PHYSICAL REVIEW D, VOLUME 59, 114014

Quarkonia and the pole mass

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(Received 15 September 1998; published 4 May 1999)

The pole mass of a heavy quark is ambiguous by an amount of order $\Lambda_{\rm QCD}$. We show that the heavy-quark potential, $V(r)$, is similarly ambiguous, but that the total static energy, $2M_{pole}+V(r)$, is unambiguous when expressed in terms of a short-distance mass. This implies that the extraction of a short-distance mass from the quarkonium spectrum is free of an ambiguity of order $\Lambda_{\rm QCD}$, in contrast with the pole mass.
[S0556-2821(99)05211-X]

PACS number(s): 14.40.Gx, 12.38.Cy, 12.39.Jh

The pole mass of a heavy quark is known to be well defined at any finite order in perturbation theory [1]. However, it is known that the pole mass of a heavy quark is an ambiguous concept at large orders in perturbation theory [2–4]. The perturbative series relating the pole mass to a short-distance mass [such as the modified minimal subtraction scheme ($\overline{\rm MS}$) mass] has coefficients which grow factorially, leading to an ambiguity of order $\Lambda_{\rm QCD}$ in the pole mass [3,4]. This factorial divergence is related to the existence of an infrared renormalon pole [5] in the Borel-transformed quark self-energy when evaluated on shell [4].

The ambiguity in the heavy-quark pole mass can be understood heuristically in terms of confinement. The pole mass of a heavy meson (bound state of a heavy quark and a light[1] antiquark) is well defined, since it is a physical quantity. To extract the pole mass of the heavy quark from the meson mass, one must subtract the binding energy, of order $\Lambda_{\rm QCD}$. However, it is impossible to define this binding energy unambiguously, since one cannot separate the light antiquark from the heavy quark, due to confinement. Hence the pole mass of a heavy quark is ambiguous by an amount of order $\Lambda_{\rm QCD}$.

Consider instead quarkonium, a bound state of a heavy quark and a heavy antiquark. If the mass of the heavy quark is sufficiently large, the strong coupling governing the heavy-quark interactions in quarkonia states with small principal quantum number is weak [6]. The quarkonium can then be handled just like positronium in electrodynamics, by summing uncrossed Coulomb ladder diagrams to yield the Schrödinger equation, and using nonrelativistic perturbation theory. Thus the low-lying quarkonia states can be described almost entirely in terms of QCD perturbation theory. It is tempting to conclude that, if the quark mass is heavy enough, the pole mass can be extracted to arbitrary precision from a perturbative calculation of the binding energy of these states, perhaps supplemented by some nonperturbative input. The goal of this article is to show that this is not the case; the heavy-quark pole mass cannot be extracted from the quarkonium spectrum with an accuracy better than order $\Lambda_{\rm QCD}$. This is an important point to clarify, since the c- and b-quark pole masses are extracted from the J/ψ [7–11] and Υ spectra [7–9,12–16].

We begin by considering the static potential between a heavy quark and a heavy antiquark in a color singlet state at leading order in QCD [17–21]:

$$V(r)=-C_F 4\pi\alpha_s\int\frac{d^3\mathbf{k}}{(2\pi)^3}\frac{e^{i\mathbf{k}\cdot\mathbf{r}}}{\mathbf{k}^2}=-C_F\frac{\alpha_s}{r}\quad (C_F=4/3), \tag{1}$$

where $\alpha_s\equiv\alpha_s(\mu)$ is the strong coupling evaluated at the scale μ. An elegant means to analyze large orders in perturbation theory is to calculate the Borel transform (with respect to $b_0\alpha_s/4\pi$) of the static potential. This calculation, performed in Ref. [22], gives[2]

$$\begin{aligned}\tilde{V}(r)&=-C_F\frac{(4\pi)^2}{b_0}\left(\frac{e^C}{\mu^2}\right)^{-u}\int\frac{d^3\mathbf{k}}{(2\pi)^3}\frac{e^{i\mathbf{k}\cdot\mathbf{r}}}{\mathbf{k}^{2(1+u)}}\\&=-C_F\frac{4}{b_0}e^{-Cu}\frac{1}{r}(\mu r)^{2u}\frac{\Gamma(\frac{1}{2}+u)\Gamma(\frac{1}{2}-u)}{\Gamma(2u+1)}\\&=C_F\frac{4}{b_0}e^{-C/2}\mu\frac{1}{(u-1/2)}+\cdots,\end{aligned} \tag{2}$$

where u is the Borel parameter, C is a renormalization-scheme-dependent constant ($C=-5/3$ in the $\overline{\rm MS}$ scheme), and

$$b_0\equiv 11-\frac{2}{3}N_f \tag{3}$$

is the one-loop beta-function coefficient. The infrared renormalon pole nearest the origin, at $u=1/2$, controls the

[1] $m<\Lambda_{\rm QCD}$.

[2] An analysis of quarkonia energies at large orders in perturbation theory based on the position-space potential $V(r)=-C_F[\alpha_s(1/r)/r]$ may be found in Ref. [23].

asymptotic behavior of the perturbation series. This pole and its residue are made explicit in the last line of Eq. (2).

The static potential is recovered from Eq. (2) by inverse Borel transformation:

$$V(r)=\int_0^\infty du\, e^{-4\pi u/(b_0\alpha_s)}\tilde{V}(r). \tag{4}$$

The evaluation of this integral is impeded by the presence of infrared renormalon poles in $\tilde{V}(r)$ on the positive real axis at all half-integer values of u. The ambiguity in the integral is dominated by the infrared renormalon pole closest to the origin, at $u=1/2$. Estimating the ambiguity as half the magnitude of the difference between deforming the integration contour above and below the pole yields

$$\delta V(r)\sim C_F\frac{4\pi}{b_0}e^{-C/2}\mu e^{-2\pi/(b_0\alpha_s)}\sim C_F\frac{4\pi}{b_0}e^{-C/2}\Lambda_{\rm QCD}, \tag{5}$$

where the one-loop renormalization-group equation for α_s has been used to obtain the final expression. The ambiguity is a constant shift of the potential, by an amount of order $\Lambda_{\rm QCD}$.

The ambiguity of order $\Lambda_{\rm QCD}$ in the static potential was first derived in Ref. [22], but no attempt was made to interpret it. We now show that this ambiguity is due to the (implicit) use of the pole mass, which is sensitive to momenta of order $\Lambda_{\rm QCD}$, in the definition of the static potential.

The relation between the heavy-quark pole mass, M_{pole}, and a short-distance mass, M, is given in Borel space by [4]

$$\begin{aligned}\tilde{M}_{pole}&=M\left(\delta(u)+\frac{C_F}{b_0}\left[\left(\frac{M^2}{\mu^2}\right)^{-u}\right.\right.\\&\qquad\left.\left.\times e^{-Cu}6(1-u)\frac{\Gamma(u)\Gamma(1-2u)}{\Gamma(3-u)}-\frac{3}{u}+R_{\Sigma_1}(u)\right]\right)\\&=-C_F\frac{2}{b_0}e^{-C/2}\mu\frac{1}{(u-\frac{1}{2})}+\cdots,\end{aligned} \tag{6}$$

where $R_{\Sigma_1}(u)$ is a renormalization-scheme-dependent regular function. The infrared renormalon pole closest to the origin is again at $u=1/2$, and its residue is displayed in the last line of Eq. (6). The ambiguity in the pole mass can be estimated as above for the static potential, and is given by [4]

$$\delta M_{pole}\sim C_F\frac{2\pi}{b_0}e^{-C/2}\Lambda_{\rm QCD}, \tag{7}$$

as is well known.

Consider now the total static energy of a heavy quark and a heavy antiquark in a color singlet state with spatial separation r, given by the sum of the static potential energy and the rest mass of the particles. In Borel space,

$$\tilde{E}_{static}(r)=2\tilde{M}_{pole}+\tilde{V}(r). \tag{8}$$

If the pole mass is eliminated in favor of a short-distance mass via Eq. (6), we see that the infrared renormalon pole at $u=1/2$ in the total static energy is cancelled. Thus the total static energy does not have an ambiguity of order $\Lambda_{\rm QCD}$ when expressed in terms of a short-distance mass.

The total static energy of the heavy-quark–antiquark pair in a color-singlet state is an unambiguous concept [24,17]. However, when one attempts to separate this energy into the sum of the static potential energy and the pole masses of the quark and antiquark [17–21], the ambiguity of order $\Lambda_{\rm QCD}$ in the pole mass results in a corresponding ambiguity in the static potential. This is the source of the ambiguity of order $\Lambda_{\rm QCD}$ in the static potential. The ambiguity disappears when the pole mass is replaced with a short-distance mass. This well-known phenomenon occurs for many other processes in QCD [3,4,25–30].

Because it will be useful to us when we consider the full dynamical (nonstatic) quarkonium calculation, let us recall that an analysis of the infrared behavior can also be carried out via a finite gluon mass λ [3,25,31]. The static potential becomes

$$V(r)=-C_F\frac{\alpha_s}{r}e^{-\lambda r}=-C_F\alpha_s\left(\frac{1}{r}-\lambda+\cdots\right). \tag{9}$$

The term of order λ corresponds to the infrared renormalon pole at $u=1/2$. The relation between the pole mass and a short-distance mass is [3,25,31]

$$M_{pole}=M-C_F\frac{\alpha_s}{2}\lambda+\cdots. \tag{10}$$

We see that the term linear in the gluon mass cancels when we calculate the total static energy, $2M_{pole}+V(r)$, in terms of a short-distance mass. This corresponds to the cancellation of the pole at $u=1/2$ in the Borel transform of the total static energy, Eq. (8).

The full dynamical (nonstatic) quarkonium calculation at leading order in the nonrelativistic approximation requires solving the Schrödinger equation

$$\left(-\frac{\nabla^2}{M_{pole}}+V(r)-E\right)G(\mathbf{r},0,E)=\delta^{(3)}(\mathbf{r}), \tag{11}$$

where $G(\mathbf{r},0,E)$ is the Schrödinger-equation Green function and $E\equiv\sqrt{s}-2M_{pole}$ is the binding energy. The total center-of-mass energy $\sqrt{s}$ is physical and unambiguous, but the binding energy E is not, since it requires subtracting twice the pole mass. Eliminating E in favor of $\sqrt{s}$ in Eq. (11) gives

$$\begin{aligned}&\left(-\frac{\nabla^2}{M_{pole}}+2M_{pole}+V(r)-\sqrt{s}\right)G(\mathbf{r},0,\sqrt{s}-2M_{pole})\\&\quad=\delta^{(3)}(\mathbf{r}).\end{aligned} \tag{12}$$

The quarkonia masses correspond to the poles of the Green function in the $\sqrt{s}$ plane. We see that the total static energy, $2M_{pole}+V(r)$, now appears in the Schrödinger equation. As shown above, the total static energy is not ambiguous by an amount of order $\Lambda_{\rm QCD}$, but due to the ambiguity in $V(r)$, the pole mass cannot be extracted from the quarkonium spectrum with an accuracy better than order Λ_{QCD}. However, the static potential can be made free of the ambiguity of order Λ_{QCD} if the pole mass is expressed in terms of a short-distance mass and if twice the difference between the pole and the short-distance mass is absorbed into the static potential,

$$\left(-\frac{\nabla^2}{M_{pole}}+2M+\hat{V}(r)-\sqrt{s}\right)\hat{G}(\mathbf{r},0,\sqrt{s}-2M)=\delta^{(3)}(\mathbf{r}), \tag{13}$$

where

$$\hat{V}(r)=V(r)+2(M_{pole}-M). \tag{14}$$

Hence the accuracy with which a short-distance mass can be extracted from the quarkonium spectrum is not limited by order Λ_{QCD}.

The pole mass also appears in the denominator of the kinetic-energy term in the Schrödinger equation. However, replacing this mass with a short-distance mass only affects terms suppressed by powers of α_s. This can be seen most easily by using the gluon mass as an infrared regulator, Eqs. (9) and (10). The linear gluon mass terms which cancel in the total static energy are of order $\alpha_s\lambda$. Since the kinetic-energy term is of order $M_{pole}\alpha_s^2$, the linear gluon mass term generated by replacing M_{pole} with a short-distance mass is of order $\alpha_s^3\lambda$.

In this paper we have shown that the heavy-quark pole mass cannot be extracted from the quarkonium spectrum with an accuracy better than order Λ_{QCD}. This is relevant for the determination of the *c*- and *b*-quark pole masses from the J/ψ [7–11] and Υ spectra [7–9,12–16]. However, the accuracy with which a properly-defined short-distance mass can be extracted from the quarkonium spectrum is not limited by order Λ_{QCD}.

Note added. The results of this paper have also been arrived at in a recent paper [32].

We are grateful for conversations with A. El-Khadra. A.H.H. was supported in part by the U. S. Department of Energy under contract No. DOE DE-FG03-90ER40546. M.S., T.S., and S.W. were supported in part by the U. S. Department of Energy under contract No. DOE DE-FG02-91ER40677. M.S. was supported in part by a grant from the UIUC Campus Research Board.

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