us see that, if the variables describing all the perturbations are regular enough to be derivable at least 4 times in $\tau=0$, then the isocurvature initial conditions are simply imposed by setting the metric and radiation perturbations to zero initially:

isocurvature initial conditions:

$$h_0=\eta_0=\delta_\gamma=\theta_\gamma=\sigma_\gamma=\delta_\nu=\theta_\nu=\sigma_\nu=0. \tag{67}$$

This can be easily seen by using essentially the Einstein equation (44); multiplying both members by a^4, deriving once, and factoring out the present critical density ρ_c, it takes the following form:

$$a^2\frac{d^3}{d\tau^3}(h+6\eta)+4a\dot{a}(\ddot{h}+6\ddot{\eta})+(2\dot{a}^2+2a\ddot{a})(\dot{h}+6\dot{\eta})$$
$$-4k^2a\dot{a}\eta-2k^2a^2\dot{\eta}=-32\pi G\rho_c(\Omega_\gamma\dot{\sigma}_\gamma+\Omega_\nu\dot{\sigma}_\nu). \tag{68}$$

Since by hypothesis h and η are derivable 4 times in $\tau=0$, $h+6\eta$ admits the following early time expansion:

$$(h+6\eta)(\tau)=\frac{d}{d\tau}(h+6\eta)_0\tau+\frac{1}{2}\frac{d^2}{d\tau^2}(h+6\eta)_0\tau^2$$
$$+\frac{1}{6}\frac{d^3}{d\tau^3}(h+6\eta)_0\tau^3$$
$$+\frac{1}{24}\frac{d^4}{d\tau^4}(h+6\eta)_0\tau^4+O(\tau^5), \tag{69}$$

since with the initial condition (67) its initial value is zero. At $\tau=0$ the only term that survives in Eq. (68) is $\dot{a}^2(\dot{h}+6\dot{\eta})$ since $\dot{a}_0^2=8\pi G\rho_c(\Omega_\gamma+\Omega_\nu)/3$. Then, by using Eqs. (48),(49),(50) one obtains

$$(\dot{h}+6\dot{\eta})_0\frac{48\pi G}{5}\rho_c(\Omega_\gamma+\Omega_\nu)=0\Rightarrow(\dot{h}+6\dot{\eta})_0=0. \tag{70}$$

In the same way, by deriving again Eq. (68) one obtains

$$(\ddot{h}+6\ddot{\eta})_0=0. \tag{71}$$

Instead, it may be easily seen that $d^3(h+6\eta)/d\tau^3$ can be different from zero; in fact, deriving Eq. (68) 3 times would bring $d^3(h+6\eta)/d\tau^3\propto k^2\dot{a}^2\dot{\eta}$, which may be different from zero by hypothesis (67). This means that, for $\tau\to 0$, $h+6\eta=O(\tau^3)$; since $\mathcal{H}=1/\tau$ to the lowest order, this is evidently enough to make $\Psi_0=(\mathcal{H}^{-1}\dot{\Psi})_0=\zeta_0=0$, showing that the initial condition (67) implies isocurvature.

It is evident that the initial condition (67) can be realized in several ways, depending on which matter component is initially perturbed or, in other words, on which δ_x is initially different from zero. In the present case a further degree of freedom arises from the presence of the scalar field, and we will analyze separately two main situations: in the first case only one matter component (CDM or baryons) is initially perturbed; in the second case the initially perturbed component is only the scalar field.

A. Isocurvature conditions from matter perturbations

Let us consider first the case in which an initial density perturbation, with amplitude δ_{c0}, resides only on the CDM component. By integrating Eq. (35), one finds

$$\delta_c=\delta_{c0}-\frac{1}{2}h. \tag{72}$$

By hypothesis, this is the only initially perturbed quantity. All the others must be set to zero at $\tau=0$. Let us search the early time behavior of the perturbations. Since all the modes are outside the horizon at early times, we first neglect all the terms proportional to k in the Einstein equations; then we expand all the quantities in powers of ϵ defined in Eq. (45) and we calculate the leading orders. In doing this, we are assuming that all the perturbation quantities admit a Taylor expansion in $\tau=0$ of course. By making use of the above criteria and of Eqs. (46),(47), the Einstein equation (41) becomes

$$\left(1+\frac{1}{4}\frac{\Omega_m}{\Omega_r}\epsilon+O(\epsilon^2)\right)\left(1+\frac{1}{2}\frac{\Omega_m}{\Omega_r}\epsilon+O(\epsilon^2)\right)\epsilon\dot{h}$$
$$=\frac{8\pi G}{\mathcal{C}^2}[\Omega_\gamma\delta_\gamma+\Omega_\nu\delta_\nu+\Omega_c\delta_c\epsilon+\Omega_b\delta_b\epsilon+O(\epsilon^2)], \quad (73)$$

and it is immediate to gain the early time behavior of h:

$$h=\delta_{c0}\frac{\Omega_c}{\Omega_r}\epsilon-\frac{3}{8}\delta_{c0}\frac{\Omega_c\Omega_m}{\Omega_r^2}\epsilon^2+O(\epsilon^3). \quad (74)$$

From the arguments exposed at the beginning of this section, up to the order ϵ^2 we have also

$$\eta=-\frac{1}{6}h. \quad (75)$$

Let us come now to the fluid perturbation quantities. As is evident from the fluid equations, the θ and σ quantities are of higher order in $k\tau$ with respect to the purely metric perturbations h and η. Therefore, their early time behavior can be written as follows:

$$\delta_\gamma=\delta_\nu=\frac{4}{3}\delta_b=-\frac{2}{3}h, \quad (76)$$

$$\theta_\gamma=\theta_\nu=\theta_b=-\frac{1}{12}\delta_{c0}\frac{\Omega_c}{\Omega_r\mathcal{C}}k^2\epsilon^2+O(\epsilon^3),$$
$$\theta_c=0, \quad \sigma_\nu=O(\epsilon^3). \quad (77)$$

The behavior of $h+6\eta$ is interesting even if of high order in τ since it is directly related to the gauge-invariant curvature by Eqs. (65),(66) and it can be obtained by solving Eq. (44):

$$h+6\eta=\frac{\mathcal{I}_1}{3}\tau^3, \quad (78)$$

where we have defined

$$\mathcal{I}_1=\frac{4\delta_{c0}\Omega_c(\Omega_\nu-5\Omega_r)\mathcal{C}k^2}{36\Omega_r+24\Omega_\nu}. \quad (79)$$

Note that $h+6\eta\propto\tau^3$, according to the isocurvature nature of the present case, as we showed in the beginning of this section. Also, Eq. (78) can be used to find the behavior of σ_ν, by using again Eq. (44).

It remains to find the early time behavior of the scalar field perturbation $\delta\phi$. This can be done by expanding $\delta\phi$ in powers of ϵ and looking at the perturbed Klein-Gordon equation once the terms proportional to k^2 have been neglected. The inhomogeneous term is $-\frac{1}{2}\dot{\phi}\dot{h}$; $\dot{h}$ is of the order of zero from Eq. (74), and $\dot{\phi}=a\phi_t$ is at least of the order of ϵ; thereby, to the lowest order in ϵ, Eq. (11) is satisfied by

$$\delta\phi=-\frac{1}{24}\phi_{t0}\delta_{c0}\frac{\Omega_c}{\Omega_r}\frac{\epsilon^3}{\mathcal{C}}, \quad \delta\dot{\phi}=-\frac{1}{8}\phi_{t0}\delta_{c0}\frac{\Omega_c}{\Omega_r}\epsilon^2. \quad (80)$$

This completes the early time behavior of all the perturbation quantities in this case of isocurvature initial conditions. All these relations can be easily generalized to the case in which the initial perturbed matter component is the baryonic one. In the next subsection we study the other interesting case: where the initial perturbed fluid component is the scalar field itself.

B. Isocurvature conditions from scalar field perturbations

Let us suppose that at the initial time $a\to 0$ the only nonzero perturbed quantity is the scalar field, in a manner such that the total gauge-invariant energy density contrast is zero, all the other perturbations being zero; this means that at least one of the two quantities $\delta_{\phi0},\theta_{\phi0}$ must be different from zero initially; from Eq. (16), the corresponding expressions for $\delta\phi_0$ and $\delta\phi_{t0}$ are

$$\delta\phi_0=\frac{1}{V'(\phi_0)}\left[\frac{1}{2}\phi_{t0}^2\left(\delta_{\phi0}-2\frac{\theta_{\phi0}}{k}\right)+V(\phi_0)\delta_{\phi0}\right],$$
$$\delta\phi_{t0}=\frac{\phi_{t0}\theta_{\phi0}}{k}. \quad (81)$$

In order to have isocurvature, for the other quantities we impose again the initial condition (67). The relevant difference with respect to the previous situation lies in the slower rise of the metric and fluid perturbations starting from their initial zero values: they will grow according to Eqs. (41)–(44) and (48)–(50), the whole perturbation-growth machinery being initially driven only by the $O(\epsilon^4)$ contribution of the scalar field through the perturbed Einstein equations, while the perturbed Klein-Gordon equation starts its dynamics from the conditions $\delta\phi_0\neq 0$, $\delta\dot{\phi}=0$ and generates the inhomogeneous term driving the evolution of h. From Eq. (41), together with Eqs. (48)–(50), it is easy to find the early time behavior of the metric perturbation h:

$$h=\frac{3}{4}\left(\frac{\delta_{\phi0}\rho_\phi}{\rho_c\Omega_r}\right)\mathcal{C}^4\tau^4+O(\tau^5). \quad (82)$$

Using the method applied in the previous sections, one finds the leading-order behaviors:

$$\delta_\gamma=\delta_\nu=\frac{4}{3}\delta_c=\frac{4}{3}\delta_b=-\frac{2}{3}h\propto\tau^4, \quad (83)$$

$$\theta_\nu,\theta_\gamma,\theta_b\propto\tau^5, \quad \sigma_\nu\propto\tau^6, \quad (84)$$

and from Eq. (44) it can be seen that

$$h+6\eta=\frac{\mathcal{I}_2}{6}\tau^6, \quad (85)$$

where

$$\mathcal{I}_2=\frac{k^2\mathcal{C}^4}{170\Omega_r+8\Omega_\nu}\left(\frac{\delta_{\phi 0}\rho_\phi}{\rho_c\Omega_r}\right)\left(-\frac{2}{10}\Omega_\nu-\frac{125}{10}\Omega_r\right). \quad (86)$$

From the above formulas we see that the perturbations regarding the metric and the ordinary fluid components rise very slowly; indeed we found a substantial failure of this model in providing a significant amount of perturbations. For this reason we will not consider this case in the numerical integrations of Sec. VII.

It is interesting to find the behavior of scalar field perturbation at early times, which moves it away from its initial value $\delta\phi_0$; this contains corrections in $(k\tau^2)$ together with a term proportional to τ^4, as can be verified by integration of Eq. (11):

$$\delta\phi=\delta\phi_0+\delta\phi^{(2)}\tau^2+\delta\phi^{(3)}\tau^3+\delta\phi^{(4)}\tau^4 + \delta\phi^{(5)}\tau^5+\delta\phi^{(6)}\tau^6+O(\tau^7), \quad (87)$$

where the expansion coefficients are given by

$$\delta\phi^{(2)}=-\frac{1}{6}\delta\phi_0 k^2, \quad \delta\phi^{(3)}=\frac{1}{72}\frac{\Omega_m}{\Omega_r}\mathcal{C}k^2\delta\phi_0,$$

$$\delta\phi^{(4)}=\frac{1}{20}\delta\phi_0\left(\frac{k^4}{6}-\mathcal{C}^2V''\right)+\frac{1}{80}\left(\frac{\Omega_m}{\Omega_r}\right)^2\mathcal{C}^2\delta\phi^{(2)} -\frac{3}{40}\frac{\Omega_m}{\Omega_r}\mathcal{C}\delta\phi^{(3)},$$

$$\delta\phi^{(5)}=-\frac{1}{15}\frac{\Omega_m}{\Omega_r}\mathcal{C}\delta\phi^{(4)}+\left[\frac{1}{80}\left(\frac{\Omega_m}{\Omega_r}\right)^2\mathcal{C}^2-k^2\right]\delta\phi^{(3)},$$

$$\delta\phi^{(6)}=-\frac{5}{84}\frac{\Omega_m}{\Omega_r}\mathcal{C}\delta\phi^{(5)}+\left[\frac{1}{84}\left(\frac{\Omega_m}{\Omega_r}\right)^2\mathcal{C}^2-k^2\right]\delta\phi^{(4)} -\mathcal{C}^2V''\delta\phi^{(2)}-\frac{3}{2}\left(\frac{\delta_{\phi 0}\rho_\phi}{\rho_c\Omega_r}\right)\mathcal{C}^5\phi_{t0}. \quad (88)$$

We considered the expansion up to the sixth order in τ because, as we will see in the next section, going to the Newtonian gauge changes the last coefficient.

In the next section we extend the results of Secs. IV and V to the conformal Newtonian gauge.

VI. RESULTS IN THE CONFORMAL NEWTONIAN GAUGE

As is well known, the synchronous gauge is a coordinate system corresponding to observers at rest with respect to the collisionless matter component. These ''Lagrangian coordinates'' are defined by the rest frame of a set of preferred observers. More physical intuition can be achieved in the conformal Newtonian gauge, where the metric tensor is diagonal. Inside the horizon, the perturbation equations reduce to the standard nonrelativistic Newtonian equations. In this section we write the results of Secs. IV and V in the Newtonian gauge.

The connection between the two gauges is realized in general by performing a coordinate transformation relating the two frames. The link between the perturbations in the two gauges is expressed in the same coordinate point instead of the same spacetime point; this is why in most cases it is interesting to know the difference of the fluctuations in the two gauges [18].

First we write down the relations between the genuine metric perturbed quantities. In the Newtonian gauge the perturbation to g_{00} exists and it is represented by the potential Ψ; the trace of g_{ij} is instead perturbed by Φ. Their relations with h and η are

$$\Psi=\frac{1}{2k^2}\left[\ddot{h}+6\ddot{\eta}+\frac{\dot{a}}{a}(\dot{h}+6\dot{\eta})\right], \quad (89)$$

$$\Phi=\frac{1}{2k^2}\frac{\dot{a}}{a}(\dot{h}+6\dot{\eta})-\eta. \quad (90)$$

They can be easily expressed for $\epsilon\ll 1$, $k\tau\ll 1$ by substituting directly the expressions for h and η contained in Secs. IV and V.

Now we concentrate on the transformations regarding fluids and scalar field. They are the contained in the stress energy tensor, which transforms as

$$T^{\mu\nu}(\tilde{x})=\frac{\partial\tilde{x}^\mu}{\partial x^\rho}\frac{\partial\tilde{x}^\nu}{\partial x^\sigma}T^{\rho\sigma}(x). \quad (91)$$

Using this and taking care to compare the perturbations in the same coordinate point, the relations between the quantities in the two gauges (synchronous and Newtonian labeled as s and N, respectively) for each fluid are

$$\delta_s=\delta_N-\mathcal{T}\frac{\dot{\rho}}{\rho}, \quad (92)$$

$$\theta_s=\theta_N-k^2\mathcal{T}, \quad (93)$$

$$p_s=p_N-p\mathcal{T}, \quad (94)$$

$$\sigma_s=\sigma_N, \quad (95)$$

where

$$\mathcal{T}=\frac{\dot{h}+6\dot{\eta}}{2k^2} \quad (96)$$

is the lapse between the synchronous and Newtonian time coordinates. Regarding the scalar field, we compute the Newtonian gauge expression of the amplitude fluctuation $\delta\phi$ by using the transformation

$$\delta\phi_s=\delta\phi_N-\dot{\phi}\mathcal{T}. \quad (97)$$

In the following subsections we write the behavior of the fluid quantities in the $\epsilon\ll 1$, $k\tau\ll 1$ regime and in the Newtonian gauge, dropping the N subscript.

A. Adiabaticity

The leading orders for matter and radiation perturbations are

$$\delta_\gamma = -\frac{40\mathcal{N}}{15+4R_\nu} = \delta_\nu = \frac{4}{3}\delta_b = \frac{4}{3}\delta_c, \tag{98}$$

$$\theta_\gamma = \theta_\nu = \theta_c = \theta_b = \frac{10\mathcal{N}}{15+4R_\nu}k^2\tau, \tag{99}$$

$$\sigma_\nu = \frac{4\mathcal{N}}{45+12R_\nu}k^2\tau^2. \tag{100}$$

The scalar field perturbation amplitude is

$$\delta\phi = \mathcal{N}\tau^2\phi_{t0}\frac{10}{15+4R_\nu}. \tag{101}$$

Note that in this case the scalar field perturbations grow in time faster ($\propto\tau^2$) than in the synchronous gauge ($\propto\tau^4$); as we point out below, this is not a feature of the isocurvature initial conditions.

B. Isocurvature from matter

Matter and radiation behave as

$$\delta_\gamma = \delta_\nu = \frac{4}{3}\delta_b = -\frac{2}{3}\delta_{c0}\frac{\Omega_c}{\Omega_r}\mathcal{N}\tau - \frac{2\mathcal{I}_1}{k^2}\tau, \tag{102}$$

$$\delta_c = \delta_{c0} - \frac{1}{2}\delta_{c0}\frac{\Omega_c}{\Omega_r}\mathcal{C}\tau - 3\frac{\mathcal{I}_1\tau}{2k^2}, \tag{103}$$

$$\theta_\gamma = \theta_\nu = \theta_b = -\frac{1}{12}\delta_{c0}\frac{\Omega_c}{\Omega_r}k^2\mathcal{C}\tau^2 + \frac{\mathcal{I}_1\tau^2}{2}, \quad \theta_c = \frac{\mathcal{I}_1\tau^2}{2}, \tag{104}$$

$$\sigma_\nu = O(\epsilon^3). \tag{105}$$

The scalar field amplitude is given by

$$\delta\phi = -\frac{1}{24}\phi_{t0}\delta_{c0}\frac{\Omega_c}{\Omega_r}\mathcal{C}^2\tau^3 + \frac{\mathcal{I}_1\mathcal{C}\tau^3}{2k^2}\phi_{t0}, \tag{106}$$

and shows that the gauge change does not touch the order of the leading power in τ, although it modifies its numerical coefficient.

C. Isocurvature from scalar field

Matter and radiation behave as

$$\delta_\gamma = \delta_\nu = \frac{4}{3}\delta_b = -\frac{1}{2}\left(\frac{\delta_{\phi0}\rho_\phi}{\rho_c\Omega_r}\right)\mathcal{C}^4\tau^4 - 2\frac{\mathcal{I}_2}{k^2}\tau^4, \tag{107}$$

$$\delta_c = -\frac{3}{8}\left(\frac{\delta_{\phi0}\rho_\phi}{\rho_c\Omega_r}\right)\mathcal{C}^4\tau^4 - \frac{3\mathcal{I}_2\tau^4}{2k^2}, \quad \theta_c = \frac{\mathcal{I}_2\tau^5}{2}, \tag{108}$$

$$\theta_\gamma = \theta_\nu = \theta_b \propto \tau^5, \quad \sigma_\nu \propto \tau^6. \tag{109}$$

The scalar field amplitude is given by

$$\delta\phi = \delta\phi_0 + \delta\phi^{(2)}\tau^2 + \delta\phi^{(3)}\tau^3 + \delta\phi^{(4)}\tau^4 + \delta\phi^{(5)}\tau^5 + \left(\delta\phi^{(6)} + \frac{\mathcal{I}_2}{2k^2}\phi_{t0}\right)\tau^6 + O(\tau^7), \tag{110}$$

therefore, in this isocurvature case, the behavior of $\delta\phi$ is affected by the gauge change only at high orders in τ, leaving the leading terms unperturbed.

In the next section we will numerically solve the linear cosmological perturbation equations with the initial conditions sets developed in Secs. IV and V.

VII. NUMERICAL INTEGRATIONS AND DISCUSSION

We performed the numerical integration applying our considerations to a scalar field model based on ultralight pseudo Nambu-Goldstone bosons; the potential associated with this field has the form [16]

$$V(\phi) = M^4[\cos(\phi/f)+1]. \tag{111}$$

Our working point corresponds to the parameter choice $f = 1.885\times10^{18}$ GeV and $M = 10^{-3}$ eV; assuming an initial kinetic energy equal to the potential one, the starting values of the scalar field and its initial time derivative are obtained by requiring that the present contribution be $\Omega_\phi = 0.6$, fixing $H_0 = 70$ km/(sec Mpc) and $\Omega_b = 0.05$. Furthermore, we have taken the primordial power spectrum to be exactly scale invariant.

Even though the main cosmological consequences of this kind of scalar field have been analyzed by many authors (see [6,3]), here we use the formulas developed in the previous sections to accurately compare the pure adiabatic and pure isocurvature regimes. Also we give particular emphasis on the behavior of entropy and curvature perturbations, again comparing their evolution starting from isocurvature (from CDM) and adiabatic initial conditions; each case is compared with a pure CDM model with the same background parameters.

First, let us consider the power spectra of the microwave background anisotropies, both temperature and polarization. They are expressed by the expansion coefficients of the two-point correlation function into Legendre polynomials (see, e.g., [20]) and admit the following expression in terms of the quantities defined in the previous sections:

$$C_l^T = 4\pi\int\frac{dk}{k}|\Delta_{Tl}(k,\tau_0)|^2, \quad C_l^P = 4\pi\int\frac{dk}{k}|\Delta_{Pl}(k,\tau_0)|^2. \tag{112}$$

The adiabatic case is shown in Fig. 1. The presence of the scalar field (solid line) produces an increase of the power of the acoustic oscillations with respect to the CDM model (dashed line); this is due to the fact that the universe is not completely matter dominated at decoupling: thus at this time

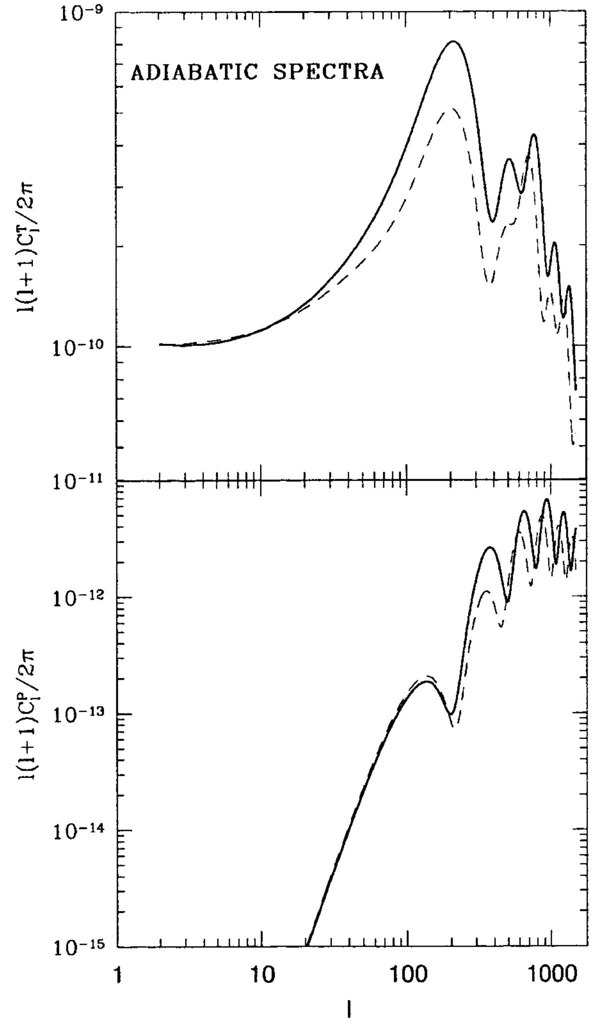

FIG. 1. Power spectra of the CMB anisotropies from adiabatic initial conditions. The background parameters are $\Omega_b=0.05$, $h=0.7$, three massless neutrino families, and $\Omega_\phi=0.6$, $\Omega_c=0.35$ (solid line), $\Omega_\phi=0$, $\Omega_c=0.95$ (dash-dotted line). Note the increase of the acoustic peaks power in the scalar field model.

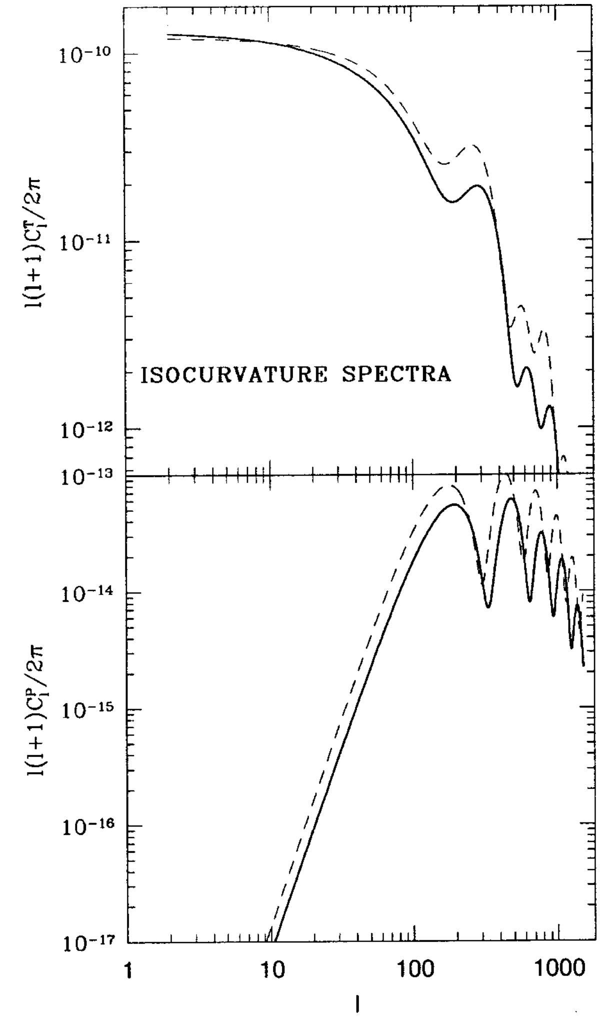

FIG. 2. Power spectra of the CMB anisotropies from isocurvature initial conditions. The background parameters are $\Omega_b=0.05$, $h=0.7$, three massless neutrino families, and $\Omega_\phi=0.6$, $\Omega_c=0.35$ (solid line), $\Omega_\phi=0$, $\Omega_c=0.95$ (dash-dotted line). Note the decrease of the acoustic peaks power in the scalar field model, an opposite behavior with respect to the adiabatic case.

the perturbations are growing faster than in the CDM models (we recall that density perturbations in adiabatic models grow as a^2 and a, respectively, in the radiation and matter eras) and this produces an early integrated Sachs-Wolfe (ISW) effect found first in [6]. Also, the position of the first peak is slightly shifted toward smaller angular scales due to the increase of the distance of the last scattering surface (projection effect). Note how these features regard both the polarization and temperature peaks. Finally, the temperature spectra show that the ISW effect is active on the smallest multipoles due to the dynamics of the scalar field in the present case; this is a distinctive feature with respect to the cosmological constant [6].

Figure 2 shows the spectrum from isocurvature perturbations. While the projection effect is the same as in the adiabatic case, now the situation regarding the amplitude of the acoustic oscillations is inverted: the peaks are lower than the ordinary models, both for polarization and temperature. This is simply due to the reduction of the matter/radiation ratio as we include ϕ by keeping $\Omega_{total}=1$; in fact, the scalar field has *no* intrinsic dynamical effect at last scattering since matter and radiation components were largely dominant: it is well known the opposite behavior of the anisotropies in adiabatic and isocurvature models as one varies $\Omega_m h^2$ (see, e.g., [20]). To better see this point, we plot in Figs. 3 and 4 the power spectra for models having fixed $\Omega_b h^2$ and $\Omega_c h^2$ but varying $\Omega_m=\Omega_b+\Omega_c$ and h by means of different Ω_ϕ. Thus we expect the same amount of perturbations in the CMB except for effects that are genuinely linked to the scalar field, such as the projection effect and the ISW effect on the smallest multipoles. This is precisely what happens for the spectra in Figs. 3 and 4. The dashed lines represent again the curves for $\Omega_\phi=0.6$ as in Figs. 1 and 2; the solid and thin lines represents $\Omega_\phi=0.5$ and 0.7, respectively. Again, the spectra show remarkably the same features for polarization and temperature, even if it should be noted how the former, arising from acoustic oscillations occurring just *at* decou-

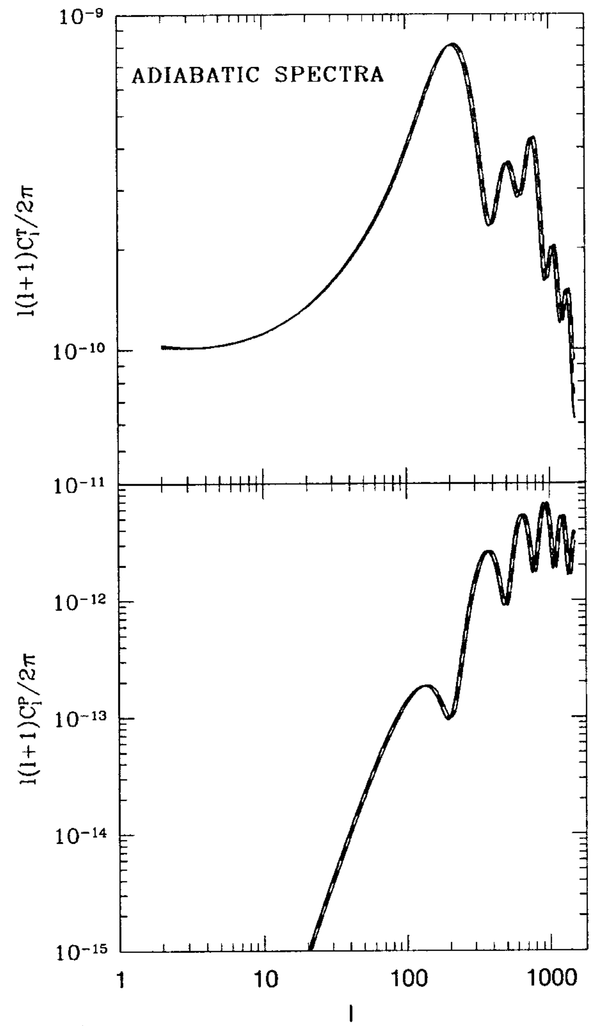

FIG. 3. Power spectra of the CMB anisotropies from adiabatic initial conditions with different Ω_ϕ and fixed $\Omega_b h^2=0.0245$ and $\Omega_c h^2=0.1715$ ($\Omega_m h^2=0.196$) as in Figs. 1 and 2. The background parameters are $\Omega_m=0.4$, $h=0.7$ (dashed line), $\Omega_m=0.3$, $h=0.81$ (thin line), and $\Omega_m=0.5$, $h=0.63$ (thick line). The amplitude of the peaks is the same, while they are slightly shifted because of the projection effect.

pling, is not influenced by the ISW effect, since it comes from the line-of-sight integration. All this shows how the perturbation power at decoupling is not touched by the subdominant scalar field; the opposite behavior in the adiabatic and isocurvature cases is explained by the decrease of matter in favor of scalar field.

More insight into the perturbation behavior may be obtained by looking at the time evolution of some significant quantities; we look at one single scale, or wave number, roughly entering in the horizon between matter and radiation equality and decoupling:

$$k=8\times10^{-2}\ \mathrm{Mpc}^{-1}. \tag{113}$$

Let us begin with the gravitational potential Ψ, in Fig. 5; the oscillatory dynamics is associated, both in the isocurvature and adiabatic cases, to the horizon crossing of the scale ex-

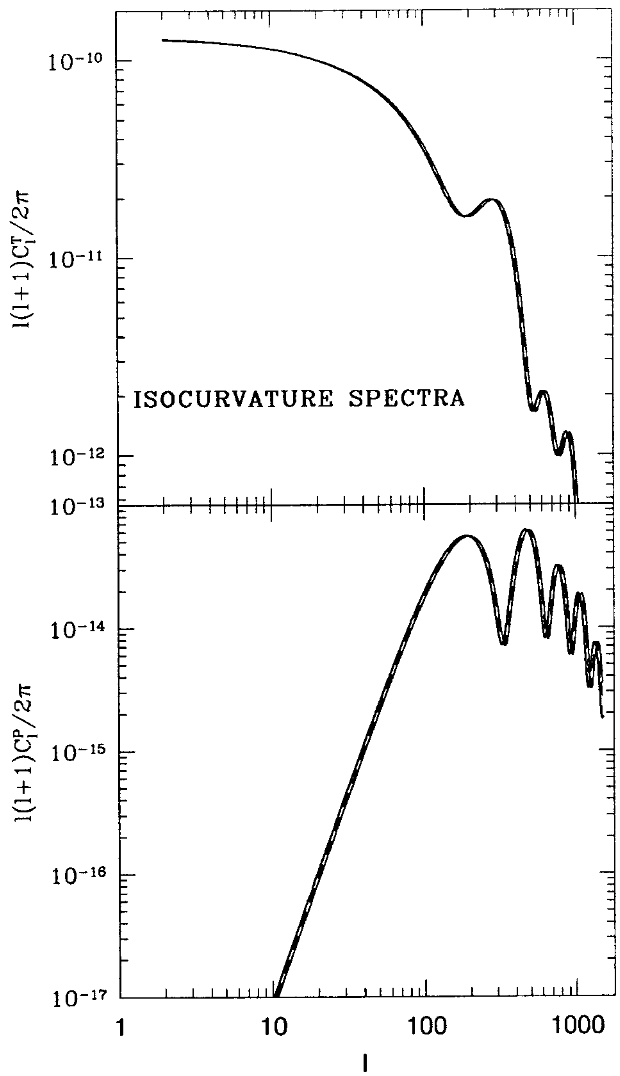

FIG. 4. Power spectra of the CMB anisotropies from isocurvature initial conditions with different Ω_ϕ and fixed $\Omega_m h^2=0.196$. The spectra show the same behavior for varying Ω_m as in Fig. 3.

amined. The amplitude of the oscillations in the scalar field models is higher for the adiabatic case and lower for the isocurvature one when compared with the corresponding CDM models. These oscillations are the source of the CMB anisotropies on subhorizon angular scales ($l\geqslant200$) through the acoustic driving effect and the early ISW effect [22], and therefore follow different behavior in the two cases.

We concentrate now on two particularly significant quantities regarding both adiabatic and isocurvature regimes, the total entropy perturbation, defined below, and the curvature ζ defined in Eq. (65); we recall that these quantities are gauge invariant. The amplitude of the total entropy perturbation is given by

$$\begin{aligned} p\Gamma &= p\Gamma_{int}+p\Gamma_{rel} \\ &= \sum_a\ (\delta p_a-c_a^2\delta\rho_a)+\sum_a\ (c_a^2-c_s^2)\delta\rho_a \\ &= \sum_a\ (\delta p_a-c_s^2\delta\rho_a), \end{aligned} \tag{114}$$

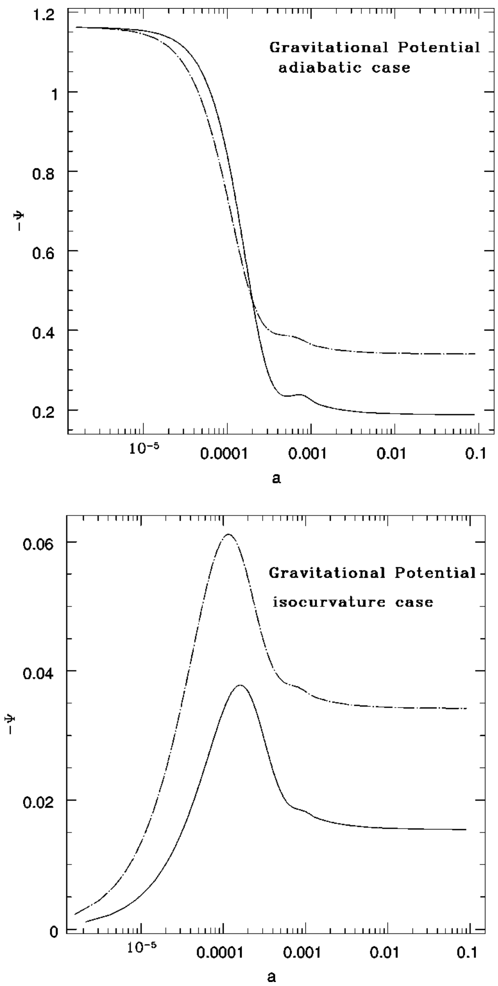

FIG. 5. Gravitational potential (in arbitrary units) at the comoving wave number $k=8\times10^{-2}$ Mpc^{-1} as a function of the time for adiabatic (top) and isocurvature (bottom) initial conditions. The background parameters are the same as in Figs. 1 and 2. The oscillatory dynamics is associated with the horizon crossing of the scale considered. In the adiabatic (isocurvature) scalar field models, the oscillation amplitudes are larger (lower) than in the corresponding CDM cases, according to the power spectra behaviors.

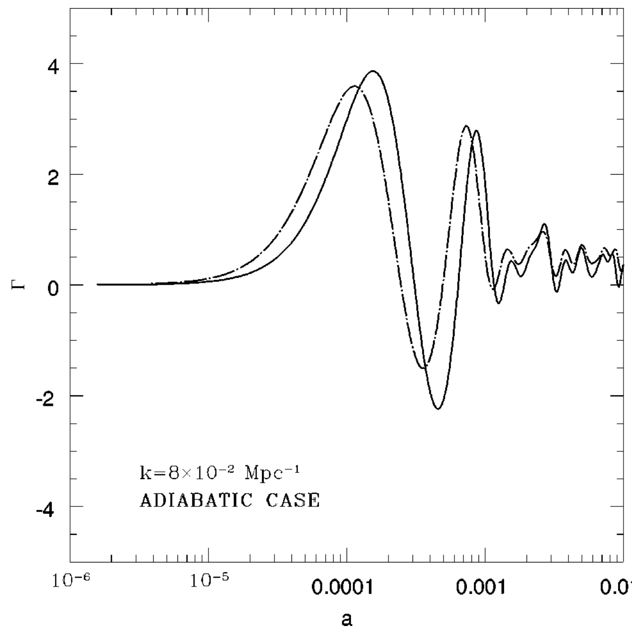

FIG. 6. Gauge-invariant entropy behavior (in arbitrary units) as a function of the time in adiabatic models for scalar field (solid line) and pure CDM (dashed line) models. Note the shift of the horizon crossing (corresponding to the oscillations) toward late times due to the effective cosmological constant.

where p is the total pressure and the sound speed must take into account the scalar field contribution $c_\phi^2=1-2V'\dot{\phi}/\dot{\rho}_\phi$ in the summation,

$$c_s^2=\frac{\sum_a h_a c_a^2}{h}, \tag{115}$$

where

$$h_a=\rho_a+p_a, \quad h=\sum_a h_a. \tag{116}$$

While the Γ_{int} term comes from the intrinsic entropy perturbation of each component, and is ultimately sourced only by the scalar field component due to the spatial and temporal variations of the scalar field equation of state, the Γ_{rel} term arises from the different dynamical behavior of the components, and it is related to the S_{ab} quantities defined in Eq. (51) by the relation [18]

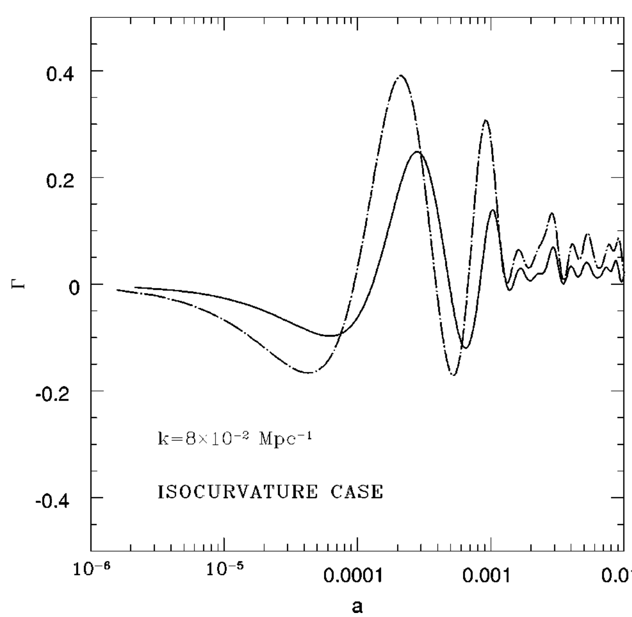

FIG. 7. Gauge-invariant entropy behavior (in arbitrary units) as a function of the time in isocurvature models for scalar field (solid line) and pure CDM (dashed line) models. Note the decrease of the oscillation amplitudes in scalar field models, due to the lack of matter with respect to the pure CDM case.

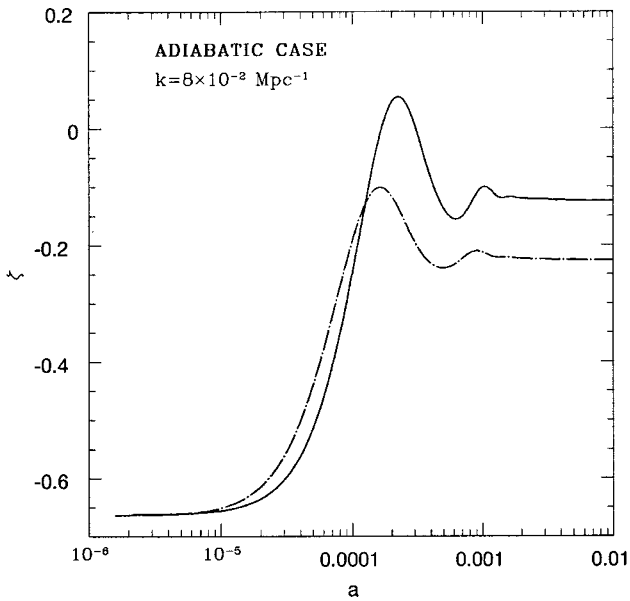

FIG. 8. Gauge-invariant curvature behavior (in arbitrary units) as a function of the time in adiabatic models for scalar field (solid line) and pure CDM (dashed line) models. Note that the curvature is nonvanishing as $a \to 0$.

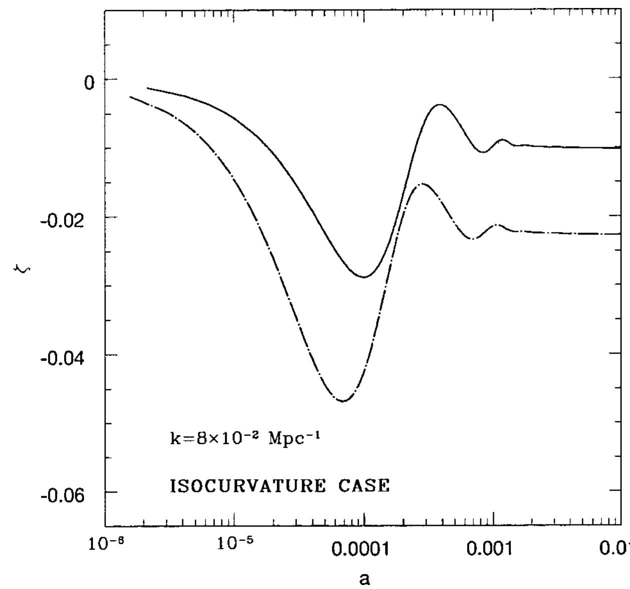

FIG. 9. Gauge-invariant curvature behavior (in arbitrary units) as a function of the time in isocurvature models for scalar field (solid line) and pure CDM (dashed line) models. Note that the curvature is vanishing as $a \to 0$.

$$p\Gamma_{rel} = \frac{1}{2} \sum_{a,b} \frac{h_a h_b}{h} (c_a^2 - c_b^2) S_{ab}. \tag{117}$$

In Figs. 6 and 7 we plot Γ vs a (solid line and arbitrary units) using isocurvature CDM and adiabatic initial conditions, respectively. We note that in both cases $\Gamma \to 0$ as $a \to 0$: in the last case, this is an obvious consequence of the definition of what adiabatic conditions are; on the other hand, taking isocurvature CDM initial conditions, we started with nonzero values of the S_{ab} relative to the CDM component and the other noncompensating components, but again $\Gamma \to 0$ as $a \to 0$. This is because at early times the cosmic fluid is radiation dominated, so that $p \propto 1/a^4$ in Eq. (114); this destroys Γ for $a \to 0$, since no radiation perturbations are present initially. Instead, the initial value of the first time derivative of Γ is different from zero only in selecting isocurvature initial conditions, since it takes contributions directly from the S_{ab} terms [18]. The behavior of entropy perturbations in both cases have been compared with those in the standard Einstein–de Sitter model $\Omega_m = 1$ (dotted line), with the same choice of Ω_b and H_0. The entropy perturbations remain nearly constant before horizon crossing; at this time the perturbation starts its oscillations that are damped in amplitude when the scale is well below the effective horizon. As an expected feature, note that in the scalar field model the peaks of the oscillations are shifted closer to the present when compared to the Einstein–de Sitter case, as the epoch of matter-radiation equality.

In Figs. 8 and 9 we plot the evolution of the gauge-invariant curvature perturbation ζ for isocurvature CDM and adiabatic initial conditions, respectively. At $a \ll 1$, this quantity is zero in the isocurvature case and nonzero in the adiabatic one, being an explicit indicator of the nature of the perturbations. Again the significant dynamics occurs in correspondence with the horizon crossing, and the latter occurs slightly later than in the CDM model due to the presence of the scalar field.

Finally, note how in all the isocurvature cases (Figs. 6 and 8) the amplitude of the oscillations is lower than in the corresponding CDM models; as we mentioned above, this is due to the reduction of the matter component in favor of the scalar field energy density. Most important, these graphs show that this is the only possible cause of this effect, since at the times of oscillations for the scale examined, roughly between equivalence and decoupling, the scalar field is very subdominant with respect to the other components.

The hypothesis of a cosmic vacuum energy stored in the potential of a scalar field enlarges naturally the possibility to gain insight into high energy physics from the traces left in the cosmic radiation and in the matter distribution. Because of the upcoming CMB experiments [23], it will be interesting to further study the cosmological imprints of these models, in the context of different theories attempting to describe the hidden sector of high energy physics.

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[1] R. Jimenez, astro-ph/9810311, 1998.
[2] G. Paturel, P. Lanoix, P. Teerikorpi, G. Theureau, L. Bottinelli, L. Gouguenheim, N. Renaud, and O. Witasse, Astron. Astrophys. (to be published), astro-ph/9810155.
[3] P.T.P. Viana and A.R. Liddle, Phys. Rev. D **57**, 674 (1998).
[4] P.G. Ferreira and M. Joyce, Phys. Rev. D **58**, 023503 (1998).
[5] E.J. Copeland, A.R. Liddle, and D. Wands, Phys. Rev. D **57**, 4686 (1998).
[6] K. Coble, S. Dodelson, and J. Friemann, Phys. Rev. D **55**, 1851 (1997).
[7] G.W. Anderson and S.M. Carroll, astro-ph/9711288, 1997.
[8] I. Zlatev, L. Wang, and P.J. Steinhardt, Phys. Rev. Lett. **82**, 896 (1999).
[9] G. Huey, L. Wang, R.R. Caldwell, and P.J. Steinhardt, Phys. Rev. D **59**, 063008 (1999).
[10] B. Ratra and A. Quillen, Mon. Not. R. Astron. Soc. **259**, 738 (1992).
[11] B. Ratra and P.J.E. Peebles, Phys. Rev. D **37**, 3406 (1988).
[12] W. Hu, Astrophys. J. (to be published), astro-ph/9801234.
[13] R.R. Caldwell, R. Dave, and P.J. Steinhardt, Phys. Rev. Lett. **80**, 1582 (1998).
[14] M. Turner and M. White, Phys. Rev. D **56**, 4439 (1997).
[15] C.P. Ma and E. Bertschinger, Astrophys. J. **455**, 7 (1995).
[16] C. Hill and G. Ross, Nucl. Phys. **B311**, 253 (1988); Phys. Lett. B **203**, 125 (1988); J. Friemann, C. T. Hill, A. Stebbins, and I. Waga, Phys. Rev. Lett. **75**, 2077 (1995); J. Estrada Vigil and L. Masperi, Mod. Phys. Lett. A **13**, 423 (1998).
[17] J.M. Bardeen, Phys. Rev. D **22**, 1882 (1980).
[18] I. Kodama and M. Sasaki, Prog. Theor. Phys. Suppl. **78**, 1 (1984).
[19] V.F. Mukhanov, H.A. Feldman, and R.H. Branderberger, Phys. Rep. **215**, 203 (1992).
[20] T. Padmanabahn, *Structure Formation in the Universe* (Cambridge University Press, Cambridge, England, 1993).
[21] S. Mollerach, Phys. Lett. B **242**, 158 (1990).
[22] W. Hu, N. Sugiyama, and J. Silk, Nature (London) **386**, 37 (1997); see also W. Hu, in *The Universe at High-Z, Large-Scale Structure and the Cosmic Microwave Background*, Proceedings of the Summer School, Laredo, Spain, 1995, edited by E. Martinez-Gonzalez and J. L. Sanz, Lecture Notes in Physics Vol. 470 (Sprnger, Berlin, 1996), p. 207, astro-ph/9511130.
[23] Microwave Anisotropy Probe home page: http://map.gsfc.nasa.gov; Planck Surveyor home page: http://astro.estec.esa.nl/SA-general/Projects/Planck