QCD at $\theta \sim \pi$

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Taking into account the terms $\sim m^2$ in the effective chiral Lagrangian, we show that, at $\theta=\pi$, the theory with two light quarks of equal mass involves two degenerate vacuum states separated by a barrier. For $N_f=3$, the energy barrier between two vacua appears already in the leading order in mass. This corresponds to the first order phase transition at $\theta=\pi$. The surface energy density of the domain wall separating two different vacua is calculated. In the immediate vicinity of the point $\theta=\pi$, two minima of the potential still exist, but one of them becomes metastable. The probability of the false vacuum decay is estimated.
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I. INTRODUCTION

It is very well known that in QCD with N_f massless quarks, chiral symmetry $SU_L(N_f)\otimes SU_R(N_f)$ is spontaneously broken down to $SU_V(N_f)$. (This is an experimental fact for $N_f=2,3$. Probably, no spontaneous chiral symmetry breaking occurs for a large enough number of flavors $N_f \sim 8-10$ [1] which will be of no concern for us here.) Spontaneous symmetry breaking means that the order parameter $\langle \overline{q_R^i} q_L^j \rangle$ $(i,j$ are flavor indices) can acquire an arbitrary direction in flavor space. Massless Goldstone particles appear. If the free quark masses are not zero, the axial chiral symmetry is broken explicitly and the minimum of the energy functional corresponds to a particular flavor orientation of the condensate. Goldstone bosons acquire small masses. In the real world with $m_u \approx 4$ MeV, $m_d \approx 7$ MeV, $m_s \approx 150$ MeV, and $\theta=0$, the vacuum state is unique.

It is interesting to study also other variants of the theory with different values of masses and θ. It has been known for a long time [2,3] that in the theory with equal light quark masses and $\theta=\pi$, there are two degenerate vacuum states. This is best seen in the framework of the effective chiral Lagrangian describing only the light pseudo-Goldstone degrees of freedom. In the leading order in mass, the effective potential is

$$V=-\Sigma\,\mathrm{Re}[\mathrm{Tr}\{\mathcal{M}e^{i\theta/N_f}U^\dagger\}] \tag{1}$$

where $U=\exp\{2i\phi^a t^a/F_\pi\}$($\phi^a$ are pseudo-Goldstone fields), $\mathcal{M}$ is the quark mass matrix, and Σ is the absolute value of the quark condensate.

Suppose $N_f=3$, $\mathcal{M}=m\hat{1}$, and $\theta=0$. The minimum of the energy is achieved at $U=\hat{1}$. For $\theta=\pi$, there are two different minima with $U=\hat{1}$ and $U=e^{2\pi i/3}\hat{1}$. They are separated by the energy barrier. The appearance of two vacuum states corresponds to spontaneous breaking of the *CP* symmetry by the Dashen mechanism [4].

The situation is, however, more confusing for $N_f=2$. The trace of a $SU(2)$ matrix is always real which means that, at $\mathcal{M}=m\hat{1}$ and $\theta=\pi$, the potential (1) does not depend on U at all. That would mean that the explicit breaking of the chiral symmetry is absent, pions are massless at this point and the phase transition would be of the second rather than of the first order. We understand, however, that the chiral symmetry of the original QCD Lagrangian *is* broken explicitly by the quark mass term for all values of θ including $\theta=\pi$, and the situation looks paradoxical.

This question was left undiscussed in the original papers [2,3], and only comparatively recently was it shown how the paradox is resolved [5]. To this end, one should take into account the terms of order $\sim m^2$ in the effective chiral Lagrangian. If doing so, the continuous vacuum degeneracy at $\theta=\pi$ is lifted, and we obtain again only two vacuum states separated by a barrier.[1]

In what follows, we confirm this finding. The main aim of the paper is to bring the analysis of Ref. [5] into contact with standard chiral theory notations and wisdom and to perform some quantitative estimates both for $N_f=3$ and $N_f=2$ for the height of the energy barrier, the surface energy density of the domain walls interpolating between two degenerate vacua at $\theta=\pi$, and for the decay rate of metastable vacuum states at the vicinity of the phase transition point.

II. THREE FLAVORS

Let us discuss first in some details the case $N_f=3$ where no complications due to higher-order terms arise. By a conjugation $U\to VUV^\dagger$, any unitary matrix U can be brought into the diagonal form $U=\mathrm{diag}(e^{i\alpha},e^{i\beta},e^{-i(\alpha+\beta)})$. When

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[1]One could, of course, anticipate it. For $\theta=0$ and in the leading order in quark mass, the mass of pions is given by the Gell-Mann–Oakes–Renner relation $F_\pi^2 M_\pi^2=(m_u+m_d)\Sigma$. A theory with $m_u=m_d$ and $\theta=\pi$ is equivalent to the theory with $m_u=-m_d$ and $\theta=0$ [only $\theta_{\mathrm{phys}}=\theta+\arg(\det\mathcal{M})$ is relevant). In that case, the pions seem to stay massless. However, the Gell-Mann–Oakes–Renner relation is true only in the leading order in m_q. Higher order corrections bring about a nonzero mass to pions.

$\mathcal{M}=m\hat{1}$, a conjugation does not change the potential (1). For diagonal U, the latter acquires the form

$$U(\alpha,\beta)=-m\Sigma\left[\cos\left(\alpha-\frac{\theta}{3}\right)+\cos\left(\beta-\frac{\theta}{3}\right)+\cos\left(\alpha+\beta+\frac{\theta}{3}\right)\right]. \quad (2)$$

The function U has six stationary points:

$$\text{I:}\quad \alpha=\beta=0,\ \text{II:}\quad \alpha=\beta=-\frac{2\pi}{3},\ \text{III:}\quad \alpha=\beta=\frac{2\pi}{3},$$

$$\text{IV:}\quad \alpha=\beta=-\frac{2\theta}{3}+\pi,\ \text{IVa:}\quad \alpha=-\alpha-\beta=-\frac{2\theta}{3}+\pi,$$

$$\text{IVb:}\quad \beta=-\alpha-\beta=-\frac{2\theta}{3}+\pi. \quad (3)$$

The points IVa and IVb are obtained from from IV by Weyl permutations and their physical properties are the same. Actually, we have here not three distinct stationary points, but the whole four-dimensional manifold $SU(3)/[SU(2)\otimes U(1)]$ of the physically equivalent stationary points related to each other by conjugation. The values of the potential at the stationary points are

$$E_{\text{I}}=-3m\Sigma\cos\frac{\theta}{3},\quad E_{\text{II}}=-3m\Sigma\cos\frac{\theta+2\pi}{3},$$

$$E_{\text{III}}=-3m\Sigma\cos\frac{\theta-2\pi}{3},\quad E_{\text{IV}}=m\Sigma\cos\theta. \quad (4)$$

Studying expressions (4) and the matrix of the second derivatives of the potential at $\theta=\pi$, one can readily see that (i) the points I and III are the degenerate minima; (ii) the point II is the maximum, and (iii) the points IV are saddle points. When θ is slighly less than π, I is a global minimum while III is still a minimum, but of a local variety. The latter coalesces with the saddle points at $\theta=\pi/2$. At this point the eigenvalues of the second derivative matrix pass zero and, at still lower values of θ, a metastable minimum does not exist. When we instead make θ larger than π, the picture is symmetric, only the minima I and III change their roles. At $\theta=0$, the picture is exactly reversed compared to what we had at $\theta=\pi$: there is one global minimum I, two degenerate maxima II and III, and a surface of saddle points IV.

Figure 1 illustrates how the stationary points of the potential are moved when the vacuum angle is changed. One can show that metastable vacua are absent at $\theta=0$ also with physical values of masses. A metastable vacuum appears at $\theta>\pi/2$ and, as θ grows, becomes more and more deep. At $\theta=\pi$, it is not metastable anymore, its energy is the same as the energy of the ''old'' vacuum. When θ becomes still larger, the new vacuum becomes stable while the old one becomes metastable. The latter disappears at $\theta=3\pi/2$. This picture corresponds physically to the phase transition of the first order at $\theta=\pi$ where the expectation value of the meson field corresponding to the true vacuum undergoes an abrupt change.

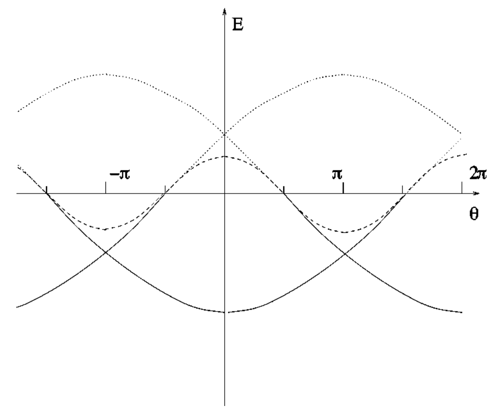

FIG. 1. Stationary point of $E(\alpha,\beta)$ for different θ. Solid lines are the minima, dotted lines are the maxima, and dashed line are the saddle points.

Two degenerate vacua at $\theta=\pi$ are separated by the domain wall.[2] To find the profile of this wall, we have to restore the kinetic term $(F_\pi^2/4)\text{Tr}\{\partial_\mu U\partial_\mu U^\dagger\}$ in the effective Lagrangian and seek for the field configurations depending only on one spatial coordinate x with the boundary conditions $U(-\infty)=1, U(\infty)=e^{2i\pi/3}$ and realizing the minimum of the energy functional

$$E=\mathcal{A}\sigma=\mathcal{A}\int_{-\infty}^{\infty}dx\left(\frac{F_\pi^2}{4}\text{Tr}\{\partial_x U\partial_x U^\dagger\}-m\Sigma\,\text{Re}[e^{i\theta/3}\text{Tr}\{U^\dagger\}]\right) \quad (5)$$

($\mathcal{A}$ is the total area factor). In our case, it suffices to seek for the solutions in the class $U=\text{diag}(e^{i\alpha(x)},e^{i\alpha(x)},e^{-2i\alpha(x)})$. Introducing $\gamma=\alpha-\pi/3$, subtracting the vacuum energy and using the Gell-Mann–Oakes–Renner relation, the expression (5) is rewritten as

$$\sigma=3F_\pi^2\int_{-\infty}^{\infty}dx\left[\frac{1}{2}\gamma'^2+\frac{M_\pi^2}{3}\left(\cos\gamma-\frac{1}{2}\right)^2\right]. \quad (6)$$

The corresponding equations of motion with the boundary conditions $\gamma(\pm\infty)=\pm\pi/3$ can be readily integrated. The first integral is

$$\gamma'=M_\pi\sqrt{2/3}\left(\cos\gamma-\frac{1}{2}\right).$$

Integrating it further, we obtain the solution

[2]The appearance of domain walls between degenerate vacua is a rather universal phenomenon. In particular such walls appear also in the supersymmetric version of QCD [6]. There are nontrivial connections between this purely field theory issue and the brane dynamics [7].

$$\cos\gamma=\frac{E^2+4E+1}{2(E^2+E+1)} \quad (7)$$

where $E=\exp\{M_\pi x/\sqrt{2}\}$. The solution (7) is centered at $x=0$ where it passes through the saddle point **IV**. There are, of course, other solutions obtained from Eq. (7) by a shift of x. Also we could have chosen the *Ansätze* $U=\mathrm{diag}(e^{i\alpha(x)},e^{-2i\alpha(x)},e^{i\alpha(x)})$ or $U=\mathrm{diag}(e^{-2i\alpha(x)},e^{i\alpha(x)},e^{i\alpha(x)})$ and obtain two other wall solutions (with the same properties) passing through the saddle points IVa and IVb. The wall surface tension is

$$\sigma=\frac{9}{\sqrt{2}}F_\pi^2 M_\pi\int_0^\infty\frac{EdE}{(E^2+E+1)^2}=3\sqrt{2}\left(1-\frac{\pi}{3\sqrt{3}}\right)M_\pi F_\pi^2. \quad (8)$$

Suppose now that $\theta=\pi+\phi$ with $0<|\phi|\ll 1$. The energies of the vacua are not degenerate anymore but are splitted apart by the value

$$\Delta\mathcal{E}\approx m\Sigma\sqrt{3}|\phi|. \quad (9)$$

A metastable vacuum should decay with the formation of bubbles of the stable phase. The quasiclassical formula for the decay rate per unit time per unit volume was derived in Ref. [8]:

$$\Gamma\propto\exp\left\{-\frac{27}{2}\pi^2\frac{\sigma^4}{(\Delta\mathcal{E})^3}\right\}, \quad (10)$$

where σ is the surface tension the bubble. Substituting here Eqs. (8), (9) [Strictly speaking, at $\theta\neq\pi$, the bubble surface tension does not coincide with Eq. (8) but is somewhat less going to zero at $\theta=\pi/2$ or $\theta=3\pi/2$; But at small $|\phi|$, the expression (8) is correct], we obtain[3]

$$\Gamma\propto\exp\left\{-\frac{CF_\pi^2}{M_\pi^2|\phi|^3}\right\} \quad (11)$$

with

$$C=2^4\times 3^5\sqrt{3}\pi^2\left[1-\frac{\pi}{3\sqrt{3}}\right]^4.$$

Note that the numerical factor C in the exponent is tremendously large, i.e., lifetime of metastable states would be tremendously large (much larger than the lifetime of the Universe) for almost all θ in the interval $(\pi/2,\ 3\pi/2)$ not too close to its boundaries (where metastable states disappear). It is a real pity that such a beautiful possibility is not realized in Nature.[4]

III. TWO FLAVORS

As was mentioned before, we have to take into account here the terms of higher order in mass in the effective potential. For $N_f=2$, it is convenient to make benefit of the fact that $SU(2)\otimes SU(2)\equiv O(4)$ and to use the four–vector notations so that $U=U_\mu\sigma_\mu=U_0+iU_i\sigma_i, U_\mu^2=1$. The 2×2 complex mass matrix involves eight real parameters which is convenient to ''organize'' in two different isotopic four-vectors[5]

$$\chi_\mu=\frac{\Sigma}{F_\pi^2}(\mathrm{Re}\ \mathrm{Tr}\{\mathcal{M}e^{i\theta/2}\},\mathrm{Im}\ \mathrm{Tr}\{\mathcal{M}e^{i\theta/2}\sigma_i\}),$$

$$\tilde{\chi}_\mu=\frac{\Sigma}{F_\pi^2}(\mathrm{Im}\ \mathrm{Tr}\{\mathcal{M}e^{i\theta/2}\},-\mathrm{Re}\ \mathrm{Tr}\{\mathcal{M}e^{i\theta/2}\sigma_i\}). \quad (12)$$

In the second order in $\chi,\tilde{\chi}$, the most general form of the potential is [10]

$$V(U_\mu)=-F_\pi^2(\chi_\mu U_\mu)-l_3(\chi_\mu U_\mu)^2-l_7(\tilde{\chi}_\mu U_\mu)^2, \quad (13)$$

where $l_{3,7}$ are some dimensionless coefficients (the coefficients $l_{1,2,4,5,6}$ multiply the structures involving the derivatives of the field U in the effective Lagrangian). The term $\propto(\chi_\mu U_\mu)(\tilde{\chi}_\mu U_\mu)$ is not allowed because it would lead to CP breaking even at $\theta=0$.

The term $\sim(\chi_\mu U_\mu)^2$ can always be neglected compared to the leading one for small masses and is not interesting. On the contrary, the term involving l_7 has a different θ dependence and, for $\mathcal{M}=m\hat{1}$ and $\theta\sim\pi$ when the leading term vanishes, determines the whole dynamics.

Before proceeding further, let us try to extract an information on the numerical value of the constant l_7. The following relation belonging to the same class as the well-known Weinberg sum rule and derived in Ref. [10] is very useful:

$$-8\left(\frac{\Sigma}{F_\pi^2}\right)^2 l_7\delta^{ik}=\int d^4x[\langle S^i(x)S^k(0)\rangle-\delta^{ik}\langle P^0(x)P^0(0)\rangle], \quad (14)$$

[3]Note the difference with the rough estimate $\ln\Gamma\sim -F_\pi^2/(M_\pi^2|\phi|^2)$ for the same quantity in Witten's paper [3]. First, one has $|\phi|^3$ rather than $|\phi|^2$ in the denominator and, second, a huge numerical factor pops up.

[4]Recently, Halperin and Zhitnitsky argued the existence of metastable states in the real QCD at $\theta=0$ [9]. However, their arguments were based on a particular model form of the effective potential incorporating also glueball degrees of freedom and involving certain cusps. The status of this potential is not quite clear yet.

[5]We use the notations of Ref. [10].

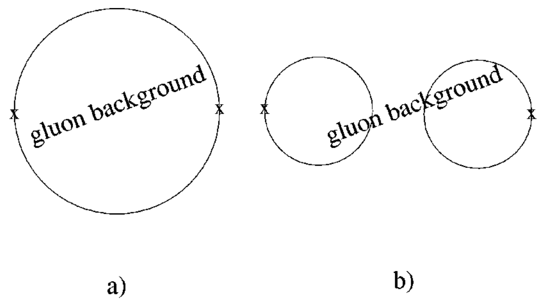

FIG. 2. Connected and disconnected contributions to the quark current correlators.

where $S^i=\overline{q}\sigma^i q$ and $P^0=i\overline{q}\gamma^5 q$. Let us calculate the right-hand side of Eq. (14) as an Euclidean functional integral. Let us first calculate the correlators in a *particular gauge field background*. Only the connected quark diagram depicted in Fig. 2(a) contributes to the correlator of the scalar isovector densities. The pseudoscalar correlator receives also a contribution from the disconnected graph in Fig. 2(b). The solid lines in Fig. 2 stand for the quark Green's functions in a particular gauge field background for which we use the spectral decomposition

$$G_A(x,y)=\sum_n \frac{\psi_n(x)\psi_n^\dagger(y)}{m-i\lambda_n}, \tag{15}$$

where λ_n are the eigenvalues of the *massless* Euclidean Dirac operator in an external gauge field and $\psi_n(x)$ are its eigenfunctions. All non-zero eigenvalues are paired: for any eigenfunction $\psi_n(x)$ with non-zero eigenvalue λ_n, $\tilde{\psi}_n(x)=\gamma^5\psi_n(x)$ is also an eigenfunction with $\tilde{\lambda}_n=-\lambda_n$. There are also zero modes. For each flavor, their number coincides with the topological charge $\nu=(1/32\pi^2)\int d^4x G^a_{\mu\nu}\tilde{G}^a_{\mu\nu}$ of the gauge field configuration.

Now, for large Euclidean volumes $mV\Sigma \gg 1$, the zero mode contribution is irrelevant for the *connected* graphs (see the discussion in Ref. [11]). On the other hand, the *only* contribution in the disconnected graph for the pseudoscalar correlator is due to zero modes. This contribution is very large $\propto \nu^2/m^2$ and is of paramount importance. Plugging in the Green's functions (15) in the correlators in Eq. (14), pairing together positive and negative λ, and integrating over gauge fields, we obtain for large volumes

$$2\left(\frac{\Sigma}{F_\pi^2}\right)^2 l_7=\frac{1}{V}\left[\left\langle \sum_n{}' \frac{m^2-\lambda_n^2}{(m^2+\lambda_n^2)^2}\right\rangle + \left\langle \sum_n{}' \frac{1}{m^2+\lambda_n^2}\right\rangle - \frac{\langle \nu^2\rangle}{m^2}\right], \tag{16}$$

where Σ_n' means the summation over positive eigenvalues only and $\langle \cdots \rangle$ stands for the gauge field averaging. The first term on the right-hand-side comes from the scalar isovector correlator, the second term is the contribution of the connected graph to the pseudoscalar correlator, and the last term is due to the disconnected graph.

Introducing the spectral density

$$\rho(\lambda)=\left\langle \sum_n \delta(\lambda-\lambda_n)\right\rangle, \tag{17}$$

the relation (16) is rewritten as

$$l_7=\frac{F_\pi^4}{\Sigma^2}\left[m^2\int_0^\infty \frac{\rho(\lambda)d\lambda}{(\lambda^2+m^2)^2}-\frac{\langle \nu^2\rangle}{2m^2V}\right]. \tag{18}$$

This relation belongs to the same class as the famous Banks and Casher relation for the fermion condensate

$$\Sigma=2m\int_0^\infty \frac{\rho(\lambda)d\lambda}{\lambda^2+m^2}=\pi\rho(0)+O(m). \tag{19}$$

For some more examples, see Refs. [12,13].

Both terms in Eq. (18) are singular $\propto 1/m$ in the chiral limit. The singularity cancels out, however. Indeed, the leading infrared contribution in the first term in square brackets is $\pi\rho(0)/(4m)\sim\Sigma/(4m)$ which is the same as for the second term due to the known result for the topological succeptibility in a theory with light quarks of the same mass [14]

$$\frac{1}{V}\langle \nu^2\rangle=\frac{m\Sigma}{N_f}+O(m^2). \tag{20}$$

The absense of singularity in the isoscalar pseudoscalar correlator means that the corresponding meson is massive: $U(1)$ problem is resolved by the 't Hooft mechanism due to fermion zero modes in topologically nontrivial gauge backgrounds. l_7 is given thereby by a constant term $\sim O(1)$ which is left out after the cancellation of singular terms. This constant is completely determined by the term $\propto m^2$ in $\langle \nu^2\rangle$. Indeed, for $N_f=2$, the spectral density is analytic at $\lambda=0$: $\rho(\lambda)=\rho(0)+\mu\lambda^2+\ldots$ [12]. The extra infrared contribution is of order $O(m)$ and can be neglected.[6] We finally obtain

$$l_7=\lim_{m\to 0}\frac{F_\pi^4}{4\Sigma^2}\left[\frac{\Sigma-2\langle \nu^2\rangle/(mV)}{m}\right]. \tag{21}$$

What can be said about the m dependence of the topological succeptibility in the next-to-leading order in mass? Let us first see what happens in a theory with two light quarks embedded in the theory involving also the third quark which is much more massive but still light enough (as is the case in the real world). The topological succeptibility in the theory with three light quarks $m_u=m_d\equiv m\ll m_s<\mu_{\rm hadr}$ is given by the expression [2,3]

[6]Note also that the spectral integral in Eq. (18) involves a logarithmic *ultraviolet* singularity at large λ where the spectral density is the same as for free fermions $\rho(\lambda)\sim\lambda^3$. It is multiplied, however, by m^2 and can be dropped out for that reason.

$$\frac{\langle \nu^2\rangle}{V}=\frac{mm_s\Sigma}{m+2m_s}=\frac{m\Sigma}{2}-\frac{m^2\Sigma}{4m_s}+O(m^3), \qquad (22)$$

i.e., the second term is *negative*[7] which means that l_7 as given by Eq. (21) is *positive*. We obtain

$$l_7=\frac{F_\pi^4}{8m_s\Sigma}\equiv\frac{F_\pi^2}{6M_\eta^2}\approx 5\times 10^{-3}. \qquad (23)$$

The same result could be obtained in a more direct way if saturating the pseudoscalar correlator in Eq. (14) by the η-meson pole [10].

The estimate (23) is quite good for the real QCD. What we are interested in here, however, is a hypothetical theory with $\theta\sim\pi$ and just two light flavors. Remarkably, an analytic result for l_7 can be obtained also in this case if the number of colors N_c is assumed to be large. For large N_c, the axial $U(1)$ symmetry is almost not affected by the anomaly which means that the η' meson is relatively light: $M_{\eta'}^2\sim\mu_{\rm hadr}^2/N_c$. We can saturate now the pseudoscalar correlator by the η' pole to obtain

$$l_7=\frac{F_\pi^2}{2M_{\eta'}^2}. \qquad (24)$$

The same result can be obtained via the relation (21). For large N_c and $N_f=2$, the topological succeptibility is known to be [15,3]

$$\frac{1}{V}\langle\nu^2\rangle=\frac{m\tau\Sigma}{2\tau+m\Sigma}=\frac{m\Sigma}{2}-\frac{m^2\Sigma^2}{4\tau}+O(m^3), \qquad (25)$$

where τ is the topoligical succeptibility in the pure Yang-Mills theory. Again, the term $\sim m^2$ in $\langle\nu^2\rangle/V$ is negative which leads to the positive l_7 which coincides with Eq. (24) due to the relation

$$F_\pi^2 M_{\eta'}^2=4\tau \qquad (26)$$

which holds in the limit $m\to 0$. We assume that l_7 is positive also for small number of colors down to $N_c=2$. Indeed, in the limit $m\to\infty$, the topological succeptibility should coincide with τ. It is natural that the series in m for small masses [the analogue of Eq. (25)] should have alternating signs. We cannot, unfortunately, formulate this statement (the positiveness of l_7) as an exact theorem though our *suspicion* is that such a theorem can somehow be proven.

We are ready now to discuss the vacua dynamics in the region $\theta\sim\pi$. Assume $U=\mathrm{diag}(e^{i\alpha},e^{-i\alpha})$. The potential (13) (with $l_3=0$) is

[7]That is quite natural, of course. The presence of an extra light quark brings about the suppression of large ν due to the extra factor m_s^ν in the fermion determinant.

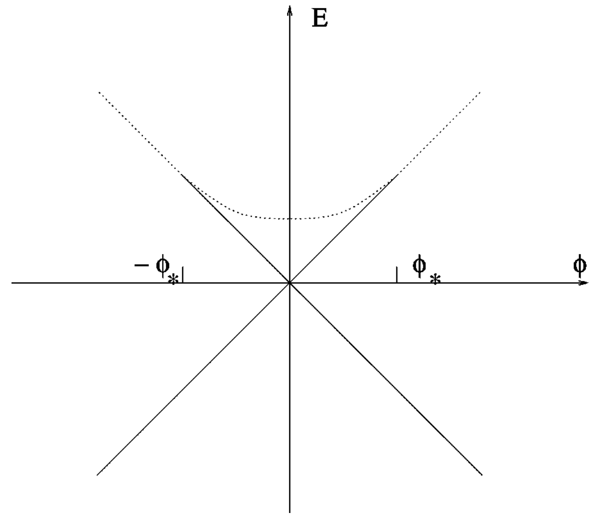

FIG. 3. Stationary point of $E(\alpha)$ for $N_f=2$ in the region of small $|\phi|=|\theta-\pi|$. Solid lines are minima and dotted lines are maxima.

$$V(\alpha)--2m\Sigma\cos\frac{\theta}{2}\cos\alpha-4l_7m^2\sin^2\frac{\theta}{2}\left(\frac{\Sigma}{F_\pi^2}\right)^2\cos^2\alpha. \qquad (27)$$

Defining $\phi=\theta-\pi$, it can be rewritten for small $|\phi|$ as

$$V(\alpha)=m\Sigma\phi\cos\alpha-4l_7m^2\left(\frac{\Sigma}{F_\pi^2}\right)^2\cos^2\alpha. \qquad (28)$$

If

$$|\phi|<\phi_*=\frac{8l_7m\Sigma}{F_\pi^4}, \qquad (29)$$

the function (28) has four stationary points:

$$\text{I}\quad \alpha=0\ \text{ with }\ E_{\mathbf{I}}=m\Sigma\phi-4l_7m^2\left(\frac{\Sigma}{F_\pi^2}\right)^2,$$

$$\text{II}\quad \alpha=\pi\ \text{ with }\ E_{\mathbf{II}}=-m\Sigma\phi-4l_7m^2\left(\frac{\Sigma}{F_\pi^2}\right)^2,$$

$$\text{III, IIIa}\quad \alpha=\pm\arccos\frac{\phi F_\pi^4}{8l_7m\Sigma}\ \text{ with }\ E_{\mathbf{III}}=\frac{F_\pi^4\phi^2}{16l_7}. \qquad (30)$$

Again, the points III, IIIa are related to each other by Weyl symmetry and we have actually the whole surface $SU(2)/U(1)\equiv S^2$ of the equivalent stationary points. Studying the second derivatives $\partial^2V/\partial\alpha^2$ for the branches (30), one readily sees that, in the region (29), the points I and II present local minima (for $\phi<0$ or $\theta<\pi$ the absolute minimum is I while II is a metastable state; for $\theta>\pi$, it is the other way round). The points III are degenerate maxima. The picture is depicted in Fig. 3. Physically, it is exactly the same as in the case $N_f=3$ and we have a first order phase transition. Only the width of the region in θ where two local minima coexist is much more narrow than in the case $N_f=3$ and goes to zero in the chiral limit $m\to 0$.

Note that with negative l_7, the picture would be reversed compared to that in Fig. 3 so that we would have in the region (29) a surface of degenerate *minima* at the points III. That corresponds to spontaneous breaking of flavor symmetry $SU_V(2)$ [5] (when $\theta\neq 0$, the contributions to the partition function coming from topologically nontrivial sectors are not positively defined, and the Vafa-Witten theorem [16] which prohibits spontaneous breaking of vector flavor symmetry at $\theta=0$ does not apply). In addition, at $\theta\neq\pi$ we would have the maxima of inequal height at the points I and II. Instead of one first order phase transition at $\theta=\pi$ we would have two consequent second order phase transitions at $\theta=\pi\pm\phi_*$. We find this picture rather unnatural and consider it as an additional argument why l_7 should be always positive.

It is not difficult now to perform the same program as for $N_f=3$ and to find the surface energy density of the domain wall at $\theta=\pi$ and the decay rate of metastable vacua at the vicinity of $\theta=\pi$. The wall configuration and the energy density at $\theta=\pi$ are obtained by minimizing the functional

$$\sigma=\int_{-\infty}^{\infty}\left[\frac{F_\pi^2}{2}(\partial_x\alpha)^2-2l_7m^2\left(\frac{\Sigma}{F_\pi^2}\right)^2(\cos 2\alpha-1)\right]dx \tag{31}$$

with the boundary conditions $\alpha(-\infty)=0$, $\alpha(\infty)=\pi$. The equations of motion have a simple solution

$$\alpha(z)=2\arctan\left[\exp\left\{\sqrt{8l_7}\,\frac{m\Sigma}{F_\pi^3}x\right\}\right]. \tag{32}$$

The surface energy density is

$$\sigma=\frac{m\Sigma}{F_\pi}\sqrt{32l_7}. \tag{33}$$

It is much lower numerically than in the case $N_f=3$ and goes to zero in the chiral limit. The rate of metastable vacuum decay at $|\phi|\ll\phi_*$ is estimated as

$$\Gamma\propto\exp\left\{-12^3\pi^2l_7^2\frac{m\Sigma}{F_\pi^4|\phi|^3}\right\}. \tag{34}$$

IV. SCHWINGER MODEL

It is very instructive to compare the situation in QCD_4 with what happens in QED_2 (the Schwinger model) with two light fermions of equal mass. The model was extensively analyzed in Ref. [17]. For zero mass, it is exactly solvable. When the mass m is not zero, but much less than the gauge coupling constant g (which carries the dimension of mass in two dimensions), a systematic expansion in the small parameter m/g (typically, in some fractional powers thereof) can be built up.

The model is *exactly* equivalent to the following bosonized model:

$$\mathcal{L}^{\rm bos}=\frac{1}{2}(\partial_\mu\phi_+)^2+\frac{1}{2}(\partial_\mu\phi_-)^2-\frac{g^2}{\pi}\left(\phi_+-\frac{\theta}{\sqrt{8\pi}}\right)^2 + C\cos\sqrt{2\pi}\phi_+\cos\sqrt{2\pi}\phi_-, \tag{35}$$

where ϕ_+ is the heavy field and ϕ_- describes light degrees of freedom. Assuming the ''conformal'' normalization for the Euclidean correlator

$$\langle :\cos\beta\phi_-(x)::\cos\beta\phi_-(0):\rangle_{C=0}=\frac{1}{2}\left(\frac{1}{|x|^{\beta^2/2\pi}}+|x|^{\beta^2/2\pi}\right) \tag{36}$$

and also

$$\langle :\cos\beta\phi_+(x):\rangle_{C=\theta=0}=1, \tag{37}$$

the constant C was calculated to be[8] [18]

$$C=\frac{mg^{1/2}e^{\gamma/2}2^{3/4}}{\pi^{5/4}}, \tag{38}$$

where γ is the Euler constant. When $m=0$, also $C=0$ so that light and heavy degrees of freedom decouple and we have just free massive Schwinger boson with the mass

$$\mu^2=\frac{2g^2}{\pi} \tag{39}$$

and a sterile massless particle. When $m\neq 0$, light ''quasi-Goldstone'' degrees of freedom acquire mass and start to interact with the heavy ones. One can write the effective Lagrangian for the light degrees of freedom which has largely the same status as the effective chiral Lagrangian in QCD. To lowest order, we can just freeze $\phi_+=\theta/\sqrt{8\pi}$, and the Lagrangian reads

$$\mathcal{L}_{\rm eff}^0=\frac{1}{2}(\partial_\mu\phi_-)^2+C\cos\frac{\theta}{2}:\cos\sqrt{2\pi}\phi_-:. \tag{40}$$

It is nothing else as the sine-Gordon model. It is exactly solved which allows one to find the vacuum energy, the fermion condensate of the original theory, and the mass spectrum. Not surprisingly, the lowest states form an isotopic triplet. A characteric mass scale of these ''pions''[9] is $\sim(C\cos\theta/2)^{2/3}\sim(m^2g\cos^2\theta/2)^{1/3}$. See Ref. [18] for the exact calculation.

When $\theta\sim\pi$, the leading term in the effective potential vanishes, however, and we are in a position to take into account higher order corrections in the Born-Oppenheimer

[8] We have changed the convention for C compared to Ref. [18] by a factor of 2.

[9] The only essential difference with the pions of QCD are that they are not true Goldstone bosons and decouple in the chiral limit. This is related to the Merman-Wigner-Coleman theorem [19] forbidding the existence of massless interacting particles and, thereby, a spontaneous breaking of a continuous symmetry in two dimensions.

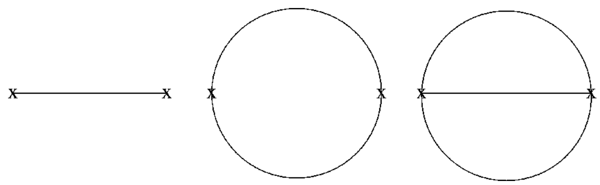

FIG. 4. Some graphs contributing in the effective lagrangian at second order. Crosses stand for the insertion of light vertex $\cos(\sqrt{2\pi}\phi_-)$ and solid lines describe heavy Schwinger bosons. The graphs with even number of the lines contribute in κ_0 and the graphs with odd number in κ_1.

parameter m/g. It is convenient to rewrite the Lagrangian (35) in terms of $\chi=\phi_+-\theta/\sqrt{8\pi}$:

$$\mathcal{L}=\frac{1}{2}(\partial_\mu\chi)^2-\frac{\mu^2}{2}\chi^2+\frac{1}{2}(\partial_\mu\phi_-)^2 + C\left[\cos\frac{\theta}{2}\cos\sqrt{2\pi}\chi-\sin\frac{\theta}{2}\sin\sqrt{2\pi}\chi\right]:\cos\sqrt{2\pi}\phi_-:. \tag{41}$$

The correction $\sim C^2\propto m^2$ in the effective potential is given by the expression

$$\Delta V(x)=-\frac{C^2}{2}:\cos\sqrt{8\pi}\phi_-(x): \left[\cos^2\frac{\theta}{2}\int d^2y|y|\langle\cos\sqrt{2\pi}\chi(y)\cos\sqrt{2\pi}\chi(0)\rangle_{C=0} + \sin^2\frac{\theta}{2}\int d^2y|y|\langle\sin\sqrt{2\pi}\chi(y)\sin\sqrt{2\pi}\chi(0)\rangle_{C=0}\right]. \tag{42}$$

This formula has the same meaning as Eq. (6.30) in the paper by Coleman [17]. We only used accurately the conformal fusion rules

$$:e^{i\beta\phi(x)}:\times:e^{i\beta\phi(0)}:=|x|^{\beta^2/2\pi}:e^{2i\beta\phi(0)}:+\cdots, \tag{43}$$

neglected the operators of higher dimension in Eq. (43) (they bring about the corrections of still higher order in m), omitted an irrelevant additive constant, and took into account *all* loops of the heavy field $\chi(x)$ drawn in Fig. 4. (Coleman only took the first graph which is much similar to the graph with the η' exchange which saturates the pseudoscalar correlator and gives the leading contribution in l_7 in QCD_4 in the large N_c limit. It is good enough for a qualitative estimate, and, as we will soon see, the account of other graphs only brings about a certain numerical factor which is very close to 1.) Expanding the integrand in $\chi(x)$ and $\chi(0)$, disregarding the ''tadpole'' contributions involving the correlators in coinciding points [they vanish due to the convention (37)], and substituting the free massive boson Green's function

$$\langle\chi(x)\chi(0)\rangle=\int\frac{e^{i\mathbf{k}\cdot\mathbf{x}}}{k^2+\mu^2}\frac{d^2k}{4\pi^2}=\frac{1}{2\pi}K_0(\mu x), \tag{44}$$

we finally obtain

$$\mathcal{L}_{\rm eff}=\frac{1}{2}(\partial_\mu\phi_-)^2+C\cos\frac{\theta}{2}:\cos\sqrt{2\pi}\phi_-: + \frac{\pi^{5/2}C^2}{\sqrt{8}g^3}\left[\kappa_0\cos^2\frac{\theta}{2}+\kappa_1\sin^2\frac{\theta}{2}\right]: \times\cos\sqrt{8\pi}\phi_-:+O(C^3), \tag{45}$$

where

$$\kappa_0=\int_0^\infty z^2dz\{\cosh[K_0(z)]-1\}=0.163,\ldots,$$

$$\kappa_1=\int_0^\infty z^2dz\sinh[K_0(z)]=1.604,\ldots. \tag{46}$$

The constant κ_0 is an exact counterpart of the constant l_3 in the effective potential (13) for QCD_4. The constant κ_1 is an exact counterpart of l_7, the quantity of primary interest for us here. We see that κ_1 is positive (and that serves as an additional argument why l_7 should be positive in QCD_4). Note that if plugging in just $K_0(z)$ instead of $\sinh[K_0(z)]$ in the integral (46) for κ_1 (*that* corresponds to taking into account only the graph with one heavy particle exchange), we would obtain $\kappa_1=\pi/2$ instead of 1.604. The effect of higher loops increases the value of κ_1 just by $\sim3\%$.

Even the analogue of the domain wall at $\theta=\pi$ exists in this nice model. In two dimensions, a domain wall is just a particle. On the classical level, it is a sine-Gordon soliton corresponding to interpolating between the points $\phi_-=0$ and $\phi_-=\sqrt{\pi/2}$. Soliton and antisoliton form an isotopic doublet.

It would be very interesting to analyze the quantum problem and to find the mass spectrum and other characteristics of the model to the order $\sim C^2$. Our impression is that it is not so easy to do at $\theta=\pi$: the problem is that $\beta=\sqrt{8\pi}$ is exactly a boundary value of the coupling. The Sine-Gordon theory with $\beta>\sqrt{8\pi}$ is sick: the Hamiltonian does not have a ground state, etc. This displays itself in the fact that the factor multiplying $:\cos\sqrt{8\pi}\phi_-:$ in Eq. (45) is dimensionless and we do not have a mass parameter out of which the soliton mass could be composed. Probably, at $\theta=\pi$, the Born-Oppenheimer expansion breaks down in this case and one has to analyze the full Lagrangian (35) by approximate methods (see Ref. [20]). The Lagrangian (45) may still be taken at the face value at other values of θ and could allow to find corrections in mass to the exact results of Ref. [18] at $\theta=0$, etc. But this is beyond the scope of the present paper.

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