Operator analysis of $l=1$ baryon masses in large N_c QCD

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We consider in detail the mass operator analysis for the nonstrange $l=1$ excited baryons in large N_c QCD. We present a straightforward procedure for constructing the large N_c baryon wave functions, and provide complete analytic expressions for the matrix elements of all the independent isosinglet mass operators. We discuss the relationship between the old-fashioned operator analyses based on nonrelativistic SU(6) symmetry and the modern large N_c approach, which has a firmer theoretical foundation. We then suggest a possible dynamical interpretation for the subset of operators preferred strongly by the data. [S0556-2821(99)07809-1]

I. INTRODUCTION

Although the QCD gauge coupling is numerically too large to permit a perturbative expansion at low energies, QCD generalized to N_c colors admits a consistent perturbative expansion in terms of $1/N_c$ [1]. Effective theories for baryons have been constructed that take into account the symmetries and power counting rules of large N_c QCD, allowing baryon observables to be computed to any desired order in the $1/N_c$ expansion. The large N_c approach has been applied with great success to the ground state baryons which fill the SU(6) **56**-plet, including studies of SU(6) spin-flavor symmetry [2–6], baryon masses [4,7–9], magnetic moments [4,8–12], and axial vector current matrix elements [2,4,8,12].

Whether the large N_c framework works equally well in describing the phenomenology of excited baryon multiplets is a question under active investigation. Recent attention has focused on the $l=1$ orbitally excited baryons, the SU(6) **70**-plet for $N_c=3$. The first application of large N_c to excited baryons was a phenomenological analysis of the strong decays [13]. This was followed by a series of more formal papers on the strong decays and axial vector current matrix elements [14,15], as well as on the matrix elements of the mass operators relevant at lowest nontrivial order [16]. Recently, the first phenomenological study of the electromagnetic transitions was presented [17,18], while a phenomenological analysis of the nonstrange $l=1$ baryon masses, including corrections up to relative order $1/N_c^2$, was undertaken by the present authors [19]. This is the subject of further consideration in the present work.

A number of issues not addressed in Ref. [19] are considered here. First, we explain how the nonstrange baryon states are constructed for arbitrary N_c. Our construction differs from that of Ref. [14], and we believe it is somewhat more transparent. After obtaining rules for simplifying the baryon operator analysis, which is essential for a proper counting of degrees of freedom in fits to observables, we present complete analytic expressions for the matrix elements of all isosinglet mass operators relevant to the orbitally excited baryons, as functions of the excited baryon quantum numbers. This presentation is relevant not only for obtaining the large

N_c results presented in our earlier work but also for identifying operator relationships holding only to leading order in $1/N_c$. For example, the matrix elements of some of the operators are linearly dependent in the $N_c \to \infty$ limit, even though the matrix elements are independent for $N_c=3$. Thus, two operators that appear naively to be of leading order in $1/N_c$ may in fact produce only one leading-order linear combination. The operator basis presented here is thus slightly improved over that of Ref. [19]. We present numerical results omitted from Ref. [19] for reasons of space, namely, fits to mass eigenvalues in which the mixing angles are predicted. We also consider the physical interpretation of our effective field theory results. It was shown in Ref. [19] that only two nontrivial operators have numerically substantial coefficients when fits to the nonstrange $l=1$ mass spectrum are performed, and this in itself is suggestive of some specific dynamical mechanism. In this work we attempt to characterize the dynamics producing these results.

This paper is organized as follows. In Sec. II we review the formulation of the large N_c operator analysis for excited baryons. In Sec. III we describe in detail the construction of the baryon states in large N_c. Sections IV and V discuss operator reduction rules and construction of the operator basis relevant to the mixed-symmetry **70**-plet states. In Sec. VI we present numerical results not included in our prior work. In Sec. VII we compare our results to model-independent analyses of the past and, in Sec. VIII, to phenomenological models. Section IX summarizes our conclusions.

II. FRAMEWORK

The observed baryons have the appropriate quantum numbers to be assigned to irreducible representations of the group SU(6)×O(3). Here SU(6) contains the spin and flavor symmetry group SU(2)×SU(3), and O(3) generates spatial rotations. We define "quarks" q as fields in the (**2**,**3**) representation of the spin-flavor group. An appropriately symmetrized collection of N_c quarks has the quantum numbers of a large N_c baryon. For $N_c=3$, states constructed in this way have the same quantum numbers as those observed in nature.

If all quarks were much heavier than Λ_{QCD}, then one

could identify the fields above as the valence quarks of the nonrelativistic quark model. Here, however, we make no such assumption. Our quark fields simply provide a convenient tensor product space in which one can define baryons with the correct total quantum numbers. The baryon wave functions can be expressed as tensors, with separate indices for the spin and flavor degrees of freedom for each quark. In the nonrelativistic quark model, all spin-flavor transformations of the baryon tensors are accomplished by acting on these indices with elements of the group SU(6), which is an exact symmetry of the theory in the limit $m_q \rightarrow \infty$, where m_q is the quark mass. In the present case, we cannot (and do not) assume that SU(6) is a good symmetry, since the quarks are light, but rather simply parametrize the complete breaking of SU(6) by allowing symmetry-breaking matrices to act on the quark spin and flavor indices. One achieves the most general breaking of quark spin and flavor symmetries by using polynomials in the SU(6) generators

$$\left(\frac{\sigma^i}{2} \otimes 1 \right), \quad \left(1 \otimes \frac{\tau^a}{2} \right), \quad \left(\frac{\sigma^i}{2} \otimes \frac{\tau^a}{2} \right), \quad (2.1)$$

where σ^i are the usual Pauli matrices. The τ^a are either Pauli or Gell-Mann matrices, depending on whether one is interested in two or three quark flavors. We focus on the two-flavor case in our operator analysis. By acting on the quark spin and flavor indices of a baryon wave function, the tensors above parametrize the breaking of the corresponding symmetries. Within a large N_c baryon multiplet, there are *always* some states for which these symmetry breaking effects are maximal. For example, consider the ground state baryons, which form a tower of states with spins ranging from 1/2 to $N_c/2$. The fact that the large N_c multiplet contains states with spins of order N_c implies that spin-spin interactions like

$$\frac{1}{N_c} S^2 \equiv \frac{1}{N_c} \sum_{\substack{\text{quarks} \\ \alpha, \beta}} \frac{\vec{\sigma}_\alpha}{2} \cdot \frac{\vec{\sigma}_\beta}{2} \quad (2.2)$$

shift some baryon mass eigenvalues at order N_c. (The reason for the $1/N_c$ prefactor is explained below.) For example, for the stretched case of a baryon with spin $N_c/2$, this matrix element evaluates to $1/N_c \times N_c/2 \times (N_c/2+1)$. On the other hand, the mean mass of the multiplet scales as N_c, since there are N_c quarks in a baryon state. Thus there are always spin-dependent splittings somewhere in the multiplet that are comparable to the average multiplet mass. While this prevents us from speaking of SU(6) as an approximate symmetry, it is nonetheless true that the breaking of this would-be symmetry is a small effect on states of small total spin. Since the physical, nonstrange baryons are chosen to have fixed total spin and isospin eigenvalues in the large N_c limit, it follows that matrix elements of σ^i/N_c and τ^a/N_c summed over all quarks are of order $1/N_c$, and hence can be treated as small numbers. Thus, this parametrization of the complete breaking of SU(6) provides an operator basis that is hierarchical in $1/N_c$ on the physical baryon states. This fact allows the construction of an effective theory for baryons that is both complete and predictive.

Let us now specify the large N_c counting rules more precisely. We define an n-body operator as one that acts on n quark lines in a large N_c baryon state. Since we work in an effective theory, we arrive at a complete operator basis independent of any specific dynamical assumptions beyond that of QCD as the underlying theory. An n-body operator has a coefficient $1/N_c^{n-1}$, reflecting the minimum $n-1$ gluon exchanges necessary to generate the operator in QCD. However, the overall effect of an operator on a given baryon observable is determined not only by the size of the operator coefficient, but also by the compensating factors of N_c that may arise when a spin-flavor generator is summed over the N_c quark lines in a baryon state. As discussed earlier, the generators σ^i and τ^a sum incoherently over N_c quark lines since the spin and isospin eigenvalues for the physical baryons are of order 1, even when one extrapolates to large N_c. The generator $\sigma^i \tau^a$, however, sums coherently, as is shown later by explicit computation [see Eq. (A1)]. Thus, the contribution of an n-body operator to a given baryon observable is of order N_c^{1+m-n}, where m is the number of times the generator $\sigma^i \tau^a$ appears. Given the set of all operators constructed by combining the generators in (2.1), linearly dependent operators of higher order can often be eliminated by use of operator reduction rules. For the ground state baryons, these rules were formalized by Dashen, Jenkins and Manohar [8]; the generalization to excited baryons is considered in some detail in Sec. IV.

The discussion above generalizes in a straightforward way to $l=1$ baryons with one orbitally excited quark. In the large N_c limit, such baryons consist of one distinguishable, excited quark in the collective potential generated by $N_c - 1$ ground state quarks. One defines separate SU(6) generators that act on the excited quark and on the nonexcited "core" quarks, respectively. In addition, one introduces the orbital angular momentum generators l^i to parametrize the breaking of O(3). Mass operators relevant to the $l=1$ baryons are formed by contracting generators in this extended set, as we discuss in Sec. IV. Again, an operator hierarchy is obtained after taking into account the factors of $1/N_c$ that appear in operator coefficients and the compensating factors of N_c that arise from coherent sums over the $O(N_c)$ ground state core quarks.

III. STATES

The defining feature of baryon states filling the mixed-symmetry negative-parity SU(6) **70**-plet is that the sole unit of orbital angular momentum is carried by the excited quark relative to the other two ground-state core quarks. The core quarks are separately symmetrized on spin-flavor and spatial indices, while the $l=1$ excited quark is antisymmetrized with respect to the other two. This construction produces the **70**-dimensional representation of SU(6), and is phenomenologically relevant: Every negative-parity baryon with mass less than 2 GeV has the appropriate spin, isospin, and strangeness quantum numbers to belong to a single **70**-plet, although some of the strange baryons needed to fill the **70** have not yet been observed. If one focuses upon nonstrange states alone, as is done in this work, then the relevant mul-
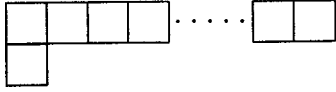
FIG. 1. Young diagram for the SU($2F$) mixed symmetry representation, the multiplet containing large N_c orbitally excited baryons with $l=1$. The top row has N_c-1 boxes.

tiplet becomes a **20** of SU(4), for which all component spin and isospin multiplets have been seen.

The mixed symmetry baryon multiplet is generalized to $N_c>3$ by symmetrizing now among N_c-1 core quarks, as indicated by the Young diagram in Fig. 1. Although this extrapolation is not unique, it is the most natural in preserving symmetry properties familiar from $N_c=3$. Total symmetry of the core is also the essential ingredient rendering the study of the orbitally excited baryons tractable in large N_c, since it greatly reduces the number of degrees of freedom. In particular, the symmetry properties of core states are completely specified by their total strangeness, spin, and isospin. For nonstrange cores, the situation is even simpler: Owing to the total symmetry of the spin-flavor state, spin and isospin are equal in this case. The core state is denoted by

$$|S_c=I_c;m_1,\alpha_1\rangle, \qquad (3.1)$$

where m's and α's here and below denote projections of spin and isospin, respectively, and the subscript c denotes core. The excited quark state is denoted

$$|1/2;m_2,\alpha_2\rangle. \qquad (3.2)$$

Finally, the orbital O(3) eigenstate is labeled in obvious notation by

$$|l,m_l\rangle. \qquad (3.3)$$

Of course, physical states are labeled by total spin J,J_3 and isospin I,I_3. The states we construct here also admit separate specification of the total spin S carried by the quarks. Nonstrange mixed-symmetry SU(6) states with one spin-1/2, isospin-1/2 quark singled out have total quark spin and isospin related by $S=I$ or $I\pm1$, with each of S and I in the range $1/2$–$N_c/2$. The sole exception is that there are no mixed-symmetry $S=I=N_c/2$ states. Let us define $\rho\equiv S-I=\pm1,0$ and $\eta/2\equiv I_c-I=\pm1/2$. Then obtaining the desired state by coupling the spins and isospins is achieved, by construction, by the use of Clebsch-Gordan coefficients:

$$|JJ_3;II_3(l,S=I+\rho)\rangle$$
$$= \sum_{m_l,m_1,\alpha_1,\eta} \begin{pmatrix} l & S & J \\ m_l & m & J_3 \end{pmatrix}$$
$$\times \begin{pmatrix} S_c & 1/2 & S \\ m_1 & m_2 & m \end{pmatrix} \begin{pmatrix} I_c & 1/2 & I \\ \alpha_1 & \alpha_2 & I_3 \end{pmatrix}$$
$$\times c_{\rho,\eta}|S_c=I_c=I+\eta/2;m_1,\alpha_1\rangle$$
$$\otimes|1/2;m_2,\alpha_2\rangle\otimes|l,m_l\rangle. \qquad (3.4)$$

States with strangeness are defined analogously, except that SU(3) Clebsch-Gordan coefficients appear in that case. The

only notation in this expression yet undefined is the coefficient $c_{\rho\eta}$; it simply represents that more than one irreducible SU(6) or SU(4) representation can occur in the product of the (N_c-1)-quark core and the one-quark excited state, and so the numbers $c_{\rho\eta}$ represent elements of orthogonal basis rotations. In the present case, elementary manipulations show that only the totally symmetric and mixed-symmetry representations result. Since $S-S_c=\pm1/2$, one has $c_{\pm,\mp}=0$ for any multiplet. All nonstrange states in the symmetric representation have $S=I$, and thus $c_{\pm,\pm}^{\mathrm{SYM}}=0$, $c_{\pm,\pm}^{\mathrm{MS}}=1$. The only complicated mixing occurs for $c_{0,\pm}$, and we obtain the mixing by means of a trick: The symmetric and mixed-symmetry multiplets possess different quadratic SU(4) or SU(6) Casimir invariants, and thus one may compute the value of the Casimir operator both on the full state on the left-hand side of Eq. (3.4), where it assumes a known value (see next section), or on the separate core and excited states on the right-hand side of Eq. (3.4) using the matrix elements presented in Appendix A. After a straightforward calculation, one finds, for the $S=I$ nonstrange states,

$$c_{0+}^{\mathrm{MS}}=+\sqrt{\frac{S[N_c+2(S+1)]}{N_c(2S+1)}} \quad \text{and}$$

$$c_{0-}^{\mathrm{MS}}=-\sqrt{\frac{(S+1)(N_c-2S)}{N_c(2S+1)}}, \qquad (3.5)$$

and the coefficients for symmetric states are the orthogonal combination, $c_{0+}^{\mathrm{SYM}}=-c_{0-}^{\mathrm{MS}}$, $c_{0-}^{\mathrm{SYM}}=c_{0+}^{\mathrm{MS}}$.

IV. REDUCTIONS

There are numerous operator identities or operator reduction rules which are known for the ground state baryons and which can be used to eliminate many operator products from lists of candidate independent operators. The identities are not general to all representations, but work when applied to ground state baryons. The proofs of many of them depend upon the symmetry of the ground state.

In this section we study operator reductions applicable to the mixed-symmetry **70**-plet. Technical details are provided in Appendix B. To put our findings in context, recall that the operator reductions for the ground state come from three sources. Two of them are the quadratic and cubic Casimir identities. The third comes because matrix elements of an operator between a state and its conjugate state are zero if the operator does not belong to a representation that can be found in (for the ground state with $N_c=3$) $\overline{\mathbf{56}}\otimes\mathbf{56}$. There are products of two generators of SU(6), or rather certain sums of products of these generators, that belong to representations not found in $\overline{\mathbf{56}}\otimes\mathbf{56}$. Those sums are then zero, and this is the third source of operator identities. Reference [8] investigates whether further identities can be found involving products of three generators, and shows that the answer is negative.

The basic operators that we start with are the generators of SU($2F$), given in a quark basis as

$$S^i \equiv q_\alpha^\dagger \left(\frac{\sigma^i}{2} \otimes 1 \right) q^\alpha,$$

$$T^a \equiv q_\alpha^\dagger \left(1 \otimes \frac{\tau^a}{2} \right) q^\alpha,$$

$$G^{ia} \equiv q_\alpha^\dagger \left(\frac{\sigma^i}{2} \otimes \frac{\tau^a}{2} \right) q^\alpha, \qquad (4.1)$$

where σ^i and τ^a are the spin and flavor matrices. The collected and properly normalized $SU(2F)$ adjoint representation one-body operators $S^i/\sqrt{F}$, $T^a/\sqrt{2}$, and $\sqrt{2}G^{ia}$ satisfy an $SU(2F)$ algebra like that of their underlying spin-flavor generators.[1] Other operators $\mathcal{O}$ can, since we are just interested in their group theoretical behavior, be built from products of these generators [8].

For the **70**-plet, first note that the mixed-symmetry representation consists of a symmetric core plus one excited quark. If one defines, in analogy with Eqs. (4.1), separate one-body operators S_c, T_c, G_c acting on the core and s, t, g on the excited quark line, then the operator reduction rules for the ground state [8] may be used on the core operators. The only difference is that $N_c \rightarrow N_c - 1$ in the core identities, to account for the different numbers of quarks present.

For the **70**-plet overall, we find that the quadratic Casimir identity leads to a new operator reduction rule. Unfortunately, the other two sources of identities for the ground state lead to no identities for the **70**-plet. However, there are some identities that come from considering products of three currents.

The quadratic Casimir identity for an arbitrary $SU(2F)$ representation R reads

$$\{q^\dagger \Lambda^A q, q^\dagger \Lambda^A q\} \equiv 2C_2(R)1, \qquad (4.2)$$

where Λ^A are the spin-flavor generators in the representation R. For the mixed-symmetry representations we are looking at, denoted MS_{N_c}, the Casimir invariant may be shown to be (see [20])

$$C_2(MS_{N_c}) = \frac{N_c}{4F}[N_c(2F-1) + 2F(2F-3)]. \qquad (4.3)$$

In the mixed-symmetry case, Λ^A is the sum of core and excited generators (i.e., $T = T_c + t$, etc.), and

$$C_2(S_1) = \frac{1}{4F}(4F^2 - 1),$$

$$C_2(S_{N_c-1}) = \frac{1}{4F}(N_c - 1)(N_c + 2F - 1)(2F - 1), \qquad (4.4)$$

where S_{N_c-1} is the symmetric representation with $N_c - 1$ quarks and S_1 is just the fundamental representation of a single quark. This means that the quadratic Casimir identity for MS_{N_c} can be expressed as

$$2sS_c + FtT_c + 4FgG_c = -\frac{1}{2}(N_c + 2F - 1), \qquad (4.5)$$

so that the operator gG_c may always be eliminated in favor of sS_c and tT_c.

The cubic Casimir identity reads,

$$d^{ABC}(q^\dagger \Lambda^A q)(q^\dagger \Lambda^B q)(q^\dagger \Lambda^C q) \equiv C_3(R)1. \qquad (4.6)$$

For products of two generators contracted with d^{ABC}, one may write

$$d^{ABC}(q^\dagger \Lambda^B q)(q^\dagger \Lambda^C q) \equiv \frac{C_3(R)}{C_2(R)} q^\dagger \Lambda^A q + X^A(R), \qquad (4.7)$$

where X^A is that part of the two-body combination on the left hand side annihilated by contraction with $q^\dagger \Lambda^A q$. For completely symmetric representations, $X^A = 0$, as was shown explicitly in [8]; one can show the same for completely antisymmetric representations. In such cases, one may derive a number of operator reduction rules. However, X^A need not be zero for arbitrary representations, since nothing guarantees that all spin-flavor combinations of the quark operators $(q^\dagger q)(q^\dagger q)$ reduce to a single $(q^\dagger q)$ for a representation of arbitrary symmetry properties. We have found explicitly that $X^A \neq 0$ for the mixed-symmetry representation by computing several matrix elements containing both sides of Eq. (4.7). One concludes that no two-body operator reduction rules follow from the cubic Casimir relation for the mixed-symmetry representation.

Of course, the true cubic Casimir relation (4.6) holds in general. We have investigated it for the mixed-symmetry representation, and find no new operator relations, but rather the Casimir identity

$$\frac{2}{3}\left[\frac{C_3(MS_{N_c}) - C_3(S_{N_c-1}) - C_3(S_1)}{C_2(MS_{N_c}) - C_2(S_{N_c-1}) - C_2(S_1)} \right]$$

$$= \frac{C_3(S_{N_c-1})}{C_2(S_{N_c-1})} + \frac{C_3(S_1)}{C_2(S_1)}, \qquad (4.8)$$

which can indeed be verified, using the previous quadratic Casimir identities and

$$C_3(MS_{N_c}) = \frac{N_c}{4F^2}(F-1)(N_c + 2F)[N_c(2F-1)$$

$$+ F(2F-7)],$$

$$C_3(S_{N_c-1}) = \frac{1}{4F^2}(N_c-1)(N_c+2F-1)$$

$$\times (N_c+F-1)(2F-1)(F-1),$$

$$C_3(S_1) = \frac{1}{4F^2}(F^2-1)(4F^2-1). \qquad (4.9)$$

Regarding the last source of operator identities for the symmetric case, the statement for the mixed-symmetry case is simple. All representations that appear in a product of two one-body operators also appear in the product $\overline{\mathrm{MS}} \otimes \mathrm{MS}$. No additional operator identities follow.

For the mixed-symmetry representation at the three-body level, there are two large representations (called $\bar{b}b_0$ and adj_3 in Appendix B) that annihilate the baryon states. However, one can show that the operators in our list that could have overlap with these representations are all independent when acting on the physical baryon states. Thus, no further operator reduction rules occur for the flavor-singlet mass operators.

The summary of operator reduction rules for the mixed-symmetry representation nonstrange baryons therefore reads as follows: Decompose the mixed-symmetry generators into sums of separate core and excited quark pieces as labeled above. One may apply the operator reduction rules of [8] to the core generators alone, and one may also eliminate gG_c.

V. COUNTING OPERATORS

The building blocks from which one forms operators relevant to $l=1$ baryons consists of the core operators S_c^i, T_c^a, and G_c^{ia}, the excited quark operators s^i, t^a, and g^{ia}, and the orbital angular momentum operator l^i. The mixed-symmetry representation baryons have orbital quantum number $l=1$, and therefore the only required combinations of l^i are $1(\Delta l=0)$, $l^i(\Delta l=1)$, and the $\Delta l=2$ tensor

$$l^{(2)ij} \equiv \frac{1}{2}\{l^i,l^j\} - \frac{l^2}{3}\delta^{ij}. \qquad (5.1)$$

Since the physical $N_c=3$ baryons have only two core valence quarks, one need only consider operators that involve up to two core quarks in the large N_c analysis. The operator reduction rules of [8] state that one may eliminate all core contractions on flavor indices using δ^{ab}, d^{abc}, or f^{abc}, or on spin indices in two G_c's using δ^{ij} or ϵ^{ijk}.

We construct in this paper the complete set of time-reversal even, rotationally invariant, isosinglet operators for the nonstrange excited baryons. There are 18 such independent operators. [Incidentally, for three flavors there are 20 operators. The difference between the two cases is that for two flavors one has an additional operator reduction

$$S_c^i G_c^{ia} = \frac{1}{4}(N_c+1)T_c^a. \qquad (5.2)$$

TABLE I. The 18 linearly independent spin-singlet flavor-singlet operators for $F=2$, organized by powers of $1/N_c$ in their matrix elements. For $F>2$, and ignoring possible coherence in matrix elements of T_c^a, one must include $(1/N_c^2)tS_cG_c$ and $(1/N_c^2)l^ig^{ia}S_c^jG_c^{ja}$ in row N_c^{-1}.

Order of matrix element	Operator		
N_c^1	$N_c\mathbb{1}$		
N_c^0	$ls,$	$\dfrac{1}{N_c}ltG_c,$	$\dfrac{1}{N_c}l^{(2)}gG_c,$
N_c^{-1}	$\dfrac{1}{N_c}tT_c,\quad \dfrac{1}{N_c}lS_c,\quad \dfrac{1}{N_c}lgT_c,\quad \dfrac{1}{N_c}S_c^2,\quad \dfrac{1}{N_c}sS_c,$		
	$\dfrac{1}{N_c}l^{(2)}sS_c,\quad \dfrac{1}{N_c^2}l^{(2)}t\{S_c,G_c\},\quad \dfrac{1}{N_c^2}l^ig^{ja}\{S_c^j,G_c^{ia}\}$		
N_c^{-2}	$\dfrac{1}{N_c^2}(lS_c)(tT_c),\quad \dfrac{1}{N_c^2}gS_cT_c,\quad \dfrac{1}{N_c^2}l^{(2)}S_cS_c,$		
	$\dfrac{1}{N_c^2}l^{(2)}gS_cT_c,\quad \dfrac{1}{N_c^2}\{lS_c,sS_c\},\quad \dfrac{1}{N_c^2}(ls)S_c^2$		

For more than two flavors, the operators tS_cG_c and $l^ig^{ia}S_c^jG_c^{ja}$ must be included]. For the 18 operators surviving for two flavors the explicit power of N_c for a given operator is determined by using the large N_c counting given in Sec. II. Factors of $1/N_c^{n-1}$ are included in the definition of the operators, as can be seen in Table I. The full large N_c counting of the matrix elements is $O(N_c^{1-n+m})$, where m is the number of times the coherent operator G_c^{ia} appears. (For more than two flavors, T_c^a is also potentially coherent.) In Table I we have organized the operators by the overall order of their matrix elements in the $1/N_c$ expansion. Note that the nonstrange **70**-plet baryons require 7 masses and 2 mixing angles, so that matrix elements of 9 operators of the 18 shown are necessarily linearly dependent upon the other 9 when restricted to these states.

Furthermore, the analysis here is carried out for arbitrary values of N_c, and the matrix elements of a given operator are usually not homogeneous in N_c. It can happen that matrix elements of a given set of operators are linearly independent for $N_c=3$ but dependent for other values, in particular $N_c\to\infty$. This turns out to be the case for $\langle ls\rangle$ and $\langle ltG_c\rangle$, which are both $O(N_c^0)$, but $\langle ls+4ltG_c/(N_c+1)\rangle$ is $O(1/N_c)$, so that only one of the original two truly represents an independent $O(N_c^0)$ operator. This result is dependent on the particular states (here nonstrange baryons) used for evaluating the matrix elements. Since no operator has been eliminated, such a result is not an operator reduction, but rather what we call an operator demotion.

In our analysis of the masses and mixing angles of nonstrange baryons, we begin with the leading operators $N_c\mathbb{1}$, ls, and $l^{(2)}gG_c/N_c$ (see Table I), which are independent for both $N_c=3$ and $N_c\to\infty$. We then add subsets of the

8 operators appearing at $O(1/N_c)$ in Table I plus the demoted $O(1/N_c)$ combination $ls + 4ltG_c/(N_c+1)$, in search of a complete set of 9 independent operators. A number of subsets consisting of 6 such $O(1/N_c)$ operators complete the basis acting upon the baryon states for $N_c = 3$. As one can show by considering all possibilities, at least one of these operators is linearly dependent for $N_c \to \infty$. This means that one combination of the $O(1/N_c^1)$ operators can be demoted to $O(1/N_c^2)$. Using the labels of Table II, we choose $\mathcal{O}_9 \equiv (N_c+1)/N_c\mathcal{O}_4 + \mathcal{O}_5 + 8l^i g^{ja}\{S_c^j, G_c^{ia}\}/N_c^2$, which has $O(1/N_c^2)$ matrix elements. This gives us an optimal basis, which we define as a basis where the number of demoted operator combinations is maximized.[2]

The set of operators we choose, along with their matrix elements computed for the nonstrange $l = 1$ baryon states, is presented in Table II. This set is identical to that in Ref. [19], except that we replace tT_c/N_c by sS_c/N_c, and $l^i g^{ja}\{S_c^j, G_c^{ia}\}/N_c^2$ by the demoted operator defined immediately above. Table III presents, for completeness, the matrix elements of the remaining 9 operators.

VI. NUMERICAL ANALYSIS

The nine mass parameters of the nonstrange $l = 1$ baryons appearing at $N_c = 3$ consist of diagonal elements of two isospin-3/2 states, $\Delta_{1/2}$ and $\Delta_{3/2}$, and five isospin-1/2 states, $N_{1/2}$, $N'_{1/2}$, $N_{3/2}$, $N'_{3/2}$, and $N'_{5/2}$; here the subscript indicates total baryon spin, while total quark spin is indicated by the absence (1/2) or presence (3/2) of a prime. To round out the set of mass parameters, observe that there is one mixing angle for $N'_{1/2}$-$N_{1/2}$ and one for $N'_{3/2}$-$N_{3/2}$.

In Ref. [19] we showed that fits of these nine mass parameters lead to an unexpected result: Only a few of the coefficients of the effective Hamiltonian turn out to be of a natural size (namely, about a few hundred MeV), with the rest being anomalously small or even consistent with zero. This analysis was performed with certain particular sets of operators that did not fully take into account the demotions described above, and one may wonder whether these results were a fluke resulting from an unfortunate choice of basis. In the current work we possess rules for obtaining optimal demoted sets of operators as described in the previous section, and have found that fits using a number of such different choices lead to similar results. In particular, with the same mass eigenvalues and mixing angles as in [19] and the operator basis listed in Table II, one obtains the coefficients c_i defined by the relations

$$M_j = \sum_{i=1}^{9} c_i \langle \mathcal{O}_i \rangle_j, \qquad (6.1)$$

where $j = 1, \ldots, 9$ represent mass bilinears, the rows of Table II. The results of this inversion are presented in Table IV. One sees that this fit is nearly identical to that of Table III in [19], in particular that the operators $\mathbb{1}$, $l^{(2)}gG_c/N_c$, and S_c^2/N_c again appear to be by far the most significant. Replacing tT_c by sS_c and using the demoted $1/N_c^2$ combination $\mathcal{O}_9$ has little effect except to drastically decrease coefficient uncertainties in some cases.

The chief implication of Table IV is to reinforce confidence in the fits given in Ref. [19]: The operators chosen were not completely optimal, but nevertheless represent the optimal choice quite well. Moreover, the operators used in the other fits (Tables III, IV, V) in [19] are the same as in the basis used here, and therefore direct comparisons between those fits and this work are immediate.

In fact, the only other fits we wish to present here are those in which the mixing angles are neither taken from pion decays [13] nor photoproduction data [17], but rather make use only of the seven measured mass eigenvalues and predict the mixing angles. Tables V, VI, and VII are the analogues to Tables III, IV, and V in [19], respectively. Note particularly the following features: In Table V, it is again seen that the three leading operators $\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3$ in the $1/N_c$ (orders N_c^1 and N_c^0 only) give a poor accounting for the data, even when including only mass eigenvalues; furthermore, the predicted mixing angles are nowhere near the experimental values from [13] or [17]. This is no surprise, since one expects the next order corrections to be of the same order as the mass splittings. Indeed, when three additional operators $\mathcal{O}_{4,5,6}$, with matrix elements of $O(1/N_c)$, are included (Table VI), the situation becomes much better: In addition to an excellent χ^2/N_{DF} of 0.23, one finds that the mixing angles predicted from a mass analysis naturally approach the values obtained from decays. Nevertheless, only $\mathcal{O}_1$, $\mathcal{O}_3$, and $\mathcal{O}_6$ appear significant; what if one performs a fit using only those three operators? The answer is in Table VII. Here the results are most surprising: Now the operator $\mathcal{O}_3$ actually adjusts its coefficient to give a *small* contribution; the $\chi^2/N_{\mathrm{DF}} = 0.73$ is not bad, but while the prediction for the spin-3/2 angle is excellent, the prediction for the spin-1/2 angle is off by about 2σ. Even though $\mathcal{O}_3$ now looks insignificant, it is actually required to give nonzero values to the mixing angles, for observe from Table II that $\mathcal{O}_1$ and $\mathcal{O}_6$ do not contribute to mixing.

Also, neither $\mathcal{O}_3$ nor $\mathcal{O}_6$ contribute to the mass splitting $\Delta_{3/2}$-$\Delta_{1/2}$. Among the $O(1/N_c)$ or larger operators, only the spin-orbit terms split the Δ_J. In fact, the main effect of the spin-orbit terms is to split the Δ_J states; they also contribute to the nucleon mixing, but their effect on the nucleon masses is slight because of cancellations. (The coefficients of the two spin-orbit terms have opposite signs, unlike what would be expected from a single overall spin-orbit term $l \cdot S = l \cdot s + l \cdot S_c$.) So while the spin-orbit terms are small compared to $1/N_c$ expectations, they do have some importance and one may expect that the errors in the two coefficients are corre-

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[2]Beginning with the three leading-order operators, there are numerous other choices for the remaining six that provide an operator basis that is linearly independent for $N_c = 3$ and rank 8 for $N_c \to \infty$. Using the operator definitions in Tables II and III, and letting $\mathcal{O}'_9 \equiv l^i g^{ja}\{S_c^j, G_c^{ia}\}$, one can check that all such sets contain $\mathcal{O}_6$ and $\mathcal{O}'_9$, one of $\mathcal{O}_7$ and $\mathcal{O}_{11}$, one of $\mathcal{O}_8$ and $\mathcal{O}_{12}$, and two of $\mathcal{O}_4$, $\mathcal{O}_5$, and $\mathcal{O}_{10}$. An optimal basis can be formed by taking appropriate linear combinations.

TABLE II. Matrix elements $\langle \mathcal{O}_i \rangle_j$ of 9 operators, labeled as $\mathcal{O}_1, \mathcal{O}_2, \ldots, \mathcal{O}_9$, respectively, that are linearly independent for $N_c = 3$. The third and sixth rows correspond to off-diagonal matrix elements.

	$\langle \mathcal{O}_1 \rangle$	$\langle \mathcal{O}_2 \rangle$	$\langle \mathcal{O}_3 \rangle$	$\langle \mathcal{O}_4 \rangle$	$\langle \mathcal{O}_5 \rangle$	$\langle \mathcal{O}_6 \rangle$	$\langle \mathcal{O}_7 \rangle$	$\langle \mathcal{O}_8 \rangle$	$\langle \mathcal{O}_9 \rangle$
	$N_c\langle 1 \rangle$	$\langle ls \rangle$	$\frac{1}{N_c}\langle l^{(2)}gG_c \rangle$	$\left\langle ls + \frac{4}{N_c+1} ltG_c \right\rangle$	$\frac{1}{N_c}\langle lS_c \rangle$	$\frac{1}{N_c}\langle S_c^2 \rangle$	$\frac{1}{N_c}\langle sS_c \rangle$	$\frac{1}{N_c}\langle l^{(2)}sS_c \rangle$	$\frac{N_c+1}{N_c}\langle \mathcal{O}_4 \rangle + \langle \mathcal{O}_5 \rangle + \frac{8}{N_c^2}\langle l^i g^{ja}\{S_c^j, G_c^{ia}\} \rangle$
$N_{1/2}$ $\quad N_c$		$-\frac{1}{3N_c}(2N_c-3)$	0	$+\frac{2}{N_c+1}$	$-\frac{1}{3N_c^2}(N_c+3)$	$+\frac{1}{2N_c^2}(N_c+3)$	$-\frac{1}{4N_c^2}(N_c+3)$	0	$-\frac{1}{3N_c^3}(17N_c-3)$
$N'_{1/2}$ $\quad N_c$		$-\frac{5}{6}$	$-\frac{5}{48N_c}(N_c+1)$	0	$-\frac{5}{3N_c}$	$+\frac{2}{N_c}$	$+\frac{1}{2N_c}$	$+\frac{5}{6N_c}$	$+\frac{5}{3N_c^2}$
$N'_{1/2}$-$N_{1/2}$ $\quad 0$		$-\frac{1}{3}\sqrt{\frac{N_c+3}{2N_c}}$	$-\frac{5}{48N_c}\sqrt{\frac{N_c+3}{2N_c}}(2N_c-1)$	$-\frac{1}{N_c+1}\sqrt{\frac{N_c+3}{2N_c}}$	$+\frac{1}{3N_c}\sqrt{\frac{N_c+3}{2N_c}}$	0	0	$+\frac{5}{12N_c}\sqrt{\frac{N_c+3}{2N_c}}$	$-\frac{1}{3N_c^2}\sqrt{\frac{N_c+3}{2N_c}}$
$N_{3/2}$ $\quad N_c$		$+\frac{1}{6N_c}(2N_c-3)$	0	$-\frac{1}{N_c+1}$	$+\frac{1}{6N_c^2}(N_c+3)$	$+\frac{1}{2N_c^2}(N_c+3)$	$-\frac{1}{4N_c^2}(N_c+3)$	0	$+\frac{1}{6N_c^3}(17N_c-3)$
$N'_{3/2}$ $\quad N_c$		$-\frac{1}{3}$	$+\frac{1}{12N_c}(N_c+1)$	0	$-\frac{2}{3N_c}$	$+\frac{2}{N_c}$	$+\frac{1}{2N_c}$	$-\frac{2}{3N_c}$	$+\frac{2}{3N_c^2}$
$N'_{3/2}$-$N_{3/2}$ $\quad 0$		$-\frac{1}{6}\sqrt{\frac{5(N_c+3)}{N_c}}$	$+\frac{1}{96N_c}\sqrt{\frac{5(N_c+3)}{N_c}} \times (2N_c-1)$	$-\frac{1}{2(N_c+1)}\sqrt{\frac{5(N_c+3)}{N_c}}$	$+\frac{1}{6N_c}\sqrt{\frac{5(N_c+3)}{N_c}}$	0	0	$-\frac{1}{24N_c}\sqrt{\frac{5(N_c+3)}{N_c}}$	$-\frac{1}{6N_c^2}\sqrt{\frac{5(N_c+3)}{N_c}}$
$N'_{5/2}$ $\quad N_c$		$+\frac{1}{2}$	$-\frac{1}{48N_c}(N_c+1)$	0	$+\frac{1}{N_c}$	$+\frac{2}{N_c}$	$+\frac{1}{2N_c}$	$+\frac{1}{6N_c}$	$-\frac{1}{N_c^2}$
$\Delta_{1/2}$ $\quad N_c$		$+\frac{1}{3}$	0	0	$-\frac{4}{3N_c}$	$+\frac{2}{N_c}$	$-\frac{1}{N_c}$	0	$+\frac{4}{3N_c^2}$
$\Delta_{3/2}$ $\quad N_c$		$-\frac{1}{6}$	0	0	$+\frac{2}{3N_c}$	$+\frac{2}{N_c}$	$-\frac{1}{N_c}$	0	$-\frac{2}{3N_c^2}$

TABLE III. As in Table II, for operators labeled as $\mathcal{O}_{10}, \mathcal{O}_{11}, \ldots, \mathcal{O}_{18}$.

	$\langle\mathcal{O}_{10}\rangle$	$\langle\mathcal{O}_{11}\rangle$	$\langle\mathcal{O}_{12}\rangle$	$\langle\mathcal{O}_{13}\rangle$	$\langle\mathcal{O}_{14}\rangle$	$\langle\mathcal{O}_{15}\rangle$	$\langle\mathcal{O}_{16}\rangle$	$\langle\mathcal{O}_{17}\rangle$	$\langle\mathcal{O}_{18}\rangle$
	$\frac{1}{N_c}\langle lgT_c\rangle$	$\frac{1}{N_c}\langle tT_c\rangle$	$\frac{1}{N_c^2}\langle l^{(2)}t\{S_c,G_c\}\rangle$	$\frac{1}{N_c^2}\langle (ls)S_c^2\rangle$	$\frac{1}{N_c^2}\langle\{lS_c,sS_c\}\rangle$	$\frac{1}{N_c^2}\langle (lS_c)(tT_c)\rangle$	$\frac{1}{N_c^2}\langle gS_cT_c\rangle$	$\frac{1}{N_c^2}\langle l^{(2)}S_cS_c\rangle$	$\frac{1}{N_c^2}\langle l^{(2)}gS_cT_c\rangle$
$N_{1/2}$	$-\frac{1}{12N_c^2}(N_c+3)$	$-\frac{1}{4N_c^2}(N_c+3)$	0	$+\frac{1}{6N_c^3}(N_c+3)$	$+\frac{2}{3N_c^3}(N_c+3)$	$+\frac{1}{3N_c^3}(N_c+3)$	$+\frac{1}{4N_c^3}(N_c+3)$	0	0
$N'_{1/2}$	$+\frac{5}{6N_c}$	$-\frac{1}{N_c}$	$-\frac{5}{24N_c^2}(N_c+1)$	$-\frac{5}{3N_c^2}$	$-\frac{5}{3N_c^2}$	$+\frac{5}{3N_c^2}$	$-\frac{1}{2N_c^2}$	$+\frac{5}{6N_c^2}$	$-\frac{5}{6N_c^2}$
$N'_{1/2}$-$N_{1/2}$	$+\frac{1}{3N_c}\sqrt{\frac{N_c+3}{2N_c}}$	0	$+\frac{5}{24N_c^2}\sqrt{\frac{N_c+3}{2N_c}}(N_c+1)$	$-\frac{2}{3N_c^2}\sqrt{\frac{N_c+3}{2N_c}}$	$-\frac{1}{6N_c^2}\sqrt{\frac{N_c+3}{2N_c}}$	$-\frac{1}{3N_c^2}\sqrt{\frac{N_c+3}{2N_c}}$	0	$-\frac{5}{6N_c^2}\sqrt{\frac{N_c+3}{2N_c}}$	$-\frac{5}{12N_c^2}\sqrt{\frac{N_c+3}{2N_c}}$
$N_{3/2}$	$+\frac{1}{24N_c^2}(N_c+3)$	$-\frac{1}{4N_c^2}(N_c+3)$	0	$-\frac{1}{12N_c^3}(N_c+3)$	$-\frac{1}{3N_c^3}(N_c+3)$	$-\frac{1}{6N_c^3}(N_c+3)$	$+\frac{1}{4N_c^3}(N_c+3)$	0	0
$N'_{3/2}$	$+\frac{1}{3N_c}$	$-\frac{1}{N_c}$	$+\frac{1}{6N_c^2}(N_c+1)$	$-\frac{2}{3N_c^2}$	$-\frac{2}{3N_c^2}$	$+\frac{2}{3N_c^2}$	$-\frac{1}{2N_c^2}$	$-\frac{2}{3N_c^2}$	$+\frac{2}{3N_c^2}$
$N'_{3/2}$-$N_{3/2}$	$+\frac{1}{6N_c}\sqrt{\frac{5(N_c+3)}{N_c}}$	0	$-\frac{1}{48N_c^2}\sqrt{\frac{5(N_c+3)}{N_c}}(N_c+1)$	$-\frac{1}{3N_c^2}\sqrt{\frac{5(N_c+3)}{N_c}}$	$-\frac{1}{12N_c^2}\sqrt{\frac{5(N_c+3)}{N_c}}$	$-\frac{1}{6N_c^2}\sqrt{\frac{5(N_c+3)}{N_c}}$	0	$+\frac{1}{12N_c^2}\sqrt{\frac{5(N_c+3)}{N_c}}$	$+\frac{1}{24N_c^2}\sqrt{\frac{5(N_c+3)}{N_c}}$
$N'_{5/2}$	$-\frac{1}{2N_c}$	$-\frac{1}{N_c}$	$-\frac{1}{24N_c^2}(N_c+1)$	$+\frac{1}{N_c^2}$	$+\frac{1}{N_c^2}$	$-\frac{1}{N_c^2}$	$-\frac{1}{2N_c^2}$	$+\frac{1}{6N_c^2}$	$-\frac{1}{6N_c^2}$
$\Delta_{1/2}$	$+\frac{1}{6N_c}$	$+\frac{1}{2N_c}$	0	$+\frac{2}{3N_c^2}$	$+\frac{8}{3N_c^2}$	$-\frac{2}{3N_c^2}$	$-\frac{1}{2N_c^2}$	0	0
$\Delta_{3/2}$	$-\frac{1}{12N_c}$	$+\frac{1}{2N_c}$	0	$-\frac{1}{3N_c^2}$	$-\frac{4}{3N_c^2}$	$+\frac{1}{3N_c^2}$	$-\frac{1}{2N_c^2}$	0	0

TABLE IV. Operator coefficients in GeV, assuming the complete set of Table II. The vertical divisions separate operators whose contributions to the baryon masses are of orders N_c^1, N_c^0, N_c^{-1}, and N_c^{-2}, respectively.

c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9
$+0.463$	-0.036	$+0.369$	$+0.087$	$+0.086$	$+0.438$	-0.040	$+0.048$	$+0.001$
±0.020	±0.041	±0.208	±0.097	±0.080	±0.102	±0.074	±0.172	±0.084

lated. Reduced experimental error bars on the Δ_J states would clarify the role of and need for the spin-orbit terms.

We conclude from the results presented here and in Ref. [19] that the large N_c operator analysis reproduces both the experimentally measured masses and the mixing angles extracted from the strong and electromagnetic decays. We have shown here that fits to the mass eigenvalues alone may be used to predict these angles successfully, and have found that this result holds, to varying degree, in both six and three operator fits. These fits reveal that the χ^2 function is shallow with respect to the mixing angles, so that a small χ^2 is obtained in Table VII using only three operators, even when the mixing angle predictions begin to diverge from the decay analysis results. Our conclusions are unaffected by our choice of operator basis, which differs from that of Ref. [19].

VII. VINTAGE SU(6) ANALYSES

Operator analyses of baryon masses were performed long before the $1/N_c$ expansion was proposed. A main difference between modern work and the older work is that one can estimate the importance of each operator by the order in $1/N_c$ at which it contributes to the mass. Inevitably, there are other differences as well. In this section, we contrast what we have done with some of the early work.

Greenberg and Resnikoff [21] (GR) led the way in performing an analysis based on SU(6), and were later joined by Horgan and Dalitz [22] (HD). Additionally, there was work on numerical fits to the baryon mass spectrum separately from those papers that laid out the operators. At a minimum in this context, we should mention the work in Refs. [23–26]. The last of these papers also corrected some small (as it turned out) numerical errors in the previous analyses. All the analyses make the assumption that only one- and two-body operators enter. For the nonstrange members of the **70**-plet,

the early analyses found 5 independent operators, three of which are independent of the orbital angular momentum l, one linear in l, and one quadratic in l. Five operators is many fewer than we use. We need to explain how the differences come about. We will use the notation of GR and give a brief reprise of their logic.

Note before starting that GR and HD use wave functions that involve only relative position coordinates, whereas we use Hartree or independent particle wave functions, that is, wave functions relative to a fixed center of mass. The Hartree wave functions are exact in the $N_c \rightarrow \infty$ limit. This leads to some difference in reckoning what is a one-body, two-body, or three-body operator. For example, we consider l and $l \cdot s$ as one body operators. The equivalent in GR or HD would be a sum over quarks α of $L_\alpha \cdot \sigma_\alpha$, where L_α is interpreted [22] as the orbital angular momentum of one quark with respect to the center of mass of the others. They would consider this a three body operator, and do not use it. The differences between the older work and the present work due to this point of counting are least apparent in operators with no factors of angular momentum, and we turn first to them.

For one body terms, one needs operators that have matrix elements between the **6** and $\overline{\mathbf{6}}$ of SU(6), and one knows that

$$\mathbf{6} \otimes \overline{\mathbf{6}} = \mathbf{1} \oplus \mathbf{35}. \tag{7.1}$$

Looking for suitable spin-0 operators on the right hand side, there is only the T_1^1. The notation is $T_{\text{dimSU}(6)}^{\text{dimSU}(3)}$, and we only consider SU(3) singlets since we are considering neither strangeness nor isospin breaking. For two-body operators, we first note that

$$\mathbf{6} \otimes \mathbf{6} = \mathbf{15} \oplus \mathbf{21}, \tag{7.2}$$

where the **15** is antisymmetric and the **21** is symmetric. Then we examine the product

TABLE V. Three parameter fit using operators $\mathcal{O}_{1,2,3}$, giving $\chi^2/N_{\text{DF}} = 6.89/4 = 1.72$. The operators included formally yield the lowest order nontrivial contributions to the masses in the $1/N_c$ expansion. Masses are given in MeV, angles in radians. Experimental data for angles here and below are for comparison purposes and not used for fitting. Parameters (GeV): $c_1 = 0.542 \pm 0.002$, $c_2 = 0.075 \pm 0.009$, $c_3 = -0.437 \pm 0.051$.

	Fit	Expt.		Fit	Expt.
$\Delta(1700)$	1615	1720 ± 50	$N(1520)$	1520	1523 ± 8
$\Delta(1620)$	1653	1645 ± 30	$N(1535)$	1562	1538 ± 18
$N(1675)$	1677	1678 ± 8	θ_{N1} (pred)	2.47 ± 0.04	0.61 ± 0.09
$N(1700)$	1674	1700 ± 50	θ_{N3} (pred)	2.65 ± 0.03	3.04 ± 0.15
$N(1650)$	1666	1660 ± 20			

TABLE VI. Six parameter fit using operators $\mathcal{O}_{1,\ldots,6}$, giving $\chi^2/N_{\mathrm{DF}}=0.24/1=0.24$. Masses are given in MeV, angles in radians. Parameters (GeV): $c_1=0.466\pm0.014$, $c_2=-0.030\pm0.039$, $c_3=0.304\pm0.142$, $c_4=0.068\pm0.101$, $c_5=0.062\pm0.046$, $c_6=0.424\pm0.086$.

	Fit	Expt.		Fit	Expt.
$\Delta(1700)$	1699	1720 ± 50	$N(1520)$	1522	1523 ± 8
$\Delta(1620)$	1643	1645 ± 30	$N(1535)$	1538	1538 ± 18
$N(1675)$	1678	1678 ± 8	θ_{N1} (pred)	0.53±0.29	0.61±0.09
$N(1700)$	1712	1700 ± 50	θ_{N3} (pred)	3.06±0.24	3.04±0.15
$N(1650)$	1660	1660 ± 20			

$$\mathbf{21}\otimes\overline{\mathbf{21}}=\mathbf{1}\oplus\mathbf{35}\oplus\mathbf{405} \tag{7.3}$$

for suitable operators, finding another T_1^1 and a T_{405}^1. Similarly, the product

$$\mathbf{15}\otimes\overline{\mathbf{15}}=\mathbf{1}\oplus\mathbf{35}\oplus\mathbf{189} \tag{7.4}$$

yields still one more T_1^1 and a T_{189}^1. All the T_1^1's give equivalent results for a given multiplet; so we are left with 3 independent spin-0 mass operator candidates, namely T_1^1, T_{189}^1, and T_{405}^1.

On our list, we have four one- and two-body operators that contain no orbital angular momentum. They are

$$\mathbb{1},S_c^2,s\cdot S_c \quad\text{and}\quad t\cdot T_c. \tag{7.5}$$

From our viewpoint, there is a further, tacit, assumption made by the earlier authors [21,22]: Their two-body operators do not distinguish between, in our language, two S-wave quarks and an S-wave and P-wave pair. This implies that whatever physics leads to the $\sigma_i\cdot\sigma_j$ terms in the effective mass operator would give the same coefficient for any pair of quarks, whatever their wave functions. If so, the coefficients of our S_c^2 and $s\cdot S_c$ terms would not be independent, and we would have the same number of independent spin-0 operators as GR or HD. Indeed, with explicit matrix elements given by GR, we can verify the linear dependence of our operators upon theirs or vice-versa.

Next we look for spin-1 operators that can be dotted into the orbital angular momentum to give rotationally invariant operators. GR and HD only consider angular momentum

TABLE VII. Three parameter fit using operators $\mathcal{O}_1$, $\mathcal{O}_3$, and $\mathcal{O}_6$, giving $\chi^2/N_{\mathrm{DF}}=2.93/4=0.73$. Masses are given in MeV, angles in radians. Parameters (GeV): $c_1=0.457\pm0.005$, $c_3=0.088\pm0.198$, $c_6=0.459\pm0.032$.

	Fit	Expt.		Fit	Expt.
$\Delta(1700)$	1678	1720 ± 50	$N(1520)$	1525	1523 ± 8
$\Delta(1620)$	1678	1645 ± 30	$N(1535)$	1524	1538 ± 18
$N(1675)$	1676	1678 ± 8	θ_{N1} (pred)	0.11±0.23	0.61±0.09
$N(1700)$	1688	1700 ± 50	θ_{N3} (pred)	3.11±0.07	3.04±0.15
$N(1650)$	1668	1660 ± 20			

which is the relative angular momentum of a quark pair. Since unit angular momentum requires antisymmetry, GR and HD use only the antisymmetric [in SU(6)] **15** two-quark combination, and find only the operator T_{35}^1. Perusing our list, we find 3 operators at the one- or two-body level that use l once:

$$l\cdot s,l\cdot S_c \quad\text{and}\quad lgT_c. \tag{7.6}$$

The question of how to connect our l to their angular momentum operator returns. If l is the angular momentum of one quark relative to the overall center of mass, it is a three-body operator, as discussed earlier, and thus would be discarded by the early authors. For us, l is the angular momentum with respect to the center of mass, and we can interpret part of it as the angular momentum of the excited quark with respect to one particular core quark. Then, matching to the earlier authors, $l\cdot s$ and $l\cdot S_c$ would have the same coefficient by arguments already made, if S_c is taken to refer to a quark in that pair (and if not, it would be a three-body term). Regarding our third term, again following GR or HD, we would apply it only to antisymmetric subsets of quarks, and for either purely symmetric [8] or purely antisymmetric quark states one can prove a result [a consequence of Eq. (4.7)] that

$$g^{ia}T_c^a\propto S_c^i+s^i. \tag{7.7}$$

Thus, the third spin-1 operator in Eq. (7.6) becomes dependent upon the first two.

Similarly, spin-2 operators that can be combined with the $\Delta l=2$ part of l^il^j come from the symmetric **21** in the earlier authors's analysis. Here, they find only an operator T_{405}^1. We have two operators at the two- or fewer-body level, which are

$$l^{(2)}gG_c \quad\text{and}\quad l^{(2)}sS_c. \tag{7.8}$$

But again, if we ignore differences between quarks and recall that GR or HD would only let the operators act on symmetric states, there are operator reduction rules stating[3]

$$g^{ia}G_c^{ja}\propto s^iS_c^j \tag{7.9}$$

for the spin-2 piece, and again only the core quark that appears in the pair under discussion is meant above. Hence, in this view, we would have one operator also.

Thus, if we make GR's or HD's assumptions, we get their results. However, our analysis is more general and relies only on an organizing principle suggested by the underlying theory. On the practical side, GR did not use the tensor operator in their fits, on the grounds that there was not enough data at that time to justify one more operator. We found that this operator was quite important. They did, however, find that the spin-orbit operators had small coefficients [25], a result that was confirmed by Isgur and Karl [27].

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[3]To be explicit, this is the third identity from the bottom of Table VI in [8].

VIII. DYNAMICAL INTERPRETATION

The most striking feature of our analysis is that the non-strange $l=1$ mass spectrum is described adequately by two nontrivial operators,

$$\frac{1}{N_c}l^{(2)}gG_c \quad \text{and} \quad \frac{1}{N_c}S_c^2. \tag{8.1}$$

Clearly, large N_c power counting is not sufficient by itself to explain the $l=1$ baryon masses—the underlying dynamics plays a crucial role. In this section, we simply point out that the preferred set of operators in Eq. (8.1) can be understood in a constituent quark model with a single pseudoscalar meson exchange, up to corrections of order $1/N_c^2$. The argument goes as follows:

The pion couples to the quark axial-vector current so that the $\bar{q}q\pi$ coupling introduces the spin-flavor structure $\sigma^i\tau^a$ on a given quark line. In addition, pion exchange respects the large N_c counting rules given in Sec. II. A single pion exchange between the excited quark and a core quark corresponds to the operators

$$g^{ia}G_c^{ja}l_{ij}^{(2)} \tag{8.2}$$

and

$$g^{ia}G_c^{ia} \tag{8.3}$$

while pion exchange between two core quarks yields

$$G_c^{ia}G_c^{ia}. \tag{8.4}$$

These exhaust the possible two-body operators that have the desired spin-flavor structure (since $l^{(2)}G_cG_c$ is a three-body operator). The first operator is one of the two in our preferred set. The third operator may be rewritten

$$2G_c^{ia}G_c^{ia}=C_2\cdot1-\frac{1}{2}T_c^aT_c^a-\frac{1}{2}S_c^2 \tag{8.5}$$

where C_2 is the SU(4) quadratic Casimir identity for the totally symmetric core representation [the **10** of SU(4) for $N_c=3$]. Since the core wave function involves two spin and two flavor degrees of freedom, and is totally symmetric, it is straightforward to show that $T_c^2=S_c^2$. Then Eq. (8.5) implies that one may exchange $G_c^{ia}G_c^{ia}$ in favor of the identity operator and S_c^2, the second of the two operators suggested by our fits.

The remaining operator, $g^{ia}G_c^{ia}$, is peculiar in that its matrix element between two nonstrange, mixed symmetry states is given by

$$\frac{1}{N_c}\langle gG\rangle=-\frac{N_c+1}{16N_c}+\delta_{S,I}\frac{I(I+1)}{2N_c^2}, \tag{8.6}$$

which differs from the identity only at order $1/N_c^2$. Thus, to order $1/N_c$, one may make the replacements

$$\{1,g^{ia}G_c^{ja}l_{ij}^{(2)},g^{ia}G_c^{ia},G_c^{ia}G_c^{ia}\}\Rightarrow\{1,g^{ia}G_c^{ja}l_{ij}^{(2)},S_c^2\}. \tag{8.7}$$

We conclude that the operator set suggested by the data may be understood in terms of single pion exchange between quark lines. This is consistent with the interpretation of the mass spectrum advocated by Glozman and Riska [28]. Other simple models, such as single gluon exchange, do not directly select the operators suggested by our analysis and may require others that are disfavored by the data.

IX. SUMMARY AND CONCLUSIONS

We have considered what the large N_c expansion tells about the masses of the nonstrange P-wave excited baryons. We have given the effective mass operator by enumerating all the independent operators that it could contain, and ordered those operators by their size in the $1/N_c$ expansion. We have calculated the matrix elements of each of the operators for any N_c. For the effective mass operator, we have fit the coefficients of the individual operators to the data, using the masses given by the Particle Data Group [29] and after truncating the full set of operators in suitable and reasonable ways.

We find that one can fit the masses well using selected subsets of the full list of operators, and that the good fits have mixing angles that are compatible with the mixing angles that come from analyses of the mesonic and radiative decays of these baryons [13,17,30]. Estimating the size of each operator using the $1/N_c$ scheme works, in the sense that no operator is larger than expected based on those estimates. Some operators are smaller. In fact, we can get a decent fit keeping just the unit operator, one tensor operator, and the core spin-squared operator. This is compatible with the idea that the underlying dynamics is due to effective pseudoscalar meson exchanges among the quarks [28], and not easily compatible with the idea that the masses splittings are explained by single gluon exchange.

ACKNOWLEDGMENTS

APPENDIX A: EXPLICIT MATRIX ELEMENTS

Using the notation for quantum numbers defined in Eq. (3.4), we first present matrix elements of the SU(6) generators between completely symmetric core states:

$$\langle S'_c = I'_c; m'_1, \alpha'_1 | G^{ia}_c | S_c = I_c; m_1, \alpha_1 \rangle = \frac{1}{4} \sqrt{\frac{2I_c+1}{2I'_c+1}} \sqrt{(N_c+1)^2 - (I'_c - I_c)^2 (I'_c + I_c + 1)^2} \begin{pmatrix} S_c & 1 & S'_c \\ m_1 & i & m'_1 \end{pmatrix} \begin{pmatrix} I_c & 1 & I'_c \\ \alpha_1 & a & \alpha'_1 \end{pmatrix}, \quad (A1)$$

$$\langle S'_c = I'_c; m'_1, \alpha'_1 | T^a_c | S_c = I_c; m_1, \alpha_1 \rangle = \sqrt{I_c(I_c+1)} \begin{pmatrix} I_c & 1 & I_c \\ \alpha_1 & a & \alpha'_1 \end{pmatrix} \delta_{I'_c I_c} \delta_{S'_c S_c} \delta_{m'_1 m_1}, \quad (A2)$$

$$\langle S'_c = I'_c; m'_1, \alpha'_1 | S^i_c | S_c = I_c; m_1, \alpha_1 \rangle = \sqrt{I_c(I_c+1)} \begin{pmatrix} S_c & 1 & S_c \\ m_1 & i & m'_1 \end{pmatrix} \delta_{I'_c I_c} \delta_{S'_c S_c} \delta_{\alpha'_1 \alpha_1}. \quad (A3)$$

To obtain the matrix elements of s,t,g in terms of those for S_c, T_c, G_c, simply note that the excited quark is group-theoretically equivalent to a one-quark core with spin and isospin 1/2. Thus, replace each $N_c - 1$ by 1, and each $S_c = I_c$ and $S'_c = I'_c$ by 1/2. The matrix elements of the orbital angular momentum operators are

$$\langle l' m'_l | l^i | l m_l \rangle = \sqrt{l(l+1)} \begin{pmatrix} l & 1 & l \\ m_l & i & m'_l \end{pmatrix} \delta_{l' l}, \quad (A4)$$

$$\langle l' m'_l | l^{(2)ij} | l m_l \rangle = \sqrt{\frac{l(l+1)(2l-1)(2l+3)}{6}} \delta_{l' l} \sum_\mu \begin{pmatrix} 1 & 1 & 2 \\ i & j & \mu \end{pmatrix} \begin{pmatrix} l & 2 & l \\ m_l & \mu & m'_l \end{pmatrix}. \quad (A5)$$

With these results we have computed the matrix elements of all the possible isosinglet mass operators:

$$\langle \mathbb{1} \rangle = \delta_{J'J} \delta_{J'_3 J_3} \delta_{l'l} \delta_{I'I} \delta_{I'_3 I_3} \delta_{S'S}, \quad (A6)$$

$$\langle ls \rangle = \delta_{J'J} \delta_{J'_3 J_3} \delta_{l'l} \delta_{I'I} \delta_{I'_3 I_3} (-1)^{S'-S} \frac{1}{2} \sqrt{(2S'+1)(2S+1)} \sum_{l_s = l \pm 1/2} \left[l_s(l_s+1) - l(l+1) - \frac{3}{4} \right] (2l_s+1)$$

$$\times \sum_{\eta = \pm 1} c_{\rho'\eta} c_{\rho\eta} \begin{Bmatrix} I_c & \frac{1}{2} & S' \\ l & J & l_s \end{Bmatrix} \begin{Bmatrix} I_c & \frac{1}{2} & S \\ l & J & l_s \end{Bmatrix}, \quad (A7)$$

$$\langle ltG_c \rangle = \delta_{J'J} \delta_{J'_3 J_3} \delta_{l'l} \delta_{I'I} \delta_{I'_3 I_3} (-1)^{J+I+l+S'-S+1} \frac{1}{4} \sqrt{\frac{3}{2}} \sqrt{l(l+1)(2l+1)} \sqrt{(2S'+1)(2S+1)}$$

$$\times \sum_{\eta',\eta = \pm 1} c_{\rho'\eta} c_{\rho\eta} \sqrt{(2I'_c+1)(2I_c+1)} \sqrt{(N_c+1)^2 - \left(\frac{\eta'-\eta}{2}\right)^2 (2I+1)^2}$$

$$\times \begin{Bmatrix} \frac{1}{2} & 1 & \frac{1}{2} \\ I'_c & I & I_c \end{Bmatrix} \begin{Bmatrix} l & 1 & l \\ S' & J & S \end{Bmatrix} \begin{Bmatrix} S' & 1 & S \\ I_c & \frac{1}{2} & I'_c \end{Bmatrix} (-1)^{(\eta'-\eta)/2}, \quad (A8)$$

$$\langle l^{(2)} g G_c \rangle = \delta_{J'J} \delta_{J'_3 J_3} \delta_{l'l} \delta_{I'I} \delta_{I'_3 I_3} (-1)^{J-2I+l+S} \frac{1}{8} \sqrt{\frac{15}{2}} \sqrt{l(l+1)(2l-1)(2l+1)(2l+3)}$$

$$\times \sqrt{(2S'+1)(2S+1)} \begin{Bmatrix} 2 & l & l \\ J & S' & S \end{Bmatrix} \sum_{\eta',\eta = \pm 1} c_{\rho'\eta'} c_{\rho\eta} (-1)^{(1+\eta')/2}$$

$$\times \sqrt{(2I'_c+1)(2I_c+1)} \sqrt{(N_c+1)^2 - \left(\frac{\eta'-\eta}{2}\right)^2 (2I+1)^2} \begin{Bmatrix} \frac{1}{2} & 1 & \frac{1}{2} \\ I'_c & I & I_c \end{Bmatrix} \begin{Bmatrix} I'_c & I_c & 1 \\ S' & S & 2 \\ \frac{1}{2} & \frac{1}{2} & 1 \end{Bmatrix}, \quad (A9)$$

$$\langle lS_c\rangle = \delta_{J'J}\delta_{J'_3J_3}\delta_{l''l}\delta_{I''I}\delta_{I'_3I_3}(-1)^{J-I+l+S'-S}\sqrt{l(l+1)(2l+1)}\sqrt{(2S'+1)(2S+1)}$$

$$\times\begin{Bmatrix}1 & l & l\\ J & S' & S\end{Bmatrix}\sum_{\eta=\pm1}c_{\rho'\eta}c_{\rho\eta}(-1)^{(1-\eta)/2}\sqrt{I_c(I_c+1)(2I_c+1)}\begin{Bmatrix}1 & I_c & I_c\\ \tfrac{1}{2} & S' & S\end{Bmatrix},\tag{A10}$$

$$\langle tT_c\rangle = \delta_{J'J}\delta_{J'_3J_3}\delta_{l''l}\delta_{I''I}\delta_{I'_3I_3}\delta_{S'S}\frac{1}{4}\left[4I(I+1)-3-4\sum_{\eta=\pm1}c_{\rho\eta}^2 I_c(I_c+1)\right],\tag{A11}$$

$$\langle S_c^2\rangle = \delta_{J'J}\delta_{J'_3J_3}\delta_{l''l}\delta_{I''I}\delta_{I'_3I_3}\delta_{S'S}\sum_{\eta=\pm1}c_{\rho\eta}^2 I_c(I_c+1),\tag{A12}$$

$$\langle lgT_c\rangle = \delta_{J'J}\delta_{J'_3J_3}\delta_{l''l}\delta_{I''I}\delta_{I'_3I_3}(-1)^{J+I+l+1}\frac{3}{2}\sqrt{l(l+1)(2l+1)}\sqrt{(2S'+1)(2S+1)}$$

$$\times\begin{Bmatrix}1 & l & l\\ J & S' & S\end{Bmatrix}\sum_{\eta=\pm1}c_{\rho'\eta}c_{\rho\eta}\sqrt{I_c(I_c+1)(2I_c+1)}\begin{Bmatrix}1 & \tfrac{1}{2} & \tfrac{1}{2}\\ & & \\ I & I_c & I_c\end{Bmatrix}\begin{Bmatrix}1 & \tfrac{1}{2} & \tfrac{1}{2}\\ & & \\ I_c & S' & S\end{Bmatrix},\tag{A13}$$

$$\langle l^{(2)}sS_c\rangle = \delta_{J'J}\delta_{J'_3J_3}\delta_{l''l}\delta_{I''I}\delta_{I'_3I_3}(-1)^{J+l+S}\frac{\sqrt{5}}{2}\sqrt{l(l+1)(2l-1)(2l+1)(2l+3)}\sqrt{(2S'+1)(2S+1)}\begin{Bmatrix}2 & l & l\\ J & S' & S\end{Bmatrix}$$

$$\times\sum_{\eta=\pm1}c_{\rho'\eta}c_{\rho\eta}\sqrt{I_c(I_c+1)(2I_c+1)}\begin{Bmatrix}I_c & I_c & 1\\ S' & S & 2\\ \tfrac{1}{2} & \tfrac{1}{2} & 1\end{Bmatrix},\tag{A14}$$

$$\langle l^i g^{ja}\{S_c^j,G_c^{ia}\}\rangle = \delta_{J'J}\delta_{J'_3J_3}\delta_{l''l}\delta_{I''I}\delta_{I'_3I_3}(-1)^{J-I+l}\frac{3}{8}\sqrt{l(l+1)(2l+1)}\sqrt{(2S'+1)(2S+1)}$$

$$\times\begin{Bmatrix}1 & l & l\\ J & S' & S\end{Bmatrix}\sum_{\eta',\eta=\pm1}c_{\rho'\eta'}c_{\rho\eta}\begin{Bmatrix}\tfrac{1}{2} & 1 & \tfrac{1}{2}\\ & & \\ I'_c & I & I_c\end{Bmatrix}\begin{Bmatrix}I'_c & 1 & I_c\\ & & \\ S & \tfrac{1}{2} & S'\end{Bmatrix}\sqrt{(2I'_c+1)(2I_c+1)}$$

$$\times\sqrt{(N_c+1)^2-\left(\frac{\eta'-\eta}{2}\right)^2(2I+1)^2}\left[(-1)^{S-I_c+1/2}\begin{Bmatrix}\tfrac{1}{2} & 1 & \tfrac{1}{2}\\ & & \\ I'_c & S' & I'_c\end{Bmatrix}\sqrt{I'_c(I'_c+1)(2I'_c+1)}\right.$$

$$\left.+(-1)^{S'-I'_c+1/2}\begin{Bmatrix}\tfrac{1}{2} & 1 & \tfrac{1}{2}\\ & & \\ I_c & S & I_c\end{Bmatrix}\sqrt{I_c(I_c+1)(2I_c+1)}\right],\tag{A15}$$

$$\langle l^{(2)}t\{S_c,G_c\}\rangle = \delta_{J'J}\delta_{J'_3J_3}\delta_{l''l}\delta_{I''I}\delta_{I'_3I_3}(-1)^{J-I+l+S'-S+1}\frac{1}{8}\sqrt{5l(l+1)(2l-1)(2l+1)(2l+3)}\tag{A16}$$

$$\times\sqrt{(2S'+1)(2S+1)}\begin{Bmatrix}2 & l & l\\ J & S' & S\end{Bmatrix}\sum_{\eta',\eta=\pm1}c_{\rho'\eta'}c_{\rho\eta}\sqrt{(2I'_c+1)(2I_c+1)}$$

$$\times \sqrt{(N_c+1)^2-\left(\frac{\eta'-\eta}{2}\right)^2(2I+1)^2}\begin{Bmatrix}\frac{1}{2}&1&\frac{1}{2}\\[4pt]I_c'&I&I_c\end{Bmatrix}\begin{Bmatrix}2&I_c'&I_c\\[4pt]\frac{1}{2}&S'&S\end{Bmatrix}$$

$$\times\left[\sqrt{I_c'(I_c'+1)(2I_c'+1)}\begin{Bmatrix}2&1&1\\I_c'&I_c'&I\end{Bmatrix}+\sqrt{I_c(I_c+1)(2I_c+1)}\begin{Bmatrix}2&1&1\\I_c&I_c&I_c'\end{Bmatrix}\right],$$

$$\langle sS_c\rangle=\delta_{J'J}\delta_{J_3'J_3}\delta_{l'l}\delta_{I'I}\delta_{I_3'I_3}\delta_{S'S}\frac{1}{2}\left[\left(S-\frac{1}{2}\right)\left(S+\frac{3}{2}\right)-\sum_{\eta=\pm1}c_{\rho\eta}^2 I_c(I_c+1)\right],\tag{A17}$$

$$\langle(lS_c)(tT_c)\rangle=\delta_{J'J}\delta_{J_3'J_3}\delta_{l'l}\delta_{I'I}\delta_{I_3'I_3}(-1)^{J-I+l+S'-S+1}\frac{1}{4}\sqrt{l(l+1)(2l+1)}\sqrt{(2S'+1)(2S+1)}\begin{Bmatrix}1&l&l\\J&S'&S\end{Bmatrix}$$

$$\times\sum_{\eta=\pm1}c_{\rho'\eta}c_{\rho\eta}[(2I+1)\eta+2](-1)^{(1-\eta)/2}\sqrt{I_c(I_c+1)(2I_c+1)}\begin{Bmatrix}1&I_c&I_c\\[4pt]\frac{1}{2}&S'&S\end{Bmatrix},\tag{A18}$$

$$\langle(ls)S_c^2\rangle=\delta_{J'J}\delta_{J_3'J_3}\delta_{l'l}\delta_{I'I}\delta_{I_3'I_3}(-1)^{S'-S}\frac{1}{2}\sqrt{(2S'+1)(2S+1)}\sum_{l_s=l\pm1/2}\left[l_s(l_s+1)-l(l+1)-\frac{3}{4}\right](2l_s+1)$$

$$\times\sum_{\eta=\pm1}c_{\rho'\eta}c_{\rho\eta}\begin{Bmatrix}I_c&\frac{1}{2}&S'\\[4pt]l&J&l_s\end{Bmatrix}\begin{Bmatrix}I_c&\frac{1}{2}&S\\[4pt]l&J&l_s\end{Bmatrix}I_c(I_c+1),\tag{A19}$$

$$\langle\{lS_c,sS_c\}\rangle=\delta_{J'J}\delta_{J_3'J_3}\delta_{l'l}\delta_{I'I}\delta_{I_3'I_3}(-1)^{J-I+l+S'-S}\sqrt{l(l+1)(2l+1)}\sqrt{(2S'+1)(2S+1)}$$

$$\times\begin{Bmatrix}1&l&l\\J&S'&S\end{Bmatrix}\sum_{\eta=\pm1}c_{\rho'\eta}c_{\rho\eta}\left\{\frac{1}{2}[S'(S'+1)+S(S+1)]-\frac{3}{4}-I_c(I_c+1)\right\}$$

$$\times(-1)^{(1-\eta)/2}\sqrt{I_c(I_c+1)(2I_c+1)}\begin{Bmatrix}1&I_c&I_c\\[4pt]\frac{1}{2}&S'&S\end{Bmatrix},\tag{A20}$$

$$\langle gS_cT_c\rangle=\delta_{J'J}\delta_{J_3'J_3}\delta_{l'l}\delta_{I'I}\delta_{I_3'I_3}\delta_{S'S}(-1)^{I-S}\frac{3}{2}\sum_{\eta=\pm1}c_{\rho\eta}^2 I_c(I_c+1)(2I_c+1)\begin{Bmatrix}\frac{1}{2}&1&\frac{1}{2}\\[4pt]I_c&I&I_c\end{Bmatrix}\begin{Bmatrix}\frac{1}{2}&1&\frac{1}{2}\\[4pt]I_c&S&I_c\end{Bmatrix},\tag{A21}$$

$$\langle l^{(2)}S_cS_c\rangle=\delta_{J'J}\delta_{J_3'J_3}\delta_{l'l}\delta_{I'I}\delta_{I_3'I_3}(-1)^{J+I-l+S'-S}\sqrt{\frac{5l(l+1)(2l-1)(2l+1)(2l+3)}{6}}$$

$$\times\sqrt{(2S'+1)(2S+1)}\begin{Bmatrix}2&l&l\\J&S'&S\end{Bmatrix}\sum_{\eta=\pm1}c_{\rho'\eta}c_{\rho\eta}I_c(I_c+1)(2I_c+1)(-1)^{(1-\eta)/2}$$

$$\times\begin{Bmatrix}2&I_c&I_c\\[4pt]\frac{1}{2}&S'&S\end{Bmatrix}\begin{Bmatrix}2&l&l\\I_c&I_c&I_c\end{Bmatrix},\tag{A22}$$

$$\langle l^{(2)} g S_c T_c \rangle = \delta_{J'J} \delta_{J_3' J_3} \delta_{l'l} \delta_{I'I} \delta_{I_3' I_3} (-1)^{J+2I+l+S} \sqrt{\frac{5l(l+1)(2l-1)(2l+1)(2l+3)}{6}}$$

$$\times \frac{3}{2}\sqrt{(2S'+1)(2S+1)} \begin{Bmatrix} 2 & l & l \\ J & S' & S \end{Bmatrix} \sum_{\eta=\pm 1} c_{\rho' \eta} c_{\rho \eta} I_c(I_c+1)(2I_c+1)(-1)^{(\eta+1)/2}$$

$$\times \begin{Bmatrix} \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{2} & 1 & \frac{1}{2} \\ I_c & I & I_c \end{Bmatrix} \begin{Bmatrix} I_c & I_c & 1 \\ S' & S & 2 \\ \frac{1}{2} & \frac{1}{2} & 1 \end{Bmatrix} . \tag{A23}$$

APPENDIX B: OPERATOR REDUCTIONS FOR MIXED-SYMMETRY STATES

In order to establish a connection to the operator reduction rules obtained in [8] for the completely symmetric ground-state spin-flavor multiplet, let us first develop a common notation and then review the results of the derivation presented in the earlier work.

Each irreducible representation of SU($2F$) is denoted by a Dynkin label, which is a $(2F-1)$-plet $[n_1, n_2, \ldots, n_{2F-1}]$ of non-negative integers n_r describing the Young diagram of the representation; the number of boxes in row $r(=1,2,\ldots,2F-1)$ of the diagram exceeds the number in row $r+1$ by n_r. The conjugate of a given representation is obtained by reversing the order of the integers n_r. In this notation the completely symmetric N_c-box representation S is $[N_c,0,0,\ldots,0]$, while the mixed-symmetry $l=1$ baryons fill the representation MS$=[N_c-2,1,0,0,\ldots,0]$. Since all matrix elements of operators $\mathcal{O}$ between baryons B transforming according to a given representation appear through bilinears of the form $\bar{B}\mathcal{O}B$, such operators fill the representations of $\bar{B}\otimes B$. In the case of the ground-state representation, standard techniques for combining representations show that this product is

$$\bar{S}\otimes S=[0,0,0,\ldots,N_c]\otimes[N_c,0,0,\ldots,0]$$

$$= \bigoplus_{m=0}^{N_c} [m,0,0,\ldots,m], \tag{B1}$$

while the mixed-symmetry representation product gives

$$\overline{MS}\otimes MS=[0,0,\ldots,0,1,N_c-2]\otimes[N_c-2,1,0,0,\ldots,0]$$

$$=[0,0,\ldots,0]\oplus[N_c-1,0,0,\ldots,N_c-1]$$

$$\bigoplus_{m=1}^{N_c-2} 2[m,0,0,\ldots,0,m]$$

$$\bigoplus_{m=0}^{N_c-2} [m,1,0,0,\ldots,0,1,m]$$

$$\bigoplus_{m=0}^{N_c-3} [m+2,0,0,\ldots,0,1,m]$$

$$\bigoplus_{m=0}^{N_c-3} [m,1,0,0,\ldots,0,m+2]. \tag{B2}$$

It is convenient to give these representations concise names for future reference. Label $[m,0,0,\ldots,m]$ as adj$_m$, so that adj$_0$ is the singlet representation, adj$_1$ is the adjoint representation, and adj$_2$ is called $\bar{s}s$ in [8]. Let $[m,1,0,0,\ldots,0,1,m]\equiv\bar{a}a_m$, $[m+2,0,0,\ldots,0,1,m]\equiv\bar{a}s_m$, and $[m,1,0,0,\ldots,0,m+2]\equiv\bar{s}a_m$, so that $\bar{a}a_0$, $\bar{a}s_0$, and $\bar{s}a_0$ are denoted in [8] as $\bar{a}a$, $\bar{a}s$, and $\bar{s}a$, respectively.

That the operators (4.1) satisfy an SU($2F$) algebra implies that any string of the one-body operators in Eqs. (4.1) containing a commutator is reducible to a smaller string of such operators; only anticommutators need be considered. Since one-body operators appear in the adjoint representation of SU($2F$), one need only consider combinations symmetrized on the adjoint indices, $1,\text{adj},(\text{adj}\otimes\text{adj})_S,(\text{adj}\otimes\text{adj}\otimes\text{adj})_S$, etc., which may be denoted 0-,1-, 2-, etc., body operators.

Let us turn to the question of representations that appear in symmetrized products of one-body operators, but have vanishing matrix elements for the mixed-symmetry baryon states since they do not appear in the product $\overline{MS}\otimes MS$. First note that, unlike $\bar{S}\otimes S$, all representations in

$$(\text{adj}\otimes\text{adj})_S=\text{adj}_0\oplus\text{adj}_1\oplus\text{adj}_2\oplus\bar{a}a_0, \tag{B3}$$

in particular $\bar{a}a_0$, appear in the product $\overline{MS}\otimes MS$, meaning that the mixed-symmetry representation has no similar operator identity. One therefore turns to representations in the product $(\text{adj}\otimes\text{adj}\otimes\text{adj})_S$, which are listed in Table IV of [8]. Comparing this list to Eq. (B2), one sees that the only representations not present in the latter are $[0,0,1,0,0,\ldots,0,1,0,0]\equiv\bar{b}b_0$ and adj$_3$. Products of three one-body operators that transform according to these representations should indeed be reducible when acting on mixed-symmetry baryon states. The astute reader may notice that each of these representations has a special feature: $\bar{b}b_0$ does not occur for $F<3$, and adj$_3$ only gives reduction rules for the physical case $N_c=3$. The product of three one-body operators is enough to span the space of all physical baryon observables; for this reason, we do not consider representations in the product of four or more one-body operators.

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