Cosmological tracking solutions

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A substantial fraction of the energy density of the universe may consist of quintessence in the form of a slowly rolling scalar field. Since the energy density of the scalar field generally decreases more slowly than the matter energy density, it appears that the ratio of the two densities must be set to a special, infinitesimal value in the early universe in order to have the two densities nearly coincide today. Recently, we introduced the notion of tracker fields to avoid this initial conditions problem. In the paper, we address the following questions: What is the general condition to have tracker fields? What is the relation between the matter energy density and the equation-of-state of the universe imposed by tracker solutions? And can tracker solutions help to explain why quintessence is becoming important today rather than during the early universe?
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I. INTRODUCTION

Quintessence [1] has been proposed as the missing energy component that must be added to the baryonic and matter density in order to reach the critical density [2,3]. Quintessence is a dynamical, slowly evolving, spatially inhomogeneous component with negative pressure. An example is a scalar field Q slowly rolling down its potential $V(Q)$ [1,4–11]. For a general scalar field, the pressure is $p_Q=\frac{1}{2}\dot{Q}^2-V$, and the energy density is $\rho_Q=\frac{1}{2}\dot{Q}^2+V$. For a slowly rolling scalar field, the pressure can be negative if the kinetic energy is less than the potential energy. For quintessence, the equation-of-state, defined as $w_Q=p_Q/\rho_Q$, lies between 0 and -1. Depending on $V(Q)$, w_Q may be constant, slowly varying, rapidly varying or oscillatory [1]. It is also possible for Q to smoothly transform from quintessence to a form of energy with positive pressure or vice versa. For example, $V(Q)$ may have the property that Q rolls quickly at first and, hence, has positive pressure, but then Q slows down so that the pressure becomes negative, satisfying the definition of quintessence.

A key problem with the quintessence proposal is explaining why ρ_Q and the matter energy density should be comparable today. There are two aspects to this problem. First of all, throughout the history of the universe, the two densities decrease at different rates; so it appears that the conditions in the early universe have to be set very carefully in order for the energy densities to be comparable today. We refer to this issue of initial conditions as the ''coincidence problem'' [12]. This is a generalization of the flatness problem described by Dicke and Peebles in 1979 [13]; the very same issue arises with a cosmological constant as well. A second aspect, which we call the ''fine-tuning problem,'' is that the value of the quintessence energy density (or vacuum energy or curvature) is very tiny compared to typical particle physics scales. The fine-tuning condition is forced by direct measurements; however, the initial conditions or coincidence problem depends on the theoretical candidate for the missing energy.

Recently, we introduced a form of quintessence called ''tracker fields'' which avoids the coincidence problem [14]. Tracker fields have an equation-of-motion with attractor-like solutions in the sense that a very wide range of initial conditions rapidly converge to a common, cosmic evolutionary track of $\rho_Q(t)$ and $w_Q(t)$. Technically, the tracker solution differs from a classical dynamics attractor solution because neither Ω_Q nor any other parameters are fixed in time.

The initial value of ρ_Q can vary by nearly 100 orders of magnitude without altering the cosmic history. The acceptable initial conditions include the natural possibility of equipartition after inflation—nearly equal energy density in Q as in the other 100–1000 degrees of freedom (e.g., $\Omega_{Qi}\approx 10^{-3}$). Furthermore, the resulting cosmology has desirable properties. The equation-of-state w_Q varies according to the background equation-of-state w_B. When the universe is radiation-dominated ($w_B=1/3$), then w_Q is less than or equal to 1/3 and ρ_Q decreases less rapidly than the radiation density. When the universe is matter-dominated ($w_B=0$), then w_Q is less than zero and ρ_Q decreases less rapidly than the matter density. Eventually, ρ_Q surpasses the matter density and becomes the dominant component. At this point, Q slows to a crawl and $w_Q\to -1$ as $\Omega_Q\to 1$ and the universe is driven into an accelerating phase. These properties seem to match current observations well [15].

There has been considerable work on exponential potentials, a ''borderline'' case of tracking in which w_Q is very nearly equal to w_B during the radiation- and matter-dominated epochs [10,16]. However, our focus is on cases where w_Q is significantly less than w_B. This situation is more desirable, because it enables the Q-energy to eventually overtake the background density and induce a period of accelerated expansion, which produces a cosmology more consistent with measurements of the matter density, large scale structure, and supernova observations [15,18]. Hence, our use of the term ''tracker'' is meant to refer to solutions joining a common evolutionary track, as opposed to following closely the background energy density and equation-of-state.

It is also an interesting point that some tracker solutions do not require small mass parameters to obtain a small energy density today [14,17]; whether this is a satisfactory solution to the fine-tuning problem is debatable, though. We do not explore this issue in this paper.

An important consequence of the tracker solutions is the prediction of a relation between w_Q and Ω_Q today [14]. Because tracker solutions are insensitive to initial conditions, both w_Q and Ω_Q only depend on $V(Q)$. Hence, for any given $V(Q)$, once Ω_Q is measured, w_Q is determined. In general, the closer Ω_Q is to unity, the closer w_Q is to -1. However, since $\Omega_m \geqslant 0.2$ today, there is a sufficient gap between Ω_Q and unity that w_Q cannot be so close to -1. We find that $w_Q \gtrsim -0.8$ for practical models. This w_Q-Ω_Q relation, which makes the tracker field proposal distinguishable from the cosmological constant, will be explored further in this paper.

The purpose of the present paper is to expand on our introductory article on tracker fields and the coincidence problem. In particular, we want to go beyond the specific examples of tracker potentials studied before and address in a general way the following questions (the relevant section is shown in parentheses):

What is a tracker solution (Sec. II A)?

Which potentials $V(Q)$ have tracker solutions (Secs. III A, III B, III E) and which do not (Sec. III G)?

How does convergence of diverse initial conditions to a common track occur (Sec. III C)?

What additional conditions on tracking potentials are required to make them useful and practical (Sec. III H)?

How does tracking result in the prediction of an Ω_Q-w_Q relation (Sec. IV)?

What range of w_Q today is possible for tracking solutions according to the w_Q-Ω_Q relation (Sec. IV)?

Why is the Q-field first beginning to dominate at this late stage of the universe rather than at some early stage (Sec. V)?

II. DEFINITIONS AND BASIC EQUATIONS

We shall consider a scalar field with present equation-of-state $-1 < w_Q < 0$ in a flat cosmological background (consistent with inflation). The ratio of the energy density to the critical density today is Ω_Q for the Q-field and Ω_m for the baryonic and dark matter density where $\Omega_m + \Omega_Q = 1$. We use dimensionless units where the Planck mass is $M_p = 1$.

A. Basic terminology

For clarity, we define some basic terms:

Quintessence: a time-evolving, spatially inhomogeneous energy component with negative pressure $p_Q \neq -\rho_Q$ and equation-of-state $w_Q < 0$.

Q-field: a scalar field whose energy acts as quintessence today. (Note that the Q-field may go through earlier periods of rapid evolution such that its pressure exceeds zero; so, strictly speaking, the Q-field does not act as quintessence during that early period.)

w_Q, Ω_Q, *etc.*: parameters describing the Q-fields.

Tracker field: a field whose evolution according to its equation-of-motion converges to the same solution—the *tracker solution*—for a wide range of initial conditions for the field and its time derivative.

Converging: behavior in which solutions to the Q-field equation-of-motion are drawn towards a common solution, $\tilde{Q}(t)$, for a wide range of initial conditions. We make the (fine) distinction between ''tracking'' and ''converging'' because tracker solutions typically go through some periods when solutions are not approaching one another. In particular, during transitions in the background equation-of-state, such as the transition from radiation- to matter-domination, there are brief intervals during which w_Q changes rapidly and general solutions are not drawn towards the tracker solution; fortunately, these transitory periods are too short to spoil the advantages of tracking solutions.

Hybrid models: models in which the Q-field first passes through a regime of $V(Q)$ where solutions converge and then begins an extended regime where they do not. (This differs from the case just mentioned in that the non-converging behavior endures for a long period, not just during the transition in the background equation-of-state. For example, one can imagine a potential which has converging behavior as long as the tracker field is less than a certain value, but does not have converging behavior once the field rolls past that point.) These cases can be interesting if, by the time the second regime is reached, sufficient convergence of solutions has already occurred such that Q continues along a common track in the second regime.

Family of tracker solutions: For a potential $V(Q) = M^4 \tilde{v}(Q/M)$ (where $\tilde{v}$ is a dimensionless function of Q/M), there is a family of tracker solutions parametrized by M. The value of M is determined by the measured value of Ω_m today (assuming a flat universe).

B. Tracker equation

The equation-of-motion for the Q-field is

$$\ddot{Q} + 3H\dot{Q} + V' = 0 \tag{1}$$

where

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \kappa\left(\rho_m + \rho_r + \frac{1}{2}\dot{Q}^2 + V\right) \tag{2}$$

where a is the Robertson-Walker scale factor, ρ_m is the matter density, ρ_r is the radiation energy density, and $\kappa=8\pi/3$. The definition of the equation-of-state is

$$w_Q=\frac{p}{\rho}=\frac{\frac{1}{2}\dot{Q}^2-V}{\frac{1}{2}\dot{Q}^2+V}. \tag{3}$$

It is extremely useful to combine these relations into an unfamiliar form, which is the form we would like the reader to have in mind when we refer to the ''equation-of-motion'':

$$\pm\frac{V'}{V}=3\sqrt{\frac{\kappa}{\Omega_Q}}\sqrt{1+w_Q}\left[1+\frac{1}{6}\frac{d\ln x}{d\ln a}\right] \tag{4}$$

where $x=(1+w_Q)/(1-w_Q)=\frac{1}{2}\dot{Q}^2/V$ is the ratio of the kinetic to potential energy density for Q and a prime means a derivative with respect to Q. The $\pm$ sign depends on whether $V'>0$ or $V'<0$, respectively. The tracking solution (to which general solutions converge) has the property that w_Q is nearly constant and lies between w_B and -1. For $1+w_Q=\mathcal{O}(1)$, $\dot{Q}^2\approx\Omega_Q H^2$ and the equation-of-motion, Eq. (4), dictates that

$$\frac{V'}{V}\approx\frac{1}{\sqrt{\Omega_Q}}\approx\frac{H}{\dot{Q}} \tag{5}$$

for a tracking solution; we shall refer to this as the ''tracker condition.''

An important function is $\Gamma\equiv V''V/(V')^2$, whose properties determine whether tracking solutions exist. Taking the derivative of the equation-of-motion with respect to Q and combining with the equation-of-motion itself, we obtain the equation

$$\begin{aligned}\Gamma&\equiv\frac{V''V}{(V')^2}\\&=1+\frac{w_B-w_Q}{2(1+w_Q)}-\frac{1+w_B-2w_Q}{2(1+w_Q)}\frac{\dot{x}}{6+\dot{x}}\\&\quad-\frac{2}{(1+w_Q)}\frac{\ddot{x}}{(6+\dot{x})^2}\end{aligned} \tag{6}$$

where $\dot{x}\equiv d\ln x/d\ln a$ and $\ddot{x}\equiv d^2\ln x/d\ln a^2$. We will refer to this equation as the ''tracker equation.''

III. PROPERTIES OF TRACKING SOLUTIONS AND TRACKING POTENTIALS

In the following section, we prove the key properties of tracker solutions. Each subsection leads with a summary statement of the result for those who wish to skip the detailed mathematical arguments.

A. Which potentials have converging behavior and tracker solutions and which do not?

Our central theorem is the following:

(A) Tracking behavior with $w_Q<w_B$ occurs for any potential in which $\Gamma\equiv V''V/(V')^2>1$ and is nearly constant $[|d(\Gamma-1)/Hdt|\ll|\Gamma-1|]$ over the range of plausible initial Q.

(B) Tracking behavior with $w_B<w_Q<(1/2)(1+w_B)$ occurs provided $1-(1-w_B)/(6+2w_B)<\Gamma<1$ and nearly constant.

(C) Tracking does not occur for $\Gamma<1-(1-w_B)/(6+2w_B)$.

Rather than present the proof in a formal mathematical format, which holds no special interest, we present the steps of the argument over the course of the following subsections, Secs. III B–III G, explaining along the way their physical interpretation, which is important for understanding the application to cosmology. We note that each part of the theorem takes the form ''given a condition, tracking will (or will not) occur.'' That is, we establish conditions that are sufficient, but we do not show that they are necessary. We conjecture that they may be necessary based on the stability analysis in Sec. III F, which is the heart of the proof.

Case (A) is the one relevant to tracker models of quintessence since we want $w_Q<0$ today. For the most part, unless otherwise stated, this is the case assumed in our discussion. The range of initial conditions referred to in the theorem extends from $V(Q)$ equal to the initial background energy density ρ_B down to $V(Q)$ equal to the background density at matter-radiation equality, a span of over 100 orders of magnitude.

The importance of this theorem is that testing for the existence of tracking solutions reduces to a simple condition on $V(Q)$ without having to solve the equation-of-motion directly. In particular, the condition that $\Gamma\approx$const can be evaluated by testing

$$\left|\Gamma^{-1}\frac{d(\Gamma-1)}{Hdt}\right|\approx\left|\frac{\Gamma'}{\Gamma\,(V'/V)}\right|\ll 1 \tag{7}$$

over range of Q corresponding to the allowed initial conditions; the middle expression is easily computed knowing $V(Q)$ only without having to solve an equation-of-motion. (Here we have used the tracker condition $\dot{Q}/H\approx\sqrt{\Omega_Q}\approx V'/V$ which applies for the tracker solution.) An equivalent condition is that $\Delta\Gamma/\Gamma\ll 1$, where $\Delta\Gamma$ is the difference between the maximum and minimum values of Γ over the same range of Q. The condition $\Gamma>1$ is equivalent to the constraint that $|V'/V|$ be decreasing as V decreases. These conditions encompass an extremely broad range of potentials, including inverse power-law potentials $[V(Q)=M^{4+\alpha}/Q^{\alpha}$ for $\alpha>0]$ and combinations of inverse power-law terms [e.g., $V(Q)=M^4\exp(M/Q)$]. Some potentials of this form are suggested by particle physics models with dynamical symmetry breaking or nonperturbative effects [17,19–24].

If $w_Q<w_B$, converging behavior does not occur for potentials in which $|V'/V|$ strictly increases as V decreases $(\Gamma<1)$. This category includes quadratic potentials and most examples of quintessence models in the literature. Instead of converging behavior, these models require specially tuned initial conditions to obtain an acceptable value of Ω_Q today, as discussed in Sec. III G.

For $(1/2)(1+w_B)>w_Q>w_B$, converging behavior does occur if V'/V is strictly increasing as V decreases ($\Gamma<1$); however, as we explain in Sec. III E, potentials with $w_Q>w_B$ do not produce viable cosmological models of quintessence: if $w_Q>w_B$ and $\Omega_Q\geqslant 1/2$ today (as suggested by current observation), then Ω_Q must exceed Ω_m in the past and there is no period of matter-domination or structure formation.

Hybrid potentials are possible in which converging behavior only occurs for a finite period during the early universe, provided that the time is sufficiently long to bring together solutions whose initial conditions span the range of practical interest. For example, one can construct potentials in which $|V'/V|$ decreases at first as Q rolls downhill, and then $|V'/V|$ begins to increase. Another possibility is that $|V'/V|$ increases at first and $w_Q>w_B$, as discussed in the previous paragraph. We pointed out that this condition allows convergence, but cannot be maintained up to the present for practical reasons of structure formation. However, the condition could be maintained for a long, finite period in the early universe as long as it is terminated before matter-domination.

B. Γ and the tracker solution

The central theorem is expressed in terms of conditions on the parameter, Γ. As a useful guide to the proof, we summarize qualitatively how the conditions on Γ relate to the existence and nature of the tracker solution.

The theorem states a condition on the value of Γ and its time variation. The condition on its value is important because, if the inequality is violated, the tracking condition, $V'/V\sim 1/\sqrt{\Omega_Q}$, cannot be maintained. The condition ensures that, as Q rolls downhill, both sides of the relation are decreasing. If Γ lies outside the stated bounds, one side of the tracker condition is increasing and the other decreasing as Q rolls downhill; hence, the tracking condition cannot be maintained.

The condition that Γ be constant is important for having converging behavior. If Γ is nearly constant, then the tracker equation, Eq. (6), implies that there is a solution in which x and its time-derivatives are negligible. In this case, we have that the equation-of-state for the Q-field is nearly constant:

$$w_Q\approx\frac{w_B-2(\Gamma-1)}{1+2(\Gamma-1)}. \tag{8}$$

If $\Gamma>1$, the value of w_Q must be less than w_B to satisfy the tracker equation, which means that the Q-energy redshifts more slowly than the background energy. For inverse power-law potentials, $V(Q)=M^{4+\alpha}/Q^{\alpha}$, $\Gamma=1+\alpha^{-1}$, and the relation we have derived matches the relation in our first paper [14]. For $V(Q)=M^4\exp(1/Q)$, $\Gamma=1+2Q$, which satisfies our condition for $Q\ll 1$. For $\Gamma<1$, the same relation says that the value of w_Q must be greater than w_B, but we also require that $\Gamma>1-(1-w_B)/(6+2w_B)$ in order to satisfy the criterion, $w_Q<(1/2)(1+w_B)$, necessary to have converging solutions (see Sec. III E).

If Γ varies significantly with Q, then one can find a wide and continuous distribution of Q, Ω_Q, and w_Q which satisfies the tracker equation. Hence, there is no single solution to which solutions converge.

In sum, we have succeeded in expressing our conditions for tracking solutions in terms of Γ, which depends entirely on the functional form of V. A simple computation of $V''V/(V')^2$ determines if V admits tracking solutions or not. The case of interest for quintessence is $w_Q<w_B$, in which case the condition for a tracker solution is $\Gamma>1$ and nearly constant.

C. How is the tracker solution approached beginning from different initial conditions?

We shall call the tracker solution $\tilde{Q}(t)$ and the energy density of the tracker solution as a function of time $\rho_{\tilde{Q}}(t)$. This subsection explains in rough detail how solutions converge to the tracker solution for any initial ρ_Q that lies between the initial background energy density, ρ_{Bi}, and the current critical density, ρ_{c0}. This range stretches nearly 100 orders of magnitude. In particular, we explain how the convergence to the tracker solution is different if initially $\rho_B>\rho_Q>\rho_{\tilde{Q}}$ versus $\rho_{c0}<\rho_Q<\rho_{\tilde{Q}}$. For simplicity, we confine ourselves to the case $w_Q<w_B$. We complete the proof of convergence in the next subsection.

The equation-of-motion, Eq. (4), can be rearranged as

$$\frac{1}{6}\frac{d\ln x}{d\ln a}=-\frac{1}{3\sqrt{\kappa(1+w_Q)}}\sqrt{\Omega_Q}\,\frac{V'}{V}-1\equiv\Delta(t)-1 \tag{9}$$

where we have restricted ourselves for simplicity to potentials with $V'<0$. Then, since $V'<0$, the right-hand side (RHS) of this equation is a balance between a positive semidefinite term (Δ) and a negative term. The tracker condition corresponds to the two conditions: $1-w_Q^2$ significantly different from zero and w_Q nearly constant. The latter requires near balance on the RHS above ($\Delta\approx 1$) so that $d\ln x/d\ln a$ is nearly zero.

First, consider the ''overshoot'' case in which ρ_Q is initially much greater than the tracker value $\rho_{\tilde{Q}}$. For simplicity, let us assume that Q is released from rest. The evolution goes through four stages, illustrated in Figs. 1–4. The potential for this example is $V(Q)=M^4/Q^6$. (For this and all subsequent figures, the choice of $z=10^{12}$ has been chosen as the initial time for convenience of computation and illustration; a realistic figure would have initial z corresponding to the inflationary scale.)

(1) V'/V and Ω_Q are so big initially that $\Delta\gg 1$. So

$$\frac{1}{6}\frac{d\ln x}{d\ln a}=\frac{2}{1-w_Q^2}\,\frac{d\ln w_Q}{Hdt}\gg 1 \tag{10}$$

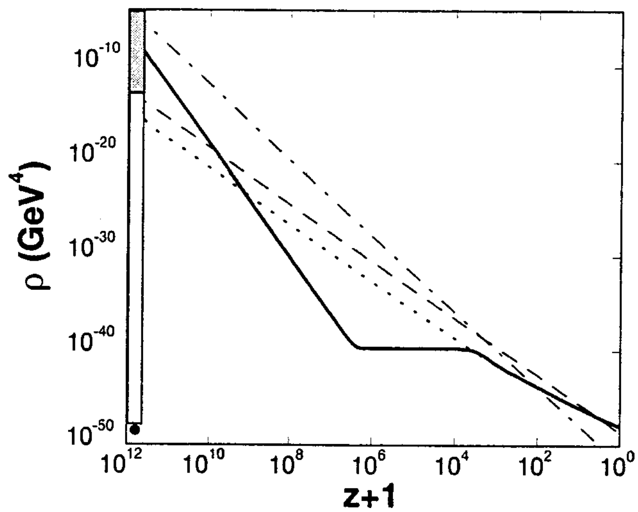

FIG. 1. Energy density versus redshift for the evolution of a tracker field. For computational convenience, $z=10^{12}$ (rather than inflation) has been arbitrarily chosen as the initial time. The white bar on left represents the range of initial ρ_Q which leads to an undershoot and the grey bar represents an overshoot, combining for a span of more than 100 orders of magnitude if we extrapolate back to inflation. The solid black circle represents the unique initial condition required if the missing energy is the vacuum energy density. The solid thick curve represents an ''overshoot'' in which ρ_Q begins from a value greater than the tracker solution value, decreases rapidly and freezes, and eventually joins the tracker solution.

and w_Q is driven towards its maximal value, $w_Q \rightarrow +1$. This means that $\dot{Q}$ becomes large and V decreases very rapidly as Q runs downhill.

(2) As Q runs downhill, V'/V and Ω_Q are decreasing. Consequently, Δ begins to decrease and ultimately reaches a value of order unity, one of the requirements for a tracker solution. However, w_Q has been driven towards $+1$; up to this point, the RHS of Eq. (9) has been positive, and so there has been no opportunity for w_Q to decrease. As a result, $1-w_Q^2$ is too small, or, more specifically, the kinetic energy is too large for Q to join the tracker solution. Hence, Q rolls farther down the potential, overshooting the tracker solution.

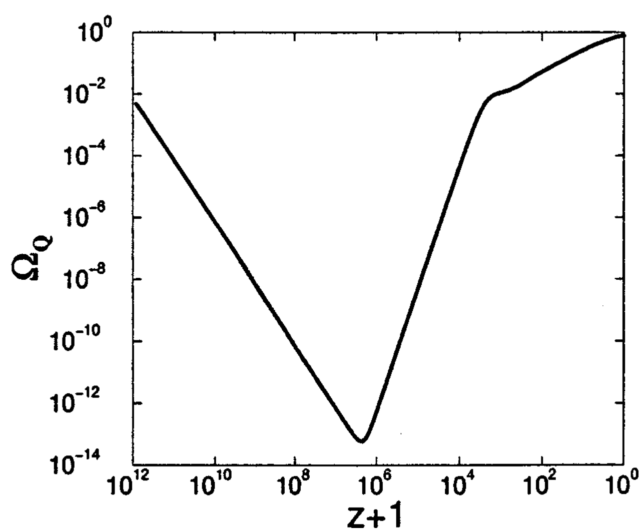

FIG. 2. A plot of Ω_Q vs redshift for the overshoot solution in Fig. 1.

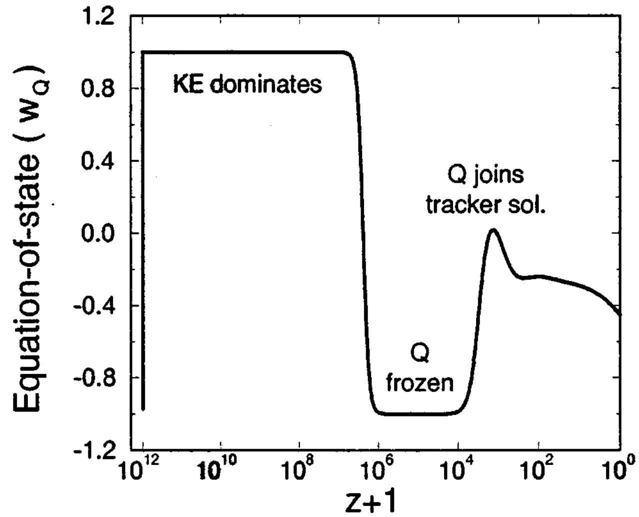

FIG. 3. A plot of w_Q vs redshift for the overshoot solution shown in Fig. 1. w_Q rushes immediately towards $+1$ and Q becomes kinetic energy dominated. The field freezes and w_Q rushes towards -1. Finally, when Q rejoins the tracker solution, w_Q increases, briefly oscillates and settles into the tracker value.

(3) Once the tracker solution is overshot, Δ becomes less than unity and the RHS of Eq. (9) changes sign. w_Q now decreases from $+1$ towards -1. One might wonder what happens when w_Q crosses through the tracker value; why does Q not track at this point? The answer is that there is now the problem that Δ is too small. So the RHS of Eq. (9) remains too negative and w_Q continues to decreases and heads towards -1.

(4) Once w_Q reaches close to -1, Q is essentially frozen

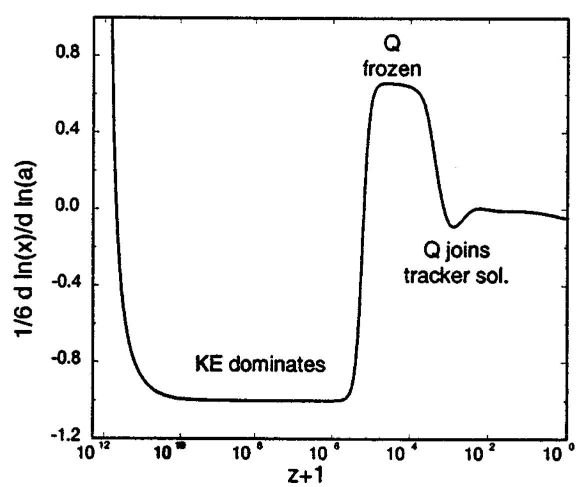

FIG. 4. A plot of $\dot{x}/6=(1/6)\, d\ln x/d\ln a$ for the overshoot solution shown in Figs. 1 and 3. At late times when Q settles into the tracker solution, $\dot{x}$ is small and w_Q is nearly constant. During the overshoot phase, $\dot{x}$ undergoes large positive and negative changes, as described in the text.

at some value Q_f and, consequently, V and V'/V are frozen. However, Δ is now increasing since Ω_Q is increasing—even though ρ_Q is nearly constant, ρ_B is decreasing. As Δ increases to order unity, the sign of Eq. (9) changes once again, w_Q increases from -1, the field runs downhill, and the sign of Eq. (9) changes yet again. After a few oscillations, the terms in Eq. (9) settle into near balance and Q is on track.

Next, consider the ''undershoot'' case in which ρ_Q is initially much less than the tracker value $\rho_{\tilde{Q}}$ and Q is released from rest. This corresponds to $Q \gg \tilde{Q}$ initially. By assumption, V and $|V'/V|$ are much smaller than the tracker value. Consequently, $1/\sqrt{\Omega_Q}$ is larger than the tracker value. The only way to satisfy the equation-of-motion, Eq. (4), is for w_Q to approach -1 so that the coefficient of $1/\sqrt{\Omega_Q}$ is nearly zero. This condition corresponds to a very small kinetic energy density or Q nearly constant. Hence, the field remains nearly ''frozen,'' and V and V'/V are nearly constant as the universe evolves. The situation is identical to beginning with step (4) above, and the scenario proceeds just as described there.

In sum, the field either drops precipitously past the tracker value and is frozen (overshoot) or it begins with a value less than the tracker solution (undershoot) and is frozen. In either case, it proceeds from the frozen state to joining the tracker solution. In the case of undershoot, the frozen value Q_f is simply the initial value of Q. For the overshoot case, Q begins by going through a kinetic energy dominated period in which $\dot{Q} \propto a^{-3}$. If the initial $\rho_{Qi} < \rho_{Bi}$, where ρ_B is the background radiation density, then $a \propto t^{1/2}$ and the frozen value of Q is

$$Q_f \approx Q_i + \sqrt{\frac{3}{4\pi}\Omega_{Q_i}} \tag{11}$$

where the subscript refers to the initial values of Q and Ω_Q. (If initially $\rho_{Qi} > \rho_{Bi}$, then $a \propto t^{1/3}$, and

$$Q_f \approx Q_i + \sqrt{\frac{3}{4\pi}}\left(1 + \frac{1}{2}\ln\frac{\rho_{Qi}}{\rho_B}\right); \tag{12}$$

in this case, Q_f is so large that Q remains frozen up to the present time.) For the overshoot case, Q_i is typically very small compared to unity and $\Omega_{Qi} = \mathcal{O}(1)$. Consequently, the frozen value Q_f depends on Ω_{Qi} only.

Initial conditions in which $\dot{Q}_i$ is non-zero do not change the scenario significantly. If $\dot{Q}_i$ is very large, then the initial behavior is kinetic energy dominated, and the evolution proceeds similar to the overshoot case. Initial fluctuations in Q also do not change the discussed behavior since they are exponentially suppressed once the potential becomes non-negligible and the field is driven towards the tracker solution (see the Appendix).

The possibility of overshoot and undershoot allows a new possibility for the case of exponential potentials recently discussed by Ferreira and Joyce [10]. The exponential potential is a special example of a tracker solution in which Ω_Q is constant during the matter dominated epoch. The practical problem with this model, as noted by Ferreira and Joyce, is Ω_Q is constrained to be small ($\Omega_Q \leq 0.15$). At the beginning of matter domination, Ω_Q must be small in order that large-scale structure be formed, but then it cannot change thereafter. Hence, it remains a small, subdominant component. This argument presumes, however, that Q is already on track at the beginning of matter domination. It is possible to tune initial conditions so that Q overshoots the tracker solution initially and does not join the tracker solution until just very recently (redshift $z=1$). Then, the constraint on Ω_Q is lifted.

D. What are the constraints on the initial value of Q and ρ_Q?

Suppose that the tracker solution corresponds to $Q=Q_0$ today and the Q has converged to a tracker solution. Then, whether ρ_Q is initially smaller than the tracker value and frozen at some $Q=Q_f$ equal to its initial value or ρ_Q is initially larger than the tracker value and falls to $Q_f \approx \sqrt{3\Omega_{Qi}/4\pi}$, it is necessary that Q_f be less than Q_0 in order that the field be tracking today. This is not a very strong constraint. Since $Q_0 = \mathcal{O}(1)$ for most tracking potentials, this only requires that $\rho_{c0} < \rho_{Qi} < \rho_{Bi}$ initially. Hence, initial conditions in which Q dominates the radiation and matter density are disallowed because Q falls so fast and drops to such a low point on the potential that it has not yet begun tracking today. However, initial conditions in which there is rough equipartition between ρ_Q and the background energy density are allowed, as well as initial values of ρ_Q ranging as low as 100 orders of magnitude smaller, comparable to the current matter density. The allowed range is impressive and spans the most physically likely possibilities.

E. Is the tracker solution stable?

What has been shown so far is that, whether the initial conditions correspond to undershoot or overshoot, Q soon reaches some frozen value Q_f which depends on the initial Q_i. Then, after some evolution, Ω_Q increases to the point where $|V'/V| \sim 1/\sqrt{\Omega_Q}$ and, according to the equation-of-motion, w_Q moves away from -1 and the field begins to roll. What remains to be shown is that solutions with w_Q not equal to the tracker solution value converge to the tracker solution. Or, equivalently, we need to show that the tracker solution is stable.

Now, consider a solution in which w_Q differs from the tracker solution value w_0 by an amount δ. Then, the master equation can be expanded to lowest order in δ and its derivatives to obtain, after some algebra,

$$\ddot{\delta} + 3\left[\frac{1}{2}(w_B+1) - w_0\right]\dot{\delta} + \frac{9}{2}(1+w_B)(1-w_0)\delta = 0 \tag{13}$$

where the overdot means $d/d\ln a$ as in the tracker equation. The solution of this equation is

$$\delta \propto a^{\gamma} \tag{14}$$

where

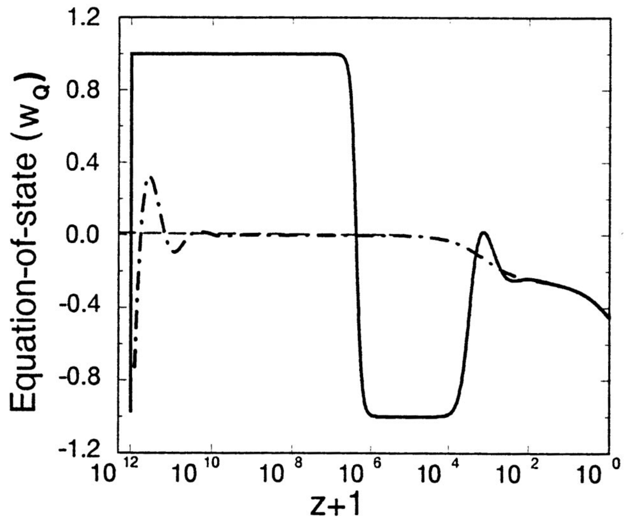

FIG. 5. The convergence of different initial conditions to the tracker solution. As derived in the text, w_Q decays exponentially fast to the tracker solution combined with small oscillations. All the curves are for $V(Q)=M^4/Q^6$. The solid curve is the overshoot case from Fig. 1. The thin dashed curve with $w\approx 0$ is the tracker solution which is overlaid for most z by the dash-dotted curve, which represents a slightly undershooting solution.

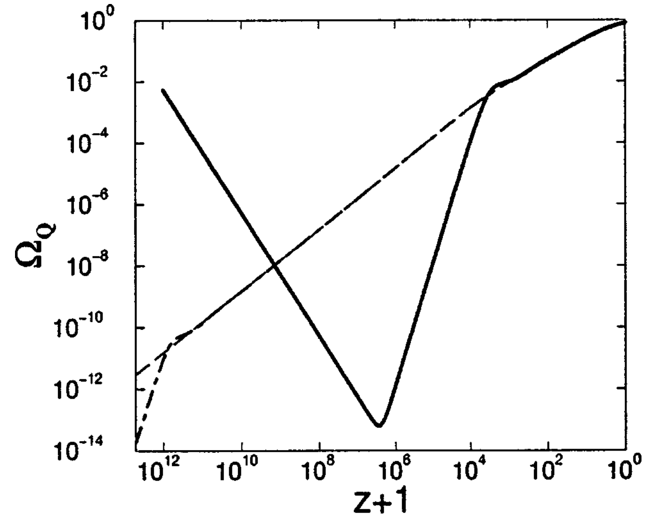

FIG. 6. A plot of Ω_Q vs redshift for the models shown in Fig. 5.

$$\gamma = -\frac{3}{2}\left[\frac{1}{2}(w_B+1)-w_0\right] \pm \frac{i}{2}\sqrt{18(1+w_B)(1-w_0)-9\left[\frac{1}{2}(w_B+1)-w_0\right]^2}. \tag{15}$$

The real part of the exponent γ is negative for w_0 between -1 and $w_0=\frac{1}{2}(1+w_B)$, which includes our entire range of interest. So without imposing any further conditions, this means that δ decays exponentially and the solution approaches the tracker solution. As δ decays, it also oscillates with a frequency described by the second term. See Figs. 5 and 6.

In deriving Eq. (15), we have assumed that Γ is strictly constant, independent of Q, which is exactly true for pure inverse power-law $[V\sim 1/Q^\alpha]$ or exponential $(V\sim\exp(\beta Q))$ potentials. The same result holds if $|d(\Gamma-1)/Hdt|/|\Gamma-1|\ll 1$ (i.e., $\Gamma-1$ varies with Q but only by a modest amount) over the plausible range of initial conditions ranging from $V(Q)\approx\rho_{Bi}$ to $V(Q)\approx\rho_{eq}$ (where ρ_{Bi} is the initial background energy density after inflation, say, and ρ_{eq} is the energy density at matter-radiation equality). The condition is equivalent to $|\Gamma'/[\Gamma\,(V'/V)]|\ll 1$. In this limit, Γ and the tracker value of w_Q change adiabatically as Q rolls downhill, satisfying the tracker equation with $\dot{x}$ being negligibly small, as discussed in the Appendix. The constraints on w_Q are the same as above. However, some important differences from the constant Γ case are pointed out in the last section.

Throughout most of our discussion in this paper, we have considered the case $w_Q<w_B$. However, our convergence criterion, $w_0<\frac{1}{2}(1+w_B)$, includes $w_Q>w_B$ or, equivalently, $1-(1-w_B)/(6+2w_B)<\Gamma<1$, as also found by Liddle and Scherrer [16]. An example is $V\sim Q^\alpha$ with $\alpha\geqslant(6+2w_B)/(1-w_B)$, $\alpha\geqslant 6$ for $w_B=0$ and $\alpha\geqslant 10$ for $w_B=1/3$. Let us suppose we reached the present Ω_Q after tracking down this potential. Because $w_Q>w_B$ in these potentials, it must be that Ω_Q exceeds Ω_B extrapolating backwards in time. Consequently, there is no period of matter-domination or structure formation, and these models have no practical interest. However, see the discussion of hybrid models below for a variation on these models that may be viable.

F. Borderline models and hybrid models

For completeness, we consider two special classes of potentials, borderline trackers in which $\Gamma=1$ and hybrid models in which $\Gamma>1$ at first and then $\Gamma<1$.

The borderline case corresponds to $V\propto\exp(\beta Q)$, which has been studied by several authors [6,7,10,11]. For this case, the tracker equation for $\dot{x}=\ddot{x}=0$ demands that $w_Q=w_B$ and, therefore, Ω_Q is constant. Hence, for this case, the tracker solution corresponds to maintaining a constant ratio of quintessence to background energy density. The only deviation occurs during the transition from radiation- to matter-domination when $\dot{x}$ becomes non-negligible, but this is a small effect.

Because Ω_Q is constant throughout the matter-dominated epoch, these models have limited practical utility. Ω_Q must be small ($\leqslant 15\%$) at the onset of matter-domination in order not to disrupt structure formation. (Quintessence suppresses the growth rate.) But then, since Ω_Q is constant, Ω_Q remains small forever. Consequently, the models require $\Omega_m>85\%$, inconsistent with a number of determinations of mass [15], and the universe never enters a period of accelerated expansion, inconsistent with recent measurements of the luminosity-redshift relation for type IA supernovas [18]. (The overshoot scenario may lift the $\Omega_m>85\%$ constraint but, as discussed in Sec. III D, introduces fine-tuning which defeats the whole purpose of the scenario.)

Hybrid models have the property that solutions converge to a tracker solution at the early phase of evolution but cease to converge after a certain point due to a change in the shape of the potential as Q rolls downhill. One can imagine a sufficiently long convergence regime that all or most plausible initial conditions have collapsed to a common tracker solution before the second regime begins. Effectively, this has the desired feature that a wide range of initial conditions lead to the same final condition. The models may be somewhat artificial in that the current cosmology is very sensitive to where the transition occurs. For example, consider the case where $w_Q < w_B$ but Γ undergoes a transition from $\Gamma > 1$ (converging) to $\Gamma < 1$. Recall that $\Gamma < 1$ corresponds to $|V'/V|$ increasing as V decreases. We have shown that, extrapolating backwards only a small interval in time, the field Q must have been frozen at a value not so different from the current value (assuming the field is rolling today and $w_Q < 0$). In this kind of hybrid model, the transition to $\Gamma < 1$ must be set so that the transition occurs so that Q is near the frozen value, which requires delicate tuning of parameters in the potential.

A different example is where $1-(1-w_B)/(6+2w_B) < \Gamma < 1$ and $w_Q > w_B$ during the early stages of the universe. We have argued that these conditions produce converging behavior but, if the conditions continue to the present, there is no period of matter-domination or structure formation (see Sec. III E). However, one can imagine hybrid models in which these conditions $[1-(1-w_B)/(6+2w_B)] < \Gamma < 1$ and $w_Q > w_B$ are satisfied for some period early in the history of the universe, providing a finite period of converging behavior. Later, as Q moves down the potential, V changes form so that $w_Q < w_B$. Viable cosmological models of this type can be constructed in which Ω_Q does not dominate the universe during the matter-dominated epoch until near the present time.

G. Why do models with increasing $|V'/V|$ and $w_Q < w_B$ fail to solve the coincidence problem?

For potentials in which $|V'/V|$ increases as V decreases ($\Gamma < 1$), the LHS of the the equation-of-motion, Eq. (4), is increasing. If $w_Q < w_B$, $1/\sqrt{\Omega_Q}$ on the RHS is decreasing as Q rolls downhill. Hence, the tracker condition, $|V'/V| \sim 1/\sqrt{\Omega_Q}$, cannot be maintained and w_Q cannot be maintained at a nearly constant value (different from -1) for any extended period.

In particular, extrapolating backwards in time, $|V'/V|$ is strictly decreasing and $1/\sqrt{\Omega_Q}$ is strictly increasing. The only way to satisfy the equation-of-motion, Eq. (4), is to have w_Q approach -1. But this corresponds to the Q-field freezing at some value $Q = Q_i$ after only a few Hubble times. The energy density of the frozen field is only slightly higher than the current energy density. Consequently, to obtain this cosmological solution, one has to have special initial conditions in the early universe (after inflation, say) that set Q precisely to Q_i and the Q-energy density to ρ_{Q_i}, a value nearly 100 orders of magnitude smaller than the background energy density. It is precisely this tuning of initial conditions (the coincidence problem) which we seek to avoid.

We note that many quintessence models ($w_Q < 0$) discussed in the literature fall into this non-tracking class and require extraordinary tuning of initial conditions. Simple examples include the harmonic potential, $V(Q) = M^2 Q^2$, and the sinusoidal potential, $V(Q) = M^4[\cos(Q/f)+1]$. In these models, the initial value of Q must be set to a specific value initially in order to obtain the measured value of Ω_Q today.

H. Some additional practical considerations

We have discovered a wide class of potentials that exhibit tracking behavior. This guarantees that a wide range of initial conditions converge to a common tracking solution, but the convergence may take longer than the age of the universe in some cases. In particular, if one assume equipartition after inflation, say, and the initial V is too far above the tracker solution, then Q falls precipitously, overshoots the tracker solution, and freezes at some $Q = Q_f$. For some potentials satisfying the condition $\Gamma \approx \text{const} > 1$, Q_f may be so large that the field does not begin to roll and track by the present epoch. If Q just started to roll by the present epoch, then it would behave exactly as a cosmological constant until now, and so the model is trivially equivalent to a Λ model. As a practical consideration, we demand that a field starting from equipartition initial conditions should start rolling by matter-domination, say, so that the model is non-trivial. This imposes a mild added constraint on potentials, $V(Q)$. For this purpose, rough estimates suffice.

Equipartition at the end of inflation, when there are hundreds or perhaps thousands of degrees of freedom in the cosmological fluid, means that the Q-field has $\Omega_i \approx 10^{-3}$. Beginning from equipartition, Q falls to some value Q_f where it freezes. According to the equation-of-motion, Q remains frozen until $\sqrt{\Omega_Q} \propto \sqrt{V}/H$ increases to where $V'/V \sim 1/\sqrt{\Omega_Q}$ or, equivalently,

$$\frac{V'^2}{V}(Q_f) > H^2(z_{eq}) = \frac{H_0^2}{a_{eq}^3} \sim \frac{1}{\Omega_0}\frac{V_0}{a_{eq}^3}. \tag{16}$$

For $V(Q) \propto 1/Q^\alpha$, this imposes the constraint

$$\frac{\alpha^2}{Q_f^{\alpha+2}} \geqslant \frac{1}{\Omega_0 Q_0^\alpha a_{eq}^3}, \tag{17}$$

where Q_0 is the present value of Q. Since today

$$\frac{V_0'^2}{V_0} \sim H_0^2 \sim \frac{8\pi}{3}\frac{V_0}{\Omega_0}, \tag{18}$$

we obtain $Q_0^2 \sim 3/8\pi\Omega_Q^0\alpha^2$. From Eq. (11), we also know that

$$\frac{4\pi}{3}Q_f^2 \sim \Omega_i \sim 10^{-3}. \tag{19}$$

Combining the above relations one gets the restriction on α

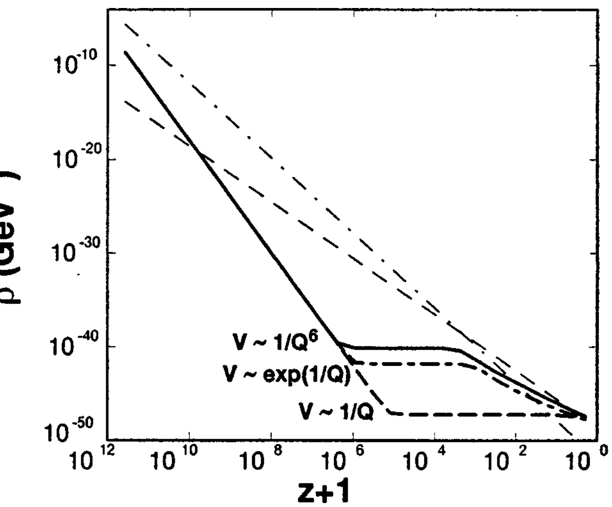

FIG. 7. A comparison of the overshoot for three different models beginning from $\Omega_i = 10^{-3}$ (equipartition). The thick solid line is for $V(Q) = M^4/Q^6$, the thick dash-dotted line is for $V = M^4[\exp(1/Q) - 1]$, and the thick long dashed line is for $V(Q) = M^4/Q$. In all three examples, Q falls rapidly downhill and freezes. In the first and second examples, Q begins to roll again and joins the tracker solution before matter-radiation equality; the third example, which violates the condition derived in the text, does not begin to roll again by the present epoch.

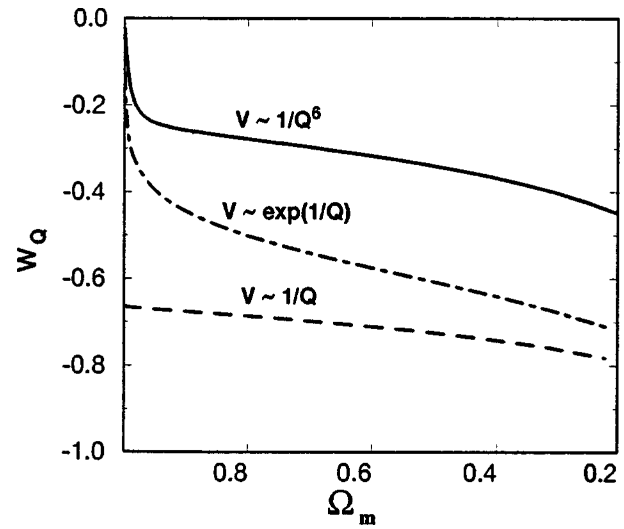

FIG. 8. The Ω_Q-w_Q relation for various potentials assuming a flat universe $\Omega_m = 1 - \Omega_Q$, where w_Q represents the present value of w_Q. The potentials and notation are the same as in Fig. 7.

$$\alpha^2 > \frac{10^{12}}{\Omega_0} \frac{\left(\dfrac{3}{4\pi}\Omega_i\right)^{(1+\alpha/2)}}{\left(\alpha^2 \Omega_0 \dfrac{3}{8\pi}\right)^{\alpha/2}} \tag{20}$$

where we have taken $a_{eq} \sim 10^{-4}$. This approximate relation leads to $\alpha \geqslant 5$. Figure 7 confirms this result showing that the $\alpha = 1$ model starts rolling much later than equality. So if one restricts to pure inverse power-law $(1/Q^\alpha)$ potentials, our constraint that Q begin from equipartition and roll before matter-radiation equality constrains us to $\alpha \geqslant 5$.

IV. Ω_Q-W_Q RELATION

An extremely important aspect of tracker solutions is the Ω_Q-w_Q relation or, equivalently, the Ω_m-w_Q relation which it forces. For any given $V = M^4 \tilde{v}(Q/M)$ (where $\tilde{v}$ is a dimensionless function of Q/M), Q and $\dot{Q}$ are totally determined independent of initial conditions by the tracker solution. The only degree of freedom is the M-parameter in the potential. M can be fixed by imposing the constraint that the universe is flat and $\Omega_m = 1 - \Omega_Q$ is determined by measurement. There is, then, no freedom left to independently vary w_Q. This is the explanation of the Ω_Q-w_Q relation, a new prediction that arises from tracker fields. The Ω_Q-w_Q relation is not unique because there remains the freedom to change the functional form of $V(Q)$. Even so, w_Q is sufficiently constrained as to be cosmologically interesting.

The general trend is that $w_Q \to -1$ as $\Omega_Q \to 1$. The fact that $\Omega_m \geqslant 0.2$ observationally means that $\Omega_Q \leqslant 0.8$ and w_Q cannot be very close to -1. How small w_Q can be is model-dependent. We are most interested in the smallest values of w_Q possible since the difference from -1 determines how difficult it is to distinguish the tracking field candidate for missing energy from cosmological constant.

In Fig. 8, we show the Ω_Q-w_Q relation for a series of pure, inverse power-law potentials, $V(Q) \propto 1/Q^\alpha$. The general trend is that w_Q increases as α decreases. The constraint given at the end of the previous section is that $\alpha \geqslant 5$ (in order that Q be rolling by matter-radiation equality beginning from equipartition initial conditions). For $\Omega_Q = 0.8$, the smallest value of w_Q is -0.52, which occurs for $\alpha = 5$. This is a large difference from -1 obtained for a cosmological constant.

However, a lower value of w_Q can be easily achieved for a more generic potential with a mixture of inverse power-laws, e.g., $V(Q) = M^4 \exp(1/Q)$. For these models, Q is small initially. If the potential is expanded in inverse powers of Q, $1/Q^\alpha$, then it is dominated in the early stages by the high-α terms. Hence, the effective value of α is much greater than 5 before matter-radiation equality, and we easily satisfy the constraint that the field be rolling before matter-radiation equality beginning from equipartition initial conditions. On the other hand, the value of Q at the present epoch is large, and the potential is dominated by the $\alpha \approx 1$ terms in its expansion. Consequently, w_Q can be even lower today than in the pure power-law case. For $\Omega_Q = 0.8$, we obtain $w_Q = -0.72$ for the exponential potential, which is in better accord with recent constraints on w_Q from supernovas [18].

It is difficult to go below this limit without artificially tuning potentials unless we relax our constraints. For example, consider the highly contrived potential $V(Q) = A/Q^{10^{-5}} + B/Q^{10}$, in which we have intentionally chosen exponents differing by six orders of magnitude in order to obtain a small w_Q today. The second term in the potential dominates before equality and ensures that the field is rolling by that point in time. The first term dominates at late times and makes the equation of state w_0 very low (because this term is very flat). For this example, we find $w_Q = -0.98$ for

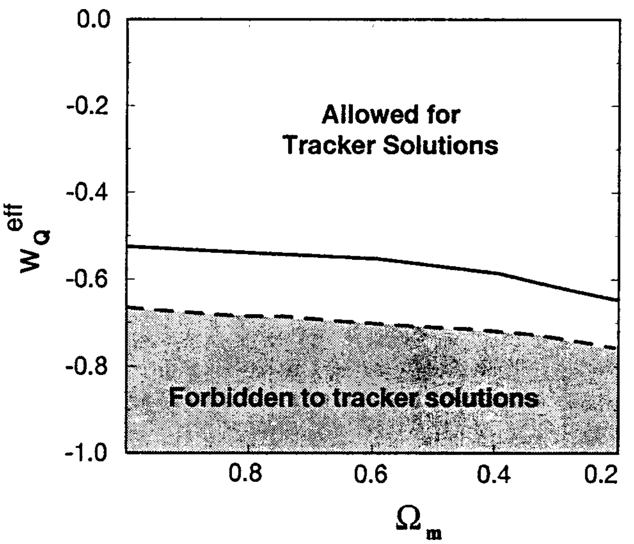

FIG. 9. A plot of w_Q^{eff} versus $\Omega_m = 1 - \Omega_Q$, showing the minimum w_Q^{eff} possible for tracker solutions. The solid line is the lower boundary assuming the constraint that the Q-field begins with equipartition initial conditions and begins rolling before matter-radiation equality. The dashed line is the lower boundary if this condition is relaxed to allow general $V \sim \Sigma c_k / Q^k$. As explained in the text, w_Q^{eff} is the value that would be measured in supernova and microwave background experiments which effectively integrate over a varying w_Q.

$\Omega_Q = 0.8$. However, we had to choose a pair of terms with exponents differing by six orders of magnitude. The lesson of this exercise (and related tests) is that the exponential potential is a reliable estimate for the minimal w_Q possible for generic, untuned potentials.

In Fig. 8, we illustrate the Ω_Q-w_Q relation for several potentials. Only the $V \sim 1/Q^6$ and $V \sim \exp(1/Q)$ potentials satisfy the condition that Q is rolling by matter-radiation equality beginning from equipartition initial conditions. In Fig. 9, we illustrate the effective w_Q^{eff} that would be measured using supernova or cosmic microwave background measurements, using the $V \sim \exp(1/Q)$ potential as defining the boundary of minimal values of w_Q possible for the tracker field case. This boundary assumes that we satisfy the strict condition that the field be rolling by matter-radiation equality beginning from equipartition initial conditions. If we relax this condition and allow a somewhat narrower range of initial conditions, then general potentials of the form $V \sim \Sigma c_k / Q^k$ (such as $V \sim 1/Q$) are allowed and w_Q can be somewhat smaller (see Fig. 8). Hence, in Fig. 9, the boundary can relax somewhat downward (dashed line) but it is difficult to obtain $w_Q < -0.8$ or $w_Q^{eff} < -0.75$. Because w_Q is evolving at recent times, the value obtained from measurements at moderate to deep redshift will differ from the current value shown in Fig. 8. For tracker potentials, the effect of integrating back in time over varying w_Q turns out to be well-mimicked by a model with constant $w_Q = w_Q^{eff}$ that has the same conformal distance to last scattering surface. For the case $\Omega_m = 0.2$, for example, Fig. 8 shows that $w_Q = -0.72$ today, but Fig. 9 shows that the measured w_Q would be $w_{eff} = -0.63$.

The result is exciting because the Ω_Q-w_Q relation and the constraint on $\Omega_m \geqslant 0.2$ today creates a sufficiently large gap between w_Q^{eff} and -1 that the tracker candidate for missing energy should be distinguishable from the cosmological constant in near-future cosmic microwave background and supernova measurements.

V. WHY IS THE UNIVERSE ACCELERATING TODAY?

We have proved in this paper that tracker potentials resolve the coincidence problem for quintessence. For a very wide range of initial conditions, cosmic evolution converges to a common track. The tracker models are similar to inflation in that they funnel a diverse range of initial conditions into a common final state. The models have only one important free parameter (M) which is fixed by the measured $\Omega_Q = 1 - \Omega_m$.

Some of the mathematical properties of tracking solutions have been noted before for $\exp(\beta Q)$, for exponential potentials with time-dependent β-coefficient and pure, inverse power-law potentials [6,7,10,11]. The present work is important because it shows that the properties are shared by a much wider class of more generic potentials. ‘‘Generic potentials’’ include, for example, all V’s which can be expanded as a finite or infinite sum of terms with inverse powers of Q, which is much more general than the special cases of a single inverse-power or a pure exponential. We use $V = M^4 \exp(1/Q)$ as an example of this more generic class, although our conclusions would remain the same for more general $V = \Sigma c_k / Q^k$.

Extending the tracker behavior to generic potentials may be important because, as we shall argue below, they have properties not shared by the special cases [$V \sim 1/Q^\alpha$ and $V \sim \exp(\beta Q)$] that relate to the puzzle of why Ω_Q only begins to dominate and initiate a period of accelerated expansion late in the history of the universe.

This is a subtle point which requires a change of approach to appreciate. Up to this point in the paper, we have imagined fixing M to guarantee that $\Omega_Q = 1 - \Omega_m$ has the measured value today. This amounts to considering one tracker solution for each $V(Q)$. Now we want to consider the entire family of tracker solutions for each given $V(Q)$ and consider whether Ω_Q is more likely to dominate late in the universe for that family of solutions or not. In particular, we want to show that, for generic $V(Q)$, the family of solutions has the property that the quintessence energy density decreases at nearly the same rate as the background radiation density in the early stages of the universe and only tends to catch up and overtake the background density late in the history of the universe when Q has rolled a considerable way down the potential. Precisely when quintessence overtakes depends on the precisely value of M, of course, but our point here is to focus on the trend of the entire family of solutions.

In general, Ω_Q is proportional to $a^{3(w_B - w_Q)} \propto t^{2(w_B - w_Q)/(1+w_B)}$, where we have shown in Eq. (8) that

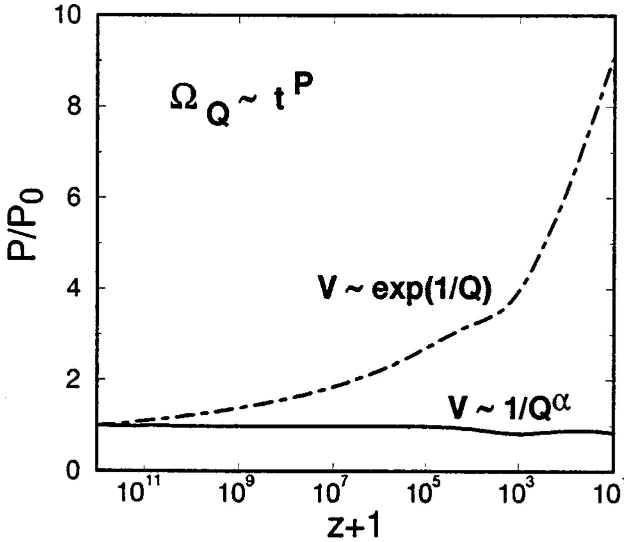

FIG. 10. A plot of P/P_0 versus t, where $\Omega_Q \propto t^P$ and P_0 is the initial value of P. The plot compares pure inverse power-law ($V \sim 1/Q^\alpha$) potentials for which P is constant with a generic potential [e.g., $V \sim \exp(1/Q)$] for which P increases with time.

$$w_B - w_Q = \frac{2(\Gamma-1)(w_B+1)}{1+2(\Gamma-1)}. \tag{21}$$

Hence, we find $\Omega_Q \propto t^P$ where

$$P = \frac{4(\Gamma-1)}{1+2(\Gamma-1)}. \tag{22}$$

For the two special cases [$V \sim 1/Q^\alpha$ and $V \sim \exp(\beta Q)$], $\Gamma - 1$ and, hence, P are constant. Consequently, Ω_Q grows as the same function of time throughout the radiation- and matter-dominated epochs for their respective family of tracker solutions. So there is no tendency for Ω_Q to grow slowly at first and then speed up later, as illustrated in Fig. 10. However, these potentials are the exception, rather than the rule.

For more general potentials, P increases as the universe ages. Consider first a potential which is the sum of two inverse power-law terms with exponents $\alpha_1 < \alpha_2$. The term with the larger power is dominant at early times when Q is small, but the term with the smaller power dominates at late times as Q rolls downhill and obtains a larger value. Hence, the effective value of α decreases and $\Gamma - 1 \propto 1/\alpha$ increases; the result is that P increases at late times. For more general potentials, such as $V \sim \exp(1/Q)$, the effective value of α decreases continuously and P increases with time. Figure 10 illustrates the comparison in the growth of P.

How does this relate to why Ω_Q dominates late in the universe? Because an increasing P means that Ω_Q grows more rapidly as the universe ages. Figure 11 compares a tracker solution for a pure inverse power-law potential ($V \sim 1/Q^6$) model with a tracker solution for $V \sim \exp(1/Q)$, where the two solutions have been chosen to begin at the same value of Ω_Q. (The start time has been chosen arbitrarily at $z = 10^{17}$ for the purposes of this illustration.) Following each curve to the right, there is a dramatic (10 orders of magnitude) difference between the time when the first solution (solid line) meets the background density versus the second solution (dot-dashed line). That is, beginning from the same Ω_Q, the first tracker solution dominates well before matter-radiation equality and the second (generic) example dominates well after matter-domination. The difference is less dramatic as α increases for the pure inverse power-law model and becomes negligible for $\alpha > 15$. Of course, the model appears more contrived. But more importantly, as α increases, the value of w_Q today (given $\Omega_m \geqslant 0.2$) approaches zero and the universe does not enter a period of acceleration by the present epoch. Hence, a significant conclusion is that the pure exponential and inverse power-law models are atypical; the generic potential has properties that illustrated in Fig. 11 that Ω_Q dominates late in the history of the universe and induces a recent period of accelerated expansion.

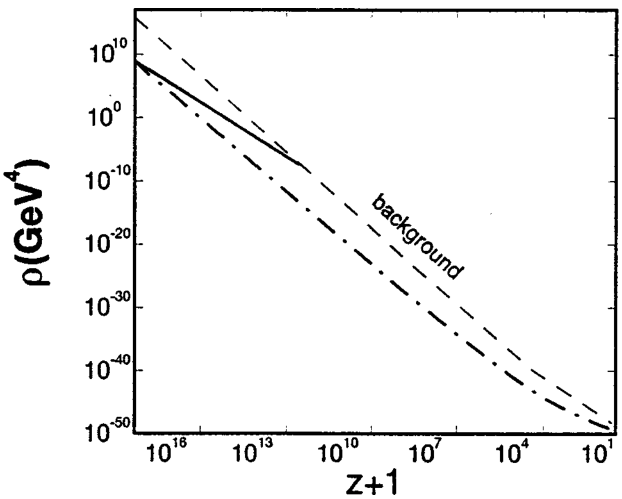

FIG. 11. A plot comparing two tracker solutions for the case of a $V \sim 1/Q^6$ potential (solid line) and a $V \sim \exp(1/Q)$ potential (dot-dashed line). The dashed line is the background density. The two tracker solutions were chosen to have the same energy density initially. The tracker solution for the generic example [$V \sim \exp(1/Q)$] reaches the background density much later than for the pure inverse-power law potential. Hence, Ω_Q is more likely to dominate late in the history of the universe in the generic case.

In sum, the general tracker behavior shown in this paper goes a long way towards resolving two key issues: the coincidence (or initial conditions) problem and why ρ_Q is dominating today rather than at some early epoch. And it leads to a new prediction—a relation between $\Omega_m = 1 - \Omega_Q$ and w_Q today that makes tracker fields distinguishable from a cosmological constant.

ACKNOWLEDGMENTS

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APPENDIX

In this appendix, we discuss the convergence to the tracker solution when $\Gamma \equiv V''V/(V')^2$ varies with Q. We wish to show that convergence occurs if the variation of Γ over the plausible range of initial conditions (varying of 100 orders of magnitude in energy density) is nearly constant. An example is $V \sim \exp(1/Q)$ for which $\Gamma = 1 + 2Q$ and $Q \ll 1$ for the plausible range of initial conditions.

The condition that Γ be nearly constant means precisely that $d(\Gamma-1)/Hdt \ll (\Gamma-1)$ or, equivalently, $|\Gamma'/[\Gamma\,(V'/V)]| \ll 1$. In this case, Γ is nearly constant over a Hubble time. Hence, we can consider an adiabatic approximation for the tracker solution in which Q_0 and w_0 satisfy the tracker equation (6) with $\dot{x}$ and $\ddot{x}$ negligibly small. Suppose w_Q and Q are both perturbed from this tracker solution by amounts δw and $\delta Q(t)$. From the definition of w_Q, we have that

$$\delta w = (1-w_0)\frac{\dot{Q}_0}{\rho_0}\,\delta\dot{Q} - (1+w_0)\frac{V_0'}{\rho_0}\,\delta Q. \qquad \text{(A1)}$$

From the equation-of-motion, Eq. (4), we know that

$$\frac{V_0'}{\rho_0} \sim \frac{V_0'}{V_0} \propto \frac{1}{\sqrt{\Omega_Q}} \propto a^{-3(w_Q-w_B)}. \qquad \text{(A2)}$$

Consequenty, Eq. (A1) implies

$$\delta w \sim \exp[-3/2(w_B-w_0)\ln a]\,\delta Q. \qquad \text{(A3)}$$

In particular, this equation shows that δw decays if δQ decays.

To show the δQ decays, we start from the more standard form of the equation-of-motion:

$$\ddot{Q} + 3H\dot{Q} + V' = 0, \qquad \text{(A4)}$$

and obtain the perturbed equation

$$\delta\ddot{Q} + 3H\,\delta\dot{Q} + V_0''\,\delta Q = 0. \qquad \text{(A5)}$$

Changing the variable to $d\tau = Hdt$, we obtain

$$\frac{d^2\delta Q}{d\tau^2} = \frac{d}{Hdt}\left(\frac{d\,\delta Q}{Hdt}\right) = \frac{1}{H^2}\ddot{Q} + m\,\frac{d\,\delta Q}{d\tau} \qquad \text{(A6)}$$

where $m \equiv (Ht)^{-1} = 3(w_B+1)/2$. Equation (A5) then becomes

$$\frac{d^2\delta Q}{d\tau^2} + (3-m)\,\frac{d\,\delta Q}{d\tau} + \frac{V_0''}{H^2}\,\delta Q = 0 \qquad \text{(A7)}$$

where

$$C \equiv \frac{V_0''}{H^2} \approx \frac{9}{4}(1-w_Q)(2+w_B+w_Q) \qquad \text{(A8)}$$

in the adiabatic approximation. The solution to Eq. (A7) is $\delta Q = A\,\exp(\beta\tau)$, where

$$\beta^2 + (3-m)\beta + C = 0 \qquad \text{(A9)}$$

with solutions

$$\beta_\pm = -\frac{B}{2}\left[1 \pm \sqrt{1 - \frac{4C}{B^2}}\right] \qquad \text{(A10)}$$

where

$$B = 3 - m = \frac{3}{2}(1-w_B). \qquad \text{(A11)}$$

Note that $B>0$ and $C>0$. Hence, we have that δQ exponentially decays, and, by the argument that preceded, $\delta w \propto \delta Q$ also decays exponentially. Combining our relations for δQ and δw, we can reproduce the result in Eq. (15) obtained for the constant Γ case.

If Q has spatial fluctuations, Eq. (A7) must be modified by a positive term proportional to $k^2\delta Q$. The effect is to increase C and modify the oscillation frequency. However, the exponential suppression of the fluctuations is retained once the field starts approaching the attractor solution. Hence, even if the initial conditions result in significant fluctuations after the field is frozen (in either the undershoot or overshoot case), the initial fluctuations are erased as Q converges to the tracker solution.

[1] R.R. Caldwell, R. Dave, and P.J. Steinhardt, Phys. Rev. Lett. **80**, 1582 (1998).

[2] See, for example, J.P. Ostriker and P.J. Steinhardt, Nature (London) **377**, 600 (1995), and references therein.

[3] M.S. Turner, G. Steigman, and L. Krauss, Phys. Rev. Lett. **52**, 2090 (1984).

[4] M.S. Turner and M. White, Phys. Rev. D **56**, R4439 (1997).

[5] N. Weiss, Phys. Lett. B **197**, 42 (1987).

[6] B. Ratra and P.J.E. Peebles, Phys. Rev. D **37**, 3406 (1988); P.J.E. Peebles and B. Ratra, Astrophys. J., Lett. Ed. **325**, L17 (1988).

[7] C. Wetterich, Astron. Astrophys. **301**, 32 (1995).

[8] J.A. Frieman *et al.*, Phys. Rev. Lett. **75**, 2077 (1995).

[9] K. Coble, S. Dodelson, and J. Frieman, Phys. Rev. D **55**, 1851 (1995).

[10] P.G. Ferreira and M. Joyce, Phys. Rev. Lett. **79**, 4740 (1997); Phys. Rev. D **58**, 023503 (1998).

[11] E.J. Copeland, A.R. Liddle, and D. Wands, Phys. Rev. D **57**, 4686 (1998).

[12] P. Steinhardt, in *Critical Problems in Physics*, edited by V.L. Fitch and D.R. Marlow (Princeton University Press, Princeton, NJ, 1997).

[13] R.H. Dicke and P.J.E. Peebles, in *General Relativity: An Einstein Centenary Survey*, edited by S.W. Hawking and W. Israel (Cambridge University Press, Cambridge, England, 1979).

[14] I. Zlatev, L. Wang, and P.J. Steinhardt, Phys. Rev. Lett. **82**, 896 (1999).

[15] L. Wang, R.R. Caldwell, J.P. Ostriker, and P.J. Steinhardt, astro-ph/9901388.

[16] A. Liddle and R. Scherrer, Phys. Rev. D **59**, 023509 (1999).

[17] P. Binetruy, Phys. Rev. D (to be published), hep-ph/9810553.

[18] S. Perlmutter *et al.*, astro-ph/9812133; A.G. Riess *et al.*, Astron. J. **116**, 1009 (1998); P.M. Garnavich *et al.*, Astrophys. J. **509**, 74 (1998).

[19] G.W. Anderson and S.M. Carroll, astro-ph/9711288.

[20] T. Barreiro, B. de Carlos, and E.J. Copeland, Phys. Rev. D **57**, 7354 (1998).

[21] P. Binetruy, M.K. Gaillard, and Y.-Y. Wu, Phys. Lett. B **412**, 288 (1997); Nucl. Phys. **B493**, 27 (1997).

[22] J.D. Barrow, Phys. Lett. B **235**, 40 (1990).

[23] C. Hill and G.G. Ross, Nucl. Phys. **B311**, 253 (1988); Phys. Lett. B **203**, 125 (1988).

[24] I. Affleck, M. Dine, and N. Seiberg, Nucl. Phys. **B256**, 557 (1985).