Dynamics of assisted inflation

Karim A. Malik and David Wands

School of Computer Science and Mathematics, University of Portsmouth, Portsmouth PO1 2EG, United Kingdom
(Received 11 December 1998; published 28 April 1999)

We investigate the dynamics of the recently proposed model of assisted inflation. In this model an arbitrary number of scalar fields with exponential potentials evolve towards an inflationary scaling solution, even if each of the individual potentials is too steep to support inflation on its own. By choosing an appropriate rotation in field space we can write down explicitly the potential for the weighted mean field along the scaling solution and for fields orthogonal to it. This demonstrates that the potential has a global minimum along the scaling solution. We show that the potential close to this attractor in the rotated field space is analogous to a hybrid inflation model, but with the vacuum energy having an exponential dependence upon a dilaton field. We present analytic solutions describing homogeneous and inhomogeneous perturbations about the attractor solution without resorting to slow-roll approximations. We discuss the curvature and isocurvature perturbation spectra produced from vacuum fluctuations during assisted inflation. [S0556-2821(99)05710-0]

PACS number(s): 98.80.Cq

I. INTRODUCTION

A single scalar field with an exponential potential is known to drive power-law inflation, where the cosmological scale factor grows as $a \propto t^p$ with $p > 1$, for sufficiently flat potentials [1–4]. Liddle, Mazumdar and Schunck [5] recently proposed a novel model of inflation driven by several scalar fields with exponential potentials. Although each separate potential,

$$V_i = V_0 \exp\left(- \sqrt{\frac{16\pi}{p_i}} \frac{\phi_i}{m_{\mathrm{Pl}}} \right), \qquad (1.1)$$

may be too steep to drive inflation by itself ($p_i < 1$), the combined effect of several such fields, with total potential energy

$$V = \sum_{i=1}^{n} V_i, \qquad (1.2)$$

leads to a power-law expansion $a \propto t^{\bar{p}}$ with [5]

$$\bar{p} = \sum_{i=1}^{n} p_i, \qquad (1.3)$$

provided $\bar{p} > 1/3$. Supergravity theories typically predict steep exponential potentials, but if many fields can cooperate to drive inflation, this may open up the possibility of obtaining inflationary solutions in such models.

Scalar fields with exponential potentials are known to possess self-similar solutions in Friedmann-Robertson-Walker models either in vacuum [2,3] or in the presence of a barotropic fluid [4,6–8]. In the presence of other matter, the scalar field is subject to additional friction, due to the larger expansion rate relative to the vacuum case. This means that a scalar field, even if it has a steep (non-inflationary) potential may still have an observable dynamical effect in a radiation or matter dominated era [9–11].

The recent paper of Liddle, Mazumdar and Schunck [5] was the first to consider the effect of additional scalar fields with independent exponential potentials. They considered n scalar fields in a spatially flat Friedmann-Robertson-Walker universe with scale factor $a(t)$. The Lagrange density for the fields is

$$\mathcal{L} = \sum_{i=1}^{n} -\frac{1}{2} (\nabla \phi_i)^2 - V_i, \qquad (1.4)$$

with each exponential potential V_i of the form given in Eq. (1.1). The cosmological expansion rate is then given by

$$H^2 = \frac{8\pi}{3 m_{\mathrm{Pl}}^2} \sum_{i=1}^{n} \left(V_i + \frac{1}{2} \dot{\phi}_i^2 \right), \qquad (1.5)$$

and the individual fields obey the field equations

$$\ddot{\phi}_i + 3H\dot{\phi}_i = -\frac{dV_i}{d\phi_i}. \qquad (1.6)$$

One can then obtain a scaling solution of the form [5]

$$\frac{\dot{\phi}_i^2}{\dot{\phi}_j^2} = \frac{V_i}{V_j} = C_{ij}. \qquad (1.7)$$

Differentiating this expression with respect to time, and using the form of the potential given in Eq. (1.1) then implies that

$$\frac{1}{\sqrt{p_i}} \dot{\phi}_i - \frac{1}{\sqrt{p_j}} \dot{\phi}_j = 0, \qquad (1.8)$$

and hence

$$C_{ij} = \frac{p_i}{p_j}. \qquad (1.9)$$
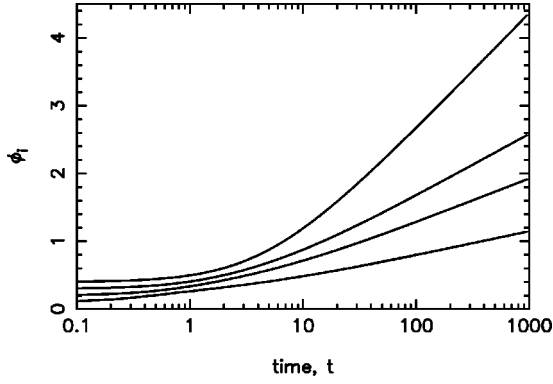
The scaling solution is thus given by [5]

    59 123501-1

FIG. 1. Evolution of four fields ϕ_1, ϕ_2, ϕ_3 and ϕ_4 (from bottom to top) during assisted inflation with $p_1=0.3$, $p_2=1$, $p_3=2$ and $p_4=7$.

$$\frac{1}{\sqrt{p_i}}\phi_i - \frac{1}{\sqrt{p_j}}\phi_j = \frac{m_{Pl}}{\sqrt{16\pi}}\ln\frac{p_j}{p_i}. \tag{1.10}$$

A numerical solution with four fields is shown in Fig. 1 as an example. In Ref. [5] the authors demonstrated the existence of a scaling solution for n scalar fields written in terms of a single re-scaled field $\tilde{\phi}=\sqrt{p/p_1}\phi_1$. The choice of ϕ_1 rather than any of the other fields is arbitrary as along the scaling solution all the ϕ_i fields are proportional to one another.

In this paper we will prove that this scaling solution is the late-time attractor by choosing a redefinition of fields (a rotation in field space) which allows us to write down the effective potential for field variations orthogonal to the scaling solution and show that this potential has a global minimum along the attractor solution. In general the full expression for an arbitrary number of fields is rather messy so we first give, in Sec. II, the simplest case where there are just two fields, and then extend this to n fields in Sec. III. The resulting inflationary potential is similar to that used in models of hybrid inflation and we show in Sec. IV that assisted inflation can be interpreted as a form of "hybrid power-law inflation." As in the case of power-law or hybrid inflation, one can obtain analytic expressions for inhomogeneous linear perturbations close to the attractor trajectory without resorting to slow-roll type approximations. Thus we are able to give exact results for the large-scale perturbation spectra due to vacuum fluctuations in the fields in Sec. V. We discuss our results in Sec. VI.

II. TWO FIELD MODEL

We will restrict our analysis initially to just two scalar fields, ϕ_1 and ϕ_2, with the Lagrange density

$$\mathcal{L}=-\frac{1}{2}(\nabla\phi_1^2)-\frac{1}{2}(\nabla\phi_2^2)-V_0\left[\exp\left(-\sqrt{\frac{16\pi}{p_1}}\frac{\phi_1}{m_{Pl}}\right)\right.$$
$$\left.+\exp\left(-\sqrt{\frac{16\pi}{p_2}}\frac{\phi_2}{m_{Pl}}\right)\right]. \tag{2.1}$$

We define the fields

$$\bar{\phi}_2=\frac{\sqrt{p_1}\phi_1+\sqrt{p_2}\phi_2}{\sqrt{p_1+p_2}}+\frac{m_{Pl}}{\sqrt{16\pi(p_1+p_2)}}$$
$$\times\left(p_1\ln\frac{p_1}{p_1+p_2}+p_2\ln\frac{p_2}{p_1+p_2}\right), \tag{2.2}$$

$$\bar{\sigma}_2=\frac{\sqrt{p_2}\phi_1-\sqrt{p_1}\phi_2}{\sqrt{p_1+p_2}}+\frac{m_{Pl}}{\sqrt{16\pi}}\sqrt{\frac{p_1p_2}{p_1+p_2}}\ln\frac{p_1}{p_2}, \tag{2.3}$$

to describe the evolution along and orthogonal to the scaling solution, respectively, by applying a Gram-Schmidt orthogonalization procedure.

The re-defined fields $\bar{\phi}_2$ and $\bar{\sigma}_2$ are orthonormal linear combinations of the original fields ϕ_1 and ϕ_2. They represent a rotation, and arbitrary shift of the origin, in field-space. Thus $\bar{\phi}_2$ and $\bar{\sigma}_2$ have canonical kinetic terms, and the Lagrange density given in Eq. (2.1) can be written as

$$\mathcal{L}=-\frac{1}{2}(\nabla\bar{\phi}_2^2)-\frac{1}{2}(\nabla\bar{\sigma}_2^2)-\bar{V}(\bar{\sigma}_2)\exp\left(-\sqrt{\frac{16\pi}{p_1+p_2}}\frac{\bar{\phi}_2}{m_{Pl}}\right), \tag{2.4}$$

where

$$\bar{V}(\bar{\sigma}_2)=V_0\left[\frac{p_1}{p_1+p_2}\exp\left(-\sqrt{\frac{16\pi}{p_1+p_2}}\sqrt{\frac{p_2}{p_1}}\frac{\bar{\sigma}_2}{m_{Pl}}\right)\right.$$
$$\left.+\frac{p_2}{p_1+p_2}\exp\left(\sqrt{\frac{16\pi}{p_1+p_2}}\sqrt{\frac{p_1}{p_2}}\frac{\bar{\sigma}_2}{m_{Pl}}\right)\right]. \tag{2.5}$$

It is easy to confirm that $\bar{V}(\bar{\sigma}_2)$ has a global minimum value V_0 at $\bar{\sigma}_2=0$, which implies that $\bar{\sigma}_2=0$ is the late time attractor, which coincides with the scaling solution given in Eq. (1.10) for two fields.

Close to the scaling solution we can expand about the minimum, to second-order in $\bar{\sigma}_2$, and we obtain

$$V(\bar{\phi}_2,\bar{\sigma}_2)\approx V_0\left[1+\frac{8\pi}{(p_1+p_2)}\frac{\bar{\sigma}_2^2}{m_{Pl}^2}\right]\exp\left(-\sqrt{\frac{16\pi}{p_1+p_2}}\frac{\bar{\phi}_2}{m_{Pl}}\right). \tag{2.6}$$

Note that the potential for the field $\bar{\sigma}_2$ has the same form as in models of hybrid inflation [12,13] where the inflaton field rolls towards the minimum of a potential with non-vanishing potential energy density V_0. Here there is in addition a "dilaton" field, $\bar{\phi}_2$, which leads to a time-dependent potential energy density as $\bar{\sigma}_2\rightarrow0$. Assisted inflation is related to hybrid inflation [12,13] in the same way that extended inflation [14] was related to Guth's old inflation model [15]. As in hybrid or extended inflation, we require a phase transition to bring inflation to an end. Otherwise the potential given by Eq. (2.6) leads to inflation into the indefinite future.

III. MANY FIELD MODEL

We will now prove that the attractor solution presented in Ref. [5] is the global attractor for an arbitrary number of fields with exponential potentials of the form given in Eq. (1.1), using proof by induction. To do this, we recursively construct the orthonormal fields and their potential.

Let us assume that we already have n fields ϕ_i with exponential potentials V_i of the form given in Eq. (1.1) and that it is possible to pick n orthonormal fields $\bar{\sigma}_2, \ldots, \bar{\sigma}_n$ and $\bar{\phi}_n$ such that the sum of the individual potentials V_i can be written as

$$\sum_{i=1}^{n} V_i = \bar{V}_n \exp\left(- \sqrt{\frac{16\pi}{\bar{p}_n}} \frac{\bar{\phi}_n}{m_{\mathrm{Pl}}} \right), \tag{3.1}$$

where we will further assume that $\bar{V}_n = \bar{V}_n(\bar{\sigma}_i)$ has a global minimum $\bar{V}_n(0) = V_0$ when $\bar{\sigma}_i = 0$ for all i from 2 to n.

It is possible to extend this form of the potential to $n+1$ fields if we consider an additional field ϕ_{n+1} with an exponential potential V_{n+1} of the form given in Eq. (1.1). Analogously to the two field case, we define

$$\bar{\phi}_{n+1} = \frac{\sqrt{\bar{p}_n}\,\bar{\phi}_n + \sqrt{p_{n+1}}\,\phi_{n+1}}{\sqrt{\bar{p}_{n+1}}}$$
$$+ \frac{m_{\mathrm{Pl}}}{\sqrt{16\pi(\bar{p}_{n+1})}}\left(\bar{p}_n \ln\frac{\bar{p}_n}{\bar{p}_{n+1}} + p_{n+1} \ln\frac{p_{n+1}}{\bar{p}_{n+1}} \right), \tag{3.2}$$

$$\bar{\sigma}_{n+1} = \frac{\sqrt{p_{n+1}}\,\bar{\phi}_n - \sqrt{\bar{p}_n}\,\phi_{n+1}}{\sqrt{\bar{p}_{n+1}}}$$
$$+ \frac{m_{\mathrm{Pl}}}{\sqrt{16\pi}}\sqrt{\frac{\bar{p}_n p_{n+1}}{\bar{p}_{n+1}}}\frac{1}{\ln}\frac{\bar{p}_n}{p_{n+1}}, \tag{3.3}$$

where

$$\bar{p}_{n+1} = \bar{p}_n + p_{n+1}. \tag{3.4}$$

Using these definitions we can show that the sum of the $n+1$ individual potentials V_i can be written as

$$\sum_{i=1}^{n+1} V_i = \bar{V}_{n+1} \exp\left(- \sqrt{\frac{16\pi}{\bar{p}_{n+1}}} \frac{\bar{\phi}_{n+1}}{m_{\mathrm{Pl}}} \right), \tag{3.5}$$

where $\bar{V}_{n+1} = \bar{V}_{n+1}(\bar{\sigma}_i)$ is given by

$$\bar{V}_{n+1} = \bar{V}_n \frac{\bar{p}_n}{\bar{p}_{n+1}} \exp\left(- \sqrt{\frac{16\pi}{\bar{p}_{n+1}}} \sqrt{\frac{p_{n+1}}{\bar{p}_n}} \frac{\bar{\sigma}_{n+1}}{m_{\mathrm{Pl}}} \right)$$
$$+ V_0 \frac{p_{n+1}}{\bar{p}_{n+1}} \exp\left(\sqrt{\frac{16\pi}{\bar{p}_{n+1}}} \sqrt{\frac{\bar{p}_n}{p_{n+1}}} \frac{\bar{\sigma}_{n+1}}{m_{\mathrm{Pl}}} \right). \tag{3.6}$$

Because we have assumed that $\bar{V}_n$ has a global minimum value $\bar{V}_n(0) = V_0$ when $\bar{\sigma}_i = 0$ for all i from 2 to n, one can verify that $\bar{V}_{n+1}$ also has a minimum value $\bar{V}_{n+1}(0) = V_0$ when $\bar{\sigma}_i = 0$, for all i from 2 to $n+1$.

However, we have already shown in Sec. II that for two fields ϕ_1 and ϕ_2, we can define two fields $\bar{\phi}_2$ and $\bar{\sigma}_2$, given in Eqs. (2.2) and (2.3) whose combined potential given in Eq. (2.4) is of the form required in Eq. (3.1), with $\bar{p}_2 = p_1 + p_2$. Hence we can write the potential in the form given in Eq. (3.1) for n fields, for all $n \geq 2$, with

$$\bar{p} \equiv \bar{p}_n = \sum_{i=1}^{n} p_i. \tag{3.7}$$

Equations (2.2) and (3.2) then lead us to the non-recursive expression for the ''weighted mean field''

$$\bar{\phi} \equiv \bar{\phi}_n = \sum_{i=1}^{n} \left(\sqrt{\frac{p_i}{\bar{p}}}\phi_i + \frac{m_{\mathrm{Pl}}}{\sqrt{16\pi\bar{p}}}p_i \ln\frac{p_i}{\bar{p}} \right), \tag{3.8}$$

which describes the evolution along the scaling solution. This is simply a rotation in field space plus an arbitrary shift, chosen to preserve the form of the potential given in Eq. (3.1). The $n-1$ fields $\bar{\sigma}_i$ describe the evolution orthogonal to the attractor trajectory.

The potential $\bar{V}_n$ has a global minimum at $\bar{\sigma}_i = 0$, which demonstrates that this is the stable late-time attractor. From Eqs. (2.5) and (3.6) we get a closed expression for $\bar{V}_n$,

$$\bar{V}_n = V_0 \left\{ \frac{p_1}{\bar{p}}\exp\left[- \frac{\sqrt{16\pi}}{m_{\mathrm{Pl}}}\sum_{i=2}^{n}\sqrt{\frac{p_i}{\bar{p}_i\bar{p}_{i-1}}}\,\bar{\sigma}_i \right] \right.$$
$$+ \sum_{i=2}^{n-1}\frac{p_i}{\bar{p}}\exp\left[\frac{\sqrt{16\pi}}{m_{\mathrm{Pl}}}\left(\sqrt{\frac{\bar{p}_{i-1}}{\bar{p}_i p_i}}\,\bar{\sigma}_i \right.\right.$$
$$\left.\left. - \sum_{j=i+1}^{n}\sqrt{\frac{p_j}{\bar{p}_j\bar{p}_{j-1}}}\,\bar{\sigma}_j \right) \right]$$
$$\left. + \frac{p_n}{\bar{p}}\exp\left[\frac{\sqrt{16\pi}}{m_{\mathrm{Pl}}}\sqrt{\frac{\bar{p}_{n-1}}{\bar{p}p_n}}\,\bar{\sigma}_n \right] \right\}. \tag{3.9}$$

Close to the attractor trajectory (to second order in $\bar{\sigma}_i$) we can write a Taylor expansion for the potential

$$\sum_{i=1}^{n} V_i = V_0 \left(1 + \frac{8\pi}{\bar{p} m_{\mathrm{Pl}}^2} \sum_{j=2}^{n} \bar{\sigma}_j^2 \right) \exp\left(-\sqrt{\frac{16\pi}{\bar{p}}} \frac{\bar{\phi}_n}{m_{\mathrm{Pl}}} \right). \tag{3.10}$$

Note that this expression is dependent only upon $\bar{p}$ and not on the individual p_i.

IV. STRINGY HYBRID INFLATION

The form of the potentials in Eqs. (2.6) and (3.10) is reminiscent of the effective potential obtained in the Einstein conformal frame from Brans-Dicke type gravity theories. The appearance of the weighted mean field, $\bar{\phi}$, as a "dilaton" field in the potential suggests that the matter Lagrangian might have a simpler form in a conformally related frame. If we work in terms of a conformally re-scaled metric

$$\tilde{g}_{\mu\nu} = \exp\left(-\sqrt{\frac{16\pi}{\bar{p}}} \frac{\bar{\phi}}{m_{\mathrm{Pl}}} \right) g_{\mu\nu}, \tag{4.1}$$

then the Lagrange density given in Eq. (2.4) becomes

$$\tilde{\mathcal{L}} = \exp\left(\sqrt{\frac{16\pi}{\bar{p}}} \frac{\bar{\phi}}{m_{\mathrm{Pl}}} \right) \times \left\{ -\frac{1}{2}(\tilde{\nabla}\bar{\phi})^2 - \sum_{i=2}^{n} \frac{1}{2}(\tilde{\nabla}\bar{\sigma}_i) - \bar{V} \right\}. \tag{4.2}$$

In this conformal related frame the field $\bar{\phi}$ is non-minimally coupled to the gravitational part of the Lagrangian. The original field equations were derived from the full action, including the Einstein-Hilbert Lagrangian of general relativity,

$$S = \int d^4x \sqrt{-g} \left[\frac{m_{\mathrm{Pl}}^2}{16\pi} R + \mathcal{L} \right], \tag{4.3}$$

where R is the Ricci scalar curvature of the metric $g_{\mu\nu}$. In terms of the conformally related metric given in Eq. (4.1) this action becomes (up to boundary terms [16])

$$S = \int d^4x \sqrt{-\tilde{g}} e^{-\Phi} \left[\frac{m_{\mathrm{Pl}}^2}{16\pi} [\tilde{R} - \omega(\tilde{\nabla}\Phi)^2] \right.$$
$$\left. -\frac{1}{2} \sum_{i=1}^{n-1} (\tilde{\nabla}\bar{\sigma}_i)^2 - \bar{V} \right], \tag{4.4}$$

where we have introduced the dimensionless dilaton field

$$\Phi = -\sqrt{\frac{16\pi}{\bar{p}}} \frac{\bar{\phi}}{m_{\mathrm{Pl}}}, \tag{4.5}$$

and the dimensionless Brans-Dicke parameter

$$\omega = \frac{\bar{p} - 3}{2}. \tag{4.6}$$

Thus the assisted inflation model is identical to $n-1$ scalar fields $\bar{\sigma}_i$ with a hybrid inflation type potential $\bar{V}(\bar{\sigma}_i)$ in a string-type gravity theory with dilaton, $\Phi \propto \bar{\phi}$. However, we note that in order to obtain power-law inflation with $\bar{p} \gg 1$ the

dimensionless constant ω must be much larger than that found in the low-energy limit of string theory where $\omega = -1$.

V. PERTURBATIONS ABOUT THE ATTRACTOR

The redefined orthonormal fields and the potential allow us to give the equations of motion for the independent degrees of freedom. If we consider only linear perturbations about the attractor then the energy density is independent of all the fields except $\bar{\phi}$, and we can solve the equation for the $\bar{\phi}$ field analytically.

The field equation for the weighted mean field is

$$\ddot{\bar{\phi}} + 3H\dot{\bar{\phi}} = \sqrt{\frac{16\pi}{\bar{p}}} \frac{V}{m_{\mathrm{Pl}}}. \tag{5.1}$$

Along the line $\bar{\sigma}_i = 0$ for all i in field space, we have

$$V = V_0 \exp\left(-\sqrt{\frac{16\pi}{\bar{p}}} \frac{\bar{\phi}}{m_{\mathrm{Pl}}} \right), \tag{5.2}$$

and the well-known power-law solution [1] with $a \propto t^{\bar{p}}$ is the late-time attractor [2,3] for this potential, where

$$\bar{\phi}(t) = \bar{\phi}_0 \ln\left(\frac{t}{t_0} \right), \tag{5.3}$$

and $\bar{\phi}_0 = m_{\mathrm{Pl}} \sqrt{\bar{p}/4\pi}$ and $t_0 = m_{\mathrm{Pl}} \sqrt{\bar{p}/8\pi V_0(3\bar{p}-1)}$.

A. Homogeneous linear perturbations

The field equations for the $\bar{\sigma}_i$ fields are

$$\ddot{\bar{\sigma}}_i + 3H\dot{\bar{\sigma}}_i + \frac{\partial V}{\partial \bar{\sigma}_i} = 0, \tag{5.4}$$

where the potential V is given by Eqs. (3.1) and (3.9), and the attractor solution corresponds to $\bar{\sigma}_i = 0$. Equation (3.10) shows that we can neglect the back-reaction of $\bar{\sigma}_i$ upon the energy density, and hence the cosmological expansion, to first-order and the field equations have the solutions

$$\bar{\sigma}_i(t) = \Sigma_{i+} t^{s_+} + \Sigma_{i-} t^{s_-}, \tag{5.5}$$

where

$$s_\pm = \frac{3\bar{p} - 1}{2} \left[-1 \pm \sqrt{\frac{3(\bar{p} - 3)}{3\bar{p} - 1}} \right], \tag{5.6}$$

for $\bar{p} > 3$, confirming that $\sigma_i = 0$ is indeed a local attractor. In the limit $\bar{p} \to \infty$ we obtain $s_i = -2$. For $1 < \bar{p} < 3$ the perturbations are under-damped and execute decaying oscillations about $\bar{\sigma}_i = 0$.

The form of the solutions given in Eq. (5.5) for $\bar{\sigma}_i(t)$ close to the attractor is the same for all the orthonormal fields $\bar{\sigma}_i$, as demonstrated in Fig. 2. Their evolution is independent
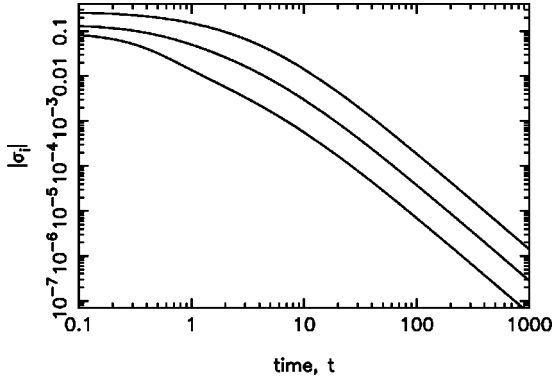
FIG. 2. Evolution of the fields $\bar{\sigma}_1$, $\bar{\sigma}_2$, and $\bar{\sigma}_3$, orthogonal to the scaling solution, in the assisted inflation model shown in Fig. 1.

of the individual p_i and determined only by the sum, $\bar{p}$, as expected from the form of the potential given in Eq. (3.10).

B. Inhomogeneous linear perturbations

Conventional hybrid inflation and power-law inflation are two of the very few models [17] in which one can obtain exact analytic expressions for the spectra of vacuum fluctuations on all scales without resorting to a slow-roll type approximation. In the case of hybrid inflation, this is only possible in the limit that the inflaton field σ approaches the minimum of its potential and we can neglect its back-reaction on the metric [18]. As the present model is so closely related to both power-law and hybrid inflation models close to the attractor, it is maybe not surprising then that we can obtain exact expressions for the evolution of inhomogeneous linear perturbations close to the scaling solution.

We will work in terms of the redefined fields $\bar{\phi}$ and $\bar{\sigma}_i$, and their perturbations on spatially flat hypersurfaces [19,20]. In the limit that $\bar{\sigma}_i \to 0$ we can neglect the back-reaction of the $\bar{\sigma}_i$ field upon the metric and the field $\bar{\phi}$. Perturbations in the field $\bar{\phi}$ then obey the usual equation for a single field driving inflation [21], and perturbations in the field $\bar{\sigma}_i$ evolve in a fixed background. Defining

$$u = a\,\delta\bar{\phi}, \tag{5.7}$$

$$v_i = a\,\delta\bar{\sigma}_i, \tag{5.8}$$

we obtain the decoupled equations of motion for perturbations with comoving wave number k,

$$u_k'' + \left(k^2 - \frac{z''}{z} \right) u_k = 0, \tag{5.9}$$

$$v_{ik}'' + \left(k^2 + a^2 \frac{d^2V}{d\bar{\sigma}_i^2} - \frac{a''}{a} \right) v_{ik} = 0, \tag{5.10}$$

where [21] $z \equiv a^2 \bar{\phi}'/a'$ and a prime denotes differentiation with respect to conformal time $\eta \equiv \int dt/a$. For power-law expansion we have $z \propto a \propto (-\eta)^{-\bar{p}/(\bar{p}-1)}$ and thus

$$\frac{a''}{a} = \frac{\bar{p}(2\bar{p}-1)}{(\bar{p}-1)^2}\,\eta^{-2}. \tag{5.11}$$

We also have $aH \propto -\bar{p}/((\bar{p}-1)\,\eta)$ which gives

$$a^2 \frac{d^2V}{d\sigma_i^2} = \frac{2(3\bar{p}-1)}{(\bar{p}-1)^2}\,\eta^{-2}, \tag{5.12}$$

where we have used the fact that $d^2V/d\bar{\sigma}_i^2 = 16\pi V/(\bar{p}m_{\mathrm{Pl}}^2)$ along the attractor. The equations of motion therefore become

$$u_k'' + \left(k^2 - \frac{\nu^2-(1/4)}{\eta^2} \right) u_k = 0, \tag{5.13}$$

$$v_{ik}'' + \left(k^2 - \frac{\lambda^2-(1/4)}{\eta^2} \right) v_{ik} = 0, \tag{5.14}$$

where

$$\nu = \frac{3}{2} + \frac{1}{\bar{p}-1}, \tag{5.15}$$

$$\lambda = \frac{3}{2}\,\frac{\sqrt{(\bar{p}-3)(\bar{p}-1/3)}}{\bar{p}-1}, \tag{5.16}$$

and the general solutions in terms of Hankel functions are

$$u_k = U_1(-k\eta)^{1/2}H_\nu^{(1)}(-k\eta) + U_2(-k\eta)^{1/2}H_\nu^{(2)}(-k\eta), \tag{5.17}$$

$$v_{ik} = V_{1i}(-k\eta)^{1/2}H_\lambda^{(1)}(-k\eta) + V_{2i}(-k\eta)^{1/2}H_\lambda^{(2)}(-k\eta). \tag{5.18}$$

Taking only positive frequency modes in the initial vacuum state for $|k\eta| \gg 1$ and normalizing requires u_k and $v_{ik} \to e^{-ik\eta}/\sqrt{2k}$, which gives the vacuum solutions

$$u_k = \frac{1}{2}(-\pi\eta)^{1/2}e^{(\pi/2)(\nu+1)i}H_\nu^{(1)}(-k\eta), \tag{5.19}$$

$$v_{ik} = \frac{1}{2}(-\pi\eta)^{1/2}e^{(\pi/2)(\lambda+1)i}H_\lambda^{(1)}(-k\eta). \tag{5.20}$$

In the opposite limit, i.e., $-k\eta \to 0$, we use the limiting form of the Hankel functions, $H_\nu^{(1)}(z) \sim -(i/\pi)\Gamma(\nu)z^{-\nu}$, and therefore on large scales, and at late times, we obtain

$$u_k \to \frac{2^{\nu-1}}{\sqrt{\pi k}}e^{i(\pi/2)\nu}(-k\eta)^{1/2-\nu}\Gamma(\nu), \tag{5.21}$$

$$v_{ik} \to \frac{2^{\lambda-1}}{\sqrt{\pi k}}e^{i(\pi/2)\lambda}(-k\eta)^{1/2-\lambda}\Gamma(\lambda). \tag{5.22}$$

The power spectrum of a Gaussian random field ψ is conventionally given by $\mathcal{P}_\psi \equiv (k^3/2\pi^2)\langle|\psi|^2\rangle$. The power spectra on large scales for the field perturbations $\delta\bar{\phi}$ and $\delta\bar{\sigma}_i$ are thus

$$\mathcal{P}_{\delta\phi}^{1/2} = \frac{C(\nu)}{\left(\nu - \frac{1}{2}\right)} \frac{H}{2\pi} (-k\eta)^{3/2 - \nu}, \qquad (5.23)$$

$$\mathcal{P}_{\delta\sigma_i}^{1/2} = \frac{C(\lambda)}{\left(\nu - \frac{1}{2}\right)} \frac{H}{2\pi} (-k\eta)^{3/2 - \lambda}, \qquad (5.24)$$

where we have used $\eta = -(\nu - 1/2)/(aH)$ and we define

$$C(\alpha) \equiv \frac{2^\alpha \Gamma(\alpha)}{2^{3/2}\Gamma\left(\frac{3}{2}\right)}. \qquad (5.25)$$

Both the weighted mean field $\bar{\phi}$ and the orthonormal fields $\bar{\sigma}_i$ are "light" fields ($m^2 < 3H^2/2$) during assisted inflation (for $\bar{p} > 3$) and thus we obtain a spectrum of fluctuations in all the fields on large scales. Note that in the de Sitter limit, $\bar{p} \to \infty$ and thus $\nu \to 3/2$ and $\lambda \to 3/2$, we have $\mathcal{P}_{\delta\phi}^{1/2} \to H/2\pi$, and $\mathcal{P}_{\delta\sigma_i}^{1/2} \to H/2\pi$.

At late times, that is $k\eta \to 0$, the perturbations in the weighted mean field, $\delta\bar{\phi}$, approach a constant, while the perturbations in the orthonormal fields, $\delta\bar{\sigma}_i$, decay in agreement with our solutions for the homogeneous perturbations given by Eqs. (5.5).

Denoting the scale dependence of the perturbation spectra by $\Delta n_x \equiv d\ln\mathcal{P}_x/d\ln k$, we obtain

$$\Delta n_{\delta\phi} = 3 - 2\nu = -\frac{2}{\bar{p} - 1}, \qquad (5.26)$$

$$\Delta n_{\delta\sigma_i} = 3 - 2\lambda$$

$$= 3\left(1 - \frac{\sqrt{(\bar{p} - 3)(\bar{p} - 1/3)}}{\bar{p} - 1}\right). \qquad (5.27)$$

VI. DISCUSSION

We have shown that the recently proposed model of assisted inflation, driven by many scalar fields with steep exponential potentials, can be better understood by performing a rotation in field space, which allows us to re-write the potential as a product of a single exponential potential for a weighted mean field, $\bar{\phi}$, and a potential $\bar{V}_n$ for the orthogonal degrees of freedom, $\bar{\sigma}_i$, which has a global minimum when $\bar{\sigma}_i = 0$. This proves that the scaling solution found in Ref. [5] is indeed the late-time attractor.

The particular form of the potential which we present for scalar fields minimally-coupled to the spacetime metric, can

also be obtained via a conformal transformation of a hybrid inflation type inflationary potential [12,13] with a non-minimally coupled, but otherwise massless, dilaton field, $\Phi \propto \bar{\phi}$. Thus we see that assisted inflation can be understood as a form of power-law hybrid inflation, where the false-vacuum energy density is diluted by the evolution of the dilaton field.

We have also been able to give exact solutions for inhomogeneous linear perturbations about the attractor trajectory in terms of our rotated fields. Perturbations in the weighted mean field $\bar{\phi}$ correspond to the perturbations in the density on the uniform curvature hypersurfaces, or equivalently, perturbations in the curvature of constant density hypersurfaces [22,20]:

$$\zeta = \frac{H\delta\bar{\phi}}{\dot{\bar{\phi}}}. \qquad (6.1)$$

These perturbations are along the attractor trajectory, and hence describe adiabatic curvature perturbations. The spectral index of the curvature perturbations on large scales is thus given from Eq. (5.26) as

$$n_s \equiv 1 + \frac{d\ln\mathcal{P}_\zeta}{d\ln k} = 1 - \frac{2}{\bar{p} - 1}, \qquad (6.2)$$

and is always negatively tilted with respect to the Harrison-Zel'dovich spectrum where $n_s = 1$. Note that in the slow-roll limit ($\bar{p} \to \infty$) we recover the result of Ref. [5].

First-order perturbations in the fields orthogonal to the weighted mean field are isocurvature perturbations during inflation. Vacuum fluctuations lead to a positively tilted spectrum. The presence of non-adiabatic perturbations can lead to more complicated evolution of the large-scale curvature perturbation than may be assumed in single-field inflation models [23–25]. However, we have shown that these perturbations decay relative to the adiabatic perturbations and hence we recover the single field limit at late times. In particular we find that the curvature perturbation ζ becomes constant on super-horizon scales during inflation. Note, however, that assisted inflation must be ended by a phase transition whose properties are not specified in the model. If this phase transition is sensitive to the isocurvature (non-adiabatic) fluctuations orthogonal to the attractor trajectory, then the curvature perturbation, ζ, during the subsequent radiation dominated era may not be simply related to the curvature perturbation during inflation.

[1] F. Lucchin and S. Matarrese, Phys. Rev. D **32**, 1316 (1985).

[2] J. J. Halliwell, Phys. Lett. B **185**, 341 (1987).

[3] A. B. Burd and J. D. Barrow, Nucl. Phys. **B308**, 929 (1988).

[4] E. J. Copeland, A. R. Liddle, and D. Wands, Phys. Rev. D **57**, 4686 (1998).

[5] A. R. Liddle, A. Mazumdar, and F. E. Schunck, Phys. Rev. D **58**, 061301 (1998).

[6] C. Wetterich, Nucl. Phys. **B302**, 668 (1988).

[7] D. Wands, E. J. Copeland, and A. R. Liddle, Ann. (N.Y.) Acad. Sci. **688**, 647 (1993).

[8] R. J. van den Hoogen, A. A. Coley, and D. Wands, Class. Quantum Grav. (to be published), gr-qc/9901014.

[9] C. Wetterich, Astron. Astrophys. **301**, 321 (1995).

[10] P. G. Ferreira and M. Joyce, Phys. Rev. Lett. **79**, 4740 (1997); Phys. Rev. D **58**, 023503 (1998).

[11] P. T. P. Viana and A. R. Liddle, Phys. Rev. D **57**, 674 (1998).

[12] A. D. Linde, Phys. Lett. B **259**, 38 (1991); Phys. Rev. D **49**, 748 (1994).

[13] E. J. Copeland, A. R. Liddle, D. H. Lyth, E. D. Stewart, and D. Wands, Phys. Rev. D **49**, 6410 (1994).

[14] D. La and P. J. Steinhardt, Phys. Rev. Lett. **62**, 376 (1989).

[15] A. H. Guth, Phys. Rev. D **23**, 347 (1981).

[16] D. Wands, Class. Quantum Grav. **11**, 269 (1994).

[17] A. R. Liddle and D. H. Lyth, Phys. Rep. **231**, 1 (1993).

[18] J. García-Bellido and D. Wands, Phys. Rev. D **54**, 7181 (1996).

[19] H. Kodama and M. Sasaki, Prog. Theor. Phys. Suppl. **78**, 1 (1984).

[20] K. A. Malik and D. Wands, gr-qc/9804046.

[21] V. F. Mukhanov, Zh. Éksp. Teor. Fiz. **94**, 1 (1988) [Sov. Phys. JETP **68**, 1297 (1988)].

[22] J. M. Bardeen, P. J. Steinhardt, and M. S. Turner, Phys. Rev. D **28**, 679 (1983).

[23] D. S. Salopek, Phys. Rev. D **52**, 5563 (1995).

[24] M. Sasaki and E. D. Stewart, Prog. Theor. Phys. **95**, 71 (1996).

[25] J. García-Bellido and D. Wands, Phys. Rev. D **53**, 5437 (1996).