Random matrix model for Wilson fermions on the lattice

Holger Hehl and Andreas Schäfer
Institut für Theoretische Physik, Universität Regensburg, Universitätsstrasse 31, D-93040 Regensburg, Germany
(Received 17 June 1998; published 19 April 1999)

We describe a random matrix model suitable for the simulation of the eigenvalues of the Dirac operator on the lattice for Wilson fermions. We compare the obtained global eigenvalue spectrum for various values of the hopping parameter κ with the lattice results of Kalkreuter. The agreement is surprisingly good. [S0556-2821(99)04009-6]

PACS number(s): 11.15.Ha, 02.60.Cb, 12.40.Ee

Recently, it became clear that the microscopic spectral properties of the lattice QCD Dirac operator are universal and can be reproduced by simple models that only share basic symmetry properties with real QCD [1,2]. Such models are provided by random matrix theory (RMT) [1,3]. This universality has recently been demonstrated for the staggered lattice Dirac operator in quenched [4] and unquenched [5] SU(2). Since the Banks-Casher formula $\Sigma=\pi\rho(0)/V$ [6] links the spectral density at zero virtuality to the chiral condensate Σ, the distribution of the small eigenvalues is of great importance for, e.g., the understanding of the chiral phase transition. RMT has also solved a long-standing problem of lattice calculations at finite chemical potential [7]. Recently, also the predictions for the energy scale at which RMT and lattice QCD start to deviate were confirmed, which among others tested the validity of the Gell-Mann–Oakes–Renner relation on the lattice [8]. While RMT makes reliable predictions only for microscopic spectral fluctuations, these successes encourage us to see if the global spectral properties can be reproduced. This is the aim of our contribution for the case of Wilson fermions. Since the universality argument only applies to microscopic properties, one cannot hope to fit global spectra except for very special cases. We analyze such a case, namely, gauge theories with infinitely strong coupling, i.e., $\beta\to 0$. Such systems can be hoped to be sufficiently chaotic as to show random matrix characteristics even on global scales. Furthermore, our model studies suggest that to describe Wilson fermions a 4×4 block structure is needed in random matrix theory. We expect this to hold true also for a description of microscopic properties at nonzero β. Random matrix models for Wilson fermions at nonzero β do not yet exist, but are of great practical importance.

We start our analysis with the Euclidean action of lattice SU(2) theory with Wilson fermions in the fundamental representation which can be written as

$$S_E=\frac{1}{2\kappa}\sum_n \psi^\dagger(n)\psi(n)-\frac{1}{2}\sum_{n,\mu}[\psi^\dagger(n)(r-\gamma_\mu)U_\mu(n)\times\psi(n+\mu)+\psi^\dagger(n+\mu)(r+\gamma_\mu)U_\mu^\dagger(n)\psi(n)]+\frac{4}{g^2}\sum_P\left\{1-\frac{1}{4}\mathrm{Tr}[U_P(n)+U_P^\dagger(n)]\right\}, \tag{1}$$

with the Wilson parameter r, which we set to 1 in the following [9]. The gauge fields A_μ are contained in the link variables U. With this action a gauge-invariant partition function can be constructed from which one can obtain vacuum expectation values of operator products in the usual way. In these partition functions we average over all gauge field configurations.

In random matrix theory we substitute the Dirac operator which includes the gauge fields by random matrices of a particular ensemble to model the very strong fluctuations of the Dirac operator when calculated with lattice gauge theory. The integration is then performed over the independent entries of the matrices [1]. With this approximation the gluons decouple from the quarks and can be integrated out; i.e., they can be neglected in random matrix models if one is only interested in quark observables. The symmetry properties of the random matrix depend on the underlying gauge group and the fermion representation. Usually in QCD we are dealing with fermions in the fundamental representation of the SU(3) gauge group, in which case one has to use the Gaussian unitary ensemble (GUE). In order to compare our results with Kalkreuter [10] who investigated the operator $\gamma_5(\not{D}+m)$ for massive Wilson fermions in an SU(2) gauge field background we have to use matrices of the Gaussian orthogonal ensemble (GOE) [11]. The Euclidean partition function with two random matrices can be written according to the above arguments as

$$Z=\int \mathcal{D}[A,B]\mathrm{e}^{-\Sigma_A A^\dagger A-\Sigma_B B^\dagger B}\int \mathcal{D}[\psi^\dagger,\psi]\mathrm{e}^{-\psi^\dagger(\not{D}+m)\psi}, \tag{2}$$

with the parameters Σ_A and Σ_B that scale the distribution variance of the Gaussian ensembles. We will now specify the operator $\not{D}+m$ in Eq. (5) and explain why we are using two different random matrices.

In the following we separate the Dirac spinors into left- and right-handed fields $\psi_{\mathrm{L,R}}=\frac{1}{2}(1\mp\gamma_5)\psi$ with the Euclidean $\gamma_5=-\gamma_5^{\mathrm{M}}$. Furthermore, we distinguish between *even* and *odd* lattice sites. These two sublattices couple rather independently (the two groups are often called ''red'' and ''black'' lattice sites in analogy to the colors of a checkerboard). The spinor field is then written as $(\psi_{\mathrm{R}}^{\mathrm{e}},\psi_{\mathrm{L}}^{\mathrm{e}},\psi_{\mathrm{R}}^{\mathrm{o}},\psi_{\mathrm{L}}^{\mathrm{o}})$, where $\psi_{\mathrm{R,L}}^{\mathrm{e,o}}=(\psi_{\mathrm{R,L}}^{\mathrm{e,o}}(1),\ldots,\psi_{\mathrm{R,L}}^{\mathrm{e,o}}(N/2))$ are vectors with respect to the lattice sites $x_\mu(1),\ldots,x_\mu(N)$.

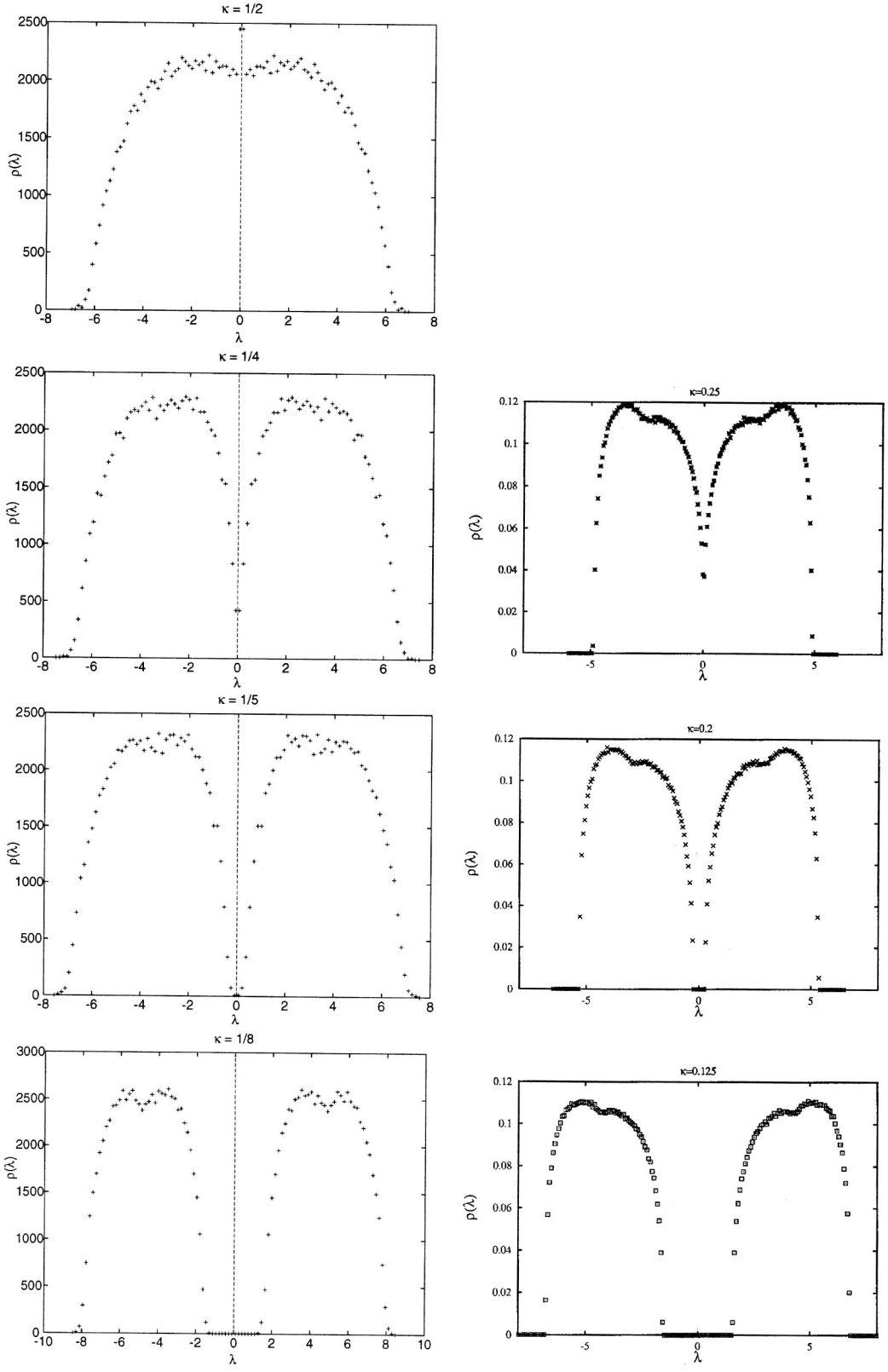

FIG. 1. Spectral densities of the massive Dirac operator for various values of the hopping parameter κ. On the right hand side are shown Kalkreuter's results for $\beta=0.0$ extracted from the first picture of Fig. 2, but with eigenvalue normalization as on the left hand side. The chosen parameters on the left are $\Sigma_A=2/25$, $\Sigma_B=8/25$. The hopping parameter is $\kappa=1/2$, 1/4, 1/5, and 1/8, respectively.

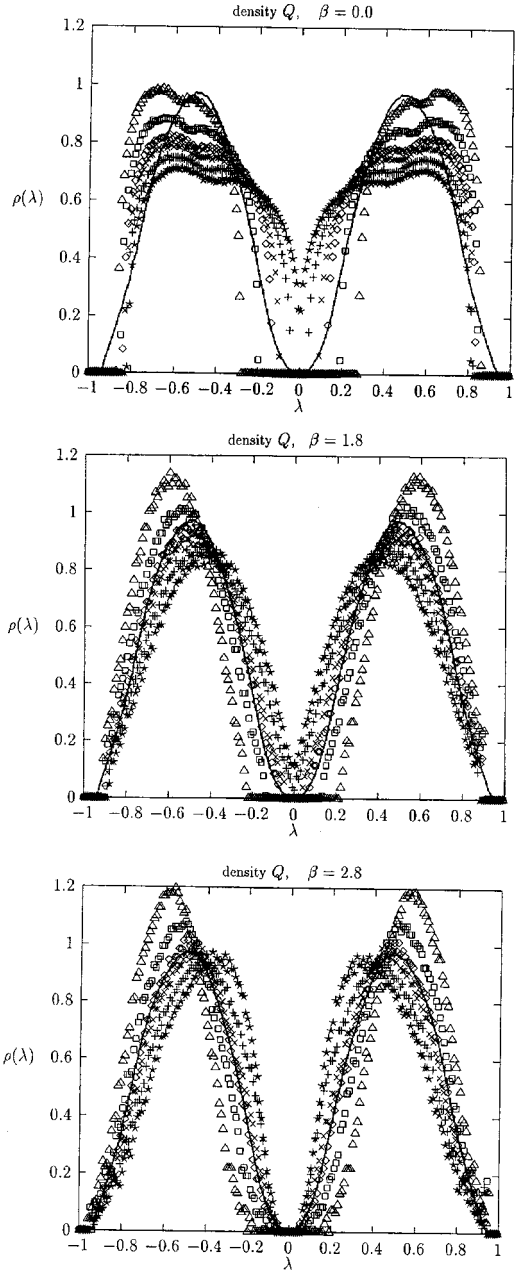

FIG. 2. Kalkreuter's results for $\beta=0.0$, 1.8, and 2.8, respectively. The eigenvalues are normalized to lie between -1 and 1. The symbols $\star$, $+$, $\times$, $\diamond$, $\square$, and $\triangle$ correspond to quenched data with κ values of 0.25, 0.20, 1/6, 0.15, 0.125, and 0.10, respectively. The solid line shows unquenched data at $\kappa=0.15$ and $\beta=2.12$.

In the above basis the mass term of Eq. (1) is simply a 4×4 block diagonal matrix and the interaction term becomes

$$-\frac{1}{2}(\psi_{\mathrm{L}}^{\mathrm{e}\dagger}\psi_{\mathrm{L}}^{\mathrm{o}}+\psi_{\mathrm{R}}^{\mathrm{e}\dagger}\psi_{\mathrm{R}}^{\mathrm{o}}+\psi_{\mathrm{L}}^{\mathrm{o}\dagger}\psi_{\mathrm{L}}^{\mathrm{e}}+\psi_{\mathrm{R}}^{\mathrm{o}\dagger}\psi_{\mathrm{R}}^{\mathrm{e}}) \tag{3a}$$

$$-\frac{1}{2}(-\psi_{\mathrm{L}}^{\mathrm{e}\dagger}\gamma_{\dot{\mu}}\psi_{\mathrm{R}}^{\mathrm{o}}-\psi_{\mathrm{R}}^{\mathrm{e}\dagger}\gamma_{\dot{\mu}}\psi_{\mathrm{L}}^{\mathrm{o}}+\psi_{\mathrm{L}}^{\mathrm{o}\dagger}\gamma_{\dot{\mu}}\psi_{\mathrm{R}}^{\mathrm{e}}+\psi_{\mathrm{R}}^{\mathrm{o}\dagger}\gamma_{\dot{\mu}}\psi_{\mathrm{L}}^{\mathrm{e}}) \tag{3b}$$

$$-\frac{\mathrm{i}}{2}ga(\psi_{\mathrm{L}}^{\mathrm{e}\dagger}A_{\dot{\mu}}\psi_{\mathrm{L}}^{\mathrm{o}}+\psi_{\mathrm{R}}^{\mathrm{e}\dagger}A_{\dot{\mu}}\psi_{\mathrm{R}}^{\mathrm{o}}-\psi_{\mathrm{L}}^{\mathrm{o}\dagger}A_{\dot{\mu}}^{\dagger}\psi_{\mathrm{L}}^{\mathrm{e}}-\psi_{\mathrm{R}}^{\mathrm{o}\dagger}A_{\dot{\mu}}^{\dagger}\psi_{\mathrm{R}}^{\mathrm{e}}) \tag{3c}$$

$$-\frac{\mathrm{i}}{2}ga(-\psi_{\mathrm{L}}^{\mathrm{e}\dagger}\not{A}\psi_{\mathrm{R}}^{\mathrm{o}}-\psi_{\mathrm{R}}^{\mathrm{e}\dagger}\not{A}\psi_{\mathrm{L}}^{\mathrm{o}}-\psi_{\mathrm{L}}^{\mathrm{o}\dagger}\not{A}^{\dagger}\psi_{\mathrm{R}}^{\mathrm{e}}-\psi_{\mathrm{R}}^{\mathrm{o}\dagger}\not{A}^{\dagger}\psi_{\mathrm{L}}^{\mathrm{e}}). \tag{3d}$$

The notation $\dot{\mu}$ indicates that μ is not a free Lorentz index but is contracted with a corresponding one that is hidden in the ψ's. We have expanded the link variables $U_{\dot{\mu}}$ to first order in $A_{\dot{\mu}}$, i.e., $U_{\dot{\mu}}=1+\mathrm{i}gaA_{\dot{\mu}}$. The term (3a) then leads to the constant block matrix

$$\begin{pmatrix}0&0&1&0\\0&0&0&1\\1&0&0&0\\0&1&0&0\end{pmatrix}. \tag{4a}$$

In (3c) we replace $A_{\dot{\mu}}$ with i/ga times the random matrix A such that we get the block matrix

$$\begin{pmatrix}0&0&-A&0\\0&0&0&-A\\-A^{\dagger}&0&0&0\\0&-A^{\dagger}&0&0\end{pmatrix}. \tag{4b}$$

In (3d) we do the same with $\not{A}$ but calling the random matrix B which gives us

$$\begin{pmatrix}0&0&0&+B\\0&0&+B&0\\0&-B^{\dagger}&0&0\\-B^{\dagger}&0&0&0\end{pmatrix}. \tag{4c}$$

In the remaining term (3b) which has the same structure as (3d) we are faced with a currentlike form, bilinear in the fields. As in (3c) different γ matrices are used depending on the lattice sites which makes this term sufficiently random to be absorbed in the matrix B. To obtain the above matrices we have rescaled the Dirac fields by $1/\sqrt{-1/2}$.

The expansion of $U_{\dot{\mu}}$ we used to argue for our assumed block structure is justified in the perturbative domain of small g. Here we are rather interested in the ergodic regime of fully developed chaoticity, which corresponds to rather large g. However, we are only interested in the general symmetry properties, which are the same in both regimes.

Adding up the above contributions (4) we arrive at the following expression for the complete Dirac operator:

$$\not{D}+m=\begin{pmatrix}\frac{1}{2\kappa}&0&1-A&B\\0&\frac{1}{2\kappa}&B&1-A\\1-A^{\dagger}&-B^{\dagger}&\frac{1}{2\kappa}&0\\-B^{\dagger}&1-A^{\dagger}&0&\frac{1}{2\kappa}\end{pmatrix}. \tag{5}$$

Neglecting the distinction between even and odd fields and looking at the chiral structure only, $\not{D}$ has the structure

of a 2×2 block matrix, namely, $\left(\begin{smallmatrix}0 & * \\ * & 0\end{smallmatrix}\right)$ since it is anticommuting with γ_5. This ensures that the eigenvalues of $\not{D}$ are distributed around zero symmetrically. Let us stress that we have in no way derived Eq. (5). We basically only give some hand-waving arguments for the form in Eq. (6) which is then tested against the numerical data. Definitely in the end more realistic models will differ from (5) but we believe that they will share the 4×4 structure.

Kalkreuter investigated in his analyses the operator $\gamma_5(\not{D}+m)$, and so we have to multiply (5) by γ_5. Here we must carefully keep in mind that the B matrix resulted from the $\not{A}$ term which implicitly contains the γ matrices. When we also define our spinor basis according to $(\psi_{\mathrm{R}}^{\mathrm{e}}, -\psi_{\mathrm{L}}^{\mathrm{e}}, \psi_{\mathrm{R}}^{\mathrm{o}}, \psi_{\mathrm{L}}^{\mathrm{o}})$ we finally arrive at

$$\gamma_5(\not{D}+m)=\begin{pmatrix} \frac{1}{2\kappa} & 0 & 1-A & B \\ 0 & -\frac{1}{2\kappa} & B & 1-A \\ 1-A^\dagger & B^\dagger & \frac{1}{2\kappa} & 0 \\ B^\dagger & 1-A^\dagger & 0 & -\frac{1}{2\kappa} \end{pmatrix}. \quad (6)$$

At first we tried to determine the spectral density of the above operator analytically. For this purpose we chose Eq. (6) as the matrix in the determinant of Eq. (2) and integrated out the $2\times\frac{1}{2}(N/2)(N/2+1)$ real matrix variables by means of the Hubbard-Stratonovich transformation which lead to a partition function with integration over only 8 complex variables. It turned out that the resulting saddle point equations are too complicated to be solved analytically so that we decided to diagonalize the operator matrix numerically.

We calculated the eigenvalues of this matrix for a large number of random variables matching the selected ensemble (GOE). We display the results in histograms for several values of the parameter $\kappa=(2m+8)^{-1}$ (see Fig. 1). Our results are on the left side of this figure. They are compared with the corresponding $\beta=0$ data of Kalkreuter taken from [12] which are shown on the right side.

One can clearly see the splitting of the spectral density $\rho(\lambda)$ into two distinct symmetric parts at a critical value of κ between 0.2 and 0.25 as well as a slight dent on top of the half-circle-like densities. The structures come about as follows. The separation of the two half-circular structures is due to the diagonal term $1/2\kappa$. The splitting of the two peaks for each half-circle-like structure results from the constant ''1'' in (5). These dependences are in good agreement with the data. The matrices A and B fluctuate according to the random matrix constants Σ_A and Σ_B [see Eq. (2)]. If Σ_A and Σ_B are large, the structure gets washed out. If Σ_A and Σ_B are different, the half-circle-like structures become asymmetric.

Since the gauge fields in the action are completely replaced by random variables, the best agreement has to be expected for $\beta=0$. The comparison shows that in this limit of extremely strong coupling RMT is actually able to fit the global eigenvalue spectrum surprisingly well. A detail which is missed is the slight difference in height of the two maxima for, e.g., positive λ. Also the falloff for large eigenvalues is steeper for the lattice results.

With decreasing coupling strength the agreement for the global spectrum becomes worse, as is seen by comparing the left hand side of Fig. 1 to the lower two pictures of Fig. 2, and one reaches the usual situation in which RMT can only describe the microscopic fluctuations.

Let us conclude: We defined for the first time a random matrix model which is suitable for the description of the eigenvalue spectrum of the Dirac operator (multiplied by γ_5) for an SU(2) gauge theory with Wilson fermions. A similar model should allow to study, e.g., the distribution properties of the lowest eigenvalues for SU(3)-Wilson fermions [13] and for the operator $(\not{D}+m)$ instead of $\gamma_5(\not{D}+m)$, which is a problem of great practical importance.

We thank T. Wettig for very helpful discussions. This work has been supported by BMBF.

[1] For a review on RMT and Dirac spectra, see the recent review by J. J. M. Verbaarschot, hep-th/9710114 and references therein.
[2] H. Leutwyler and A. V. Smilga, Phys. Rev. D **46**, 5607 (1992); E. V. Shuryak and J. J. M. Verbaarschot, Nucl. Phys. **A560**, 306 (1993).
[3] For a recent review on random matrix theory in general, see T. Guhr, A. Müller-Groeling, and H. A. Weidenmüller, Phys. Rep. **299**, 189 (1998).
[4] M. E. Berbenni-Bitsch, S. Meyer, A. Schäfer, J. J. M. Verbaarschot, and T. Wettig, Phys. Rev. Lett. **80**, 1146 (1998).
[5] M. E. Berbenni-Bitsch, S. Meyer, and T. Wettig, Phys. Rev. D **58**, 071502 (1998).
[6] T. Banks and A. Casher, Nucl. Phys. **B169**, 103 (1980).
[7] M. A. Stephanov, Nucl. Phys. B (Proc. Suppl.) **53**, 469 (1997).
[8] M. E. Berbenni-Bitsch, M. Göckeler, T. Guhr, A. D. Jackson, J.-Z. Ma, S. Meyer, A. Schäfer, H. A. Weidenmüller, T. Wettig, and T. Wilke, Phys. Lett. B **438**, 14 (1998); J. C. Osborn and J. J. M. Verbaarschot, Nucl. Phys. **B525**, 738 (1998); R. A. Janik, M. A. Nowak, G. Papp, and I. Zahed, Phys. Rev. Lett. **81**, 264 (1998).
[9] H. J. Rothe, *Lattice Gauge Theories* (World Scientific, Singapore, 1992).
[10] T. Kalkreuter, Comput. Phys. Commun. **95**, 1 (1996); T. Kalkreuter, Nucl. Phys. B (Proc. Suppl.) **49**, 168 (1996).
[11] J. J. M. Verbaarschot, Phys. Rev. Lett. **72**, 2531 (1994).
[12] J. Jurkiewicz, M. A. Nowak, and I. Zahed, Nucl. Phys. **B478**, 605 (1996) and references therein.
[13] K. Jansen and R. Sommer, Nucl. Phys. B (Proc. Suppl.) **63**, 853 (1998); R. Frezzotti and K. Jansen, *ibid.* **63**, 943 (1998) and references therein.