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Global effects due to cosmic defects in Kaluza-Klein theory

Cláudio Furtado,[1] Fernando Moraes,[1] and V. B. Bezerra[2]
[1]*Laboratório de Física Teórica e Computacional, Departamento de Física, Universidade Federal de Pernambuco, 50670-901 Recife, PE, Brazil*
[2]*Departamento de Física - CCEN, Universidade Federal da Paraíba, Caixa Postal 5008, 58051-970 João Pessoa, PB, Brazil*
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Using Kaluza-Klein theory we study the quantum mechanics of a scalar particle in the background of a magnetic cosmic string and in the background of a chiral cosmic string. We show that the wave functions, the phase shifts, and scattering amplitudes associated with the particle depend on the global features of those spacetimes. These dependences represent gravitational analogues of the well-known Aharonov-Bohm effect. In addition, we discuss the Landau levels in the presence of a cosmic string in the framework of Kaluza-Klein theory. [S0556-2821(99)03610-3]

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I. INTRODUCTION

Topological defects in spacetime can be characterized by metrics with a zero Riemann-Chistoffel curvature tensor everywhere, except on the defects, i.e., by conic-type curvature singularities [1]. Examples of such topological defects are the domain wall [2], the cosmic string [2,3], and the global monopole [4]. Cosmic strings provide a bridge between physics on microscopic and macroscopic scales. They are linear defects, analogous to vortex filaments in superfluid helium [5] and to dislocations and disclinations in condensed matter physics [6] and may have been formed in the very beginning of the universe [7].

On the other hand, Kaluza-Klein theories in five or more dimensions have experienced a renewed interest in recent years. The geometrical unification of gravitation and eletromagnetism obtained in the five-dimensional version of general relativity constructed by Kaluza [8] and Klein [9] gave some beautiful results. In a recent paper Azreg-Ainou and Clement [10] make a systematic investigation of stationary cylindrically symmetric solutions to the five-dimensional Einstein and Einstein-Gauss-Bonnet equations. In addition to the known solutions, they have found some new ones that can be characterized with five-dimensional generalizations of four-dimensional metrics which are singular on the $\rho=0$ axis. Examples of such solutions are five-dimensional generalizations of the cosmic string, the chiral cosmic string [11], and the magnetic flux string [12].

In this work, we study the quantum scalar particle in the presence of topological defects in Kaluza-Klein theory. We are mainly interested on the effects of global features of these defects on the quantum states of the particle. We solve the Klein-Gordon equation and study the scattering amplitude for the magnetic cosmic string and the chiral cosmic string backgrounds. Finally, we solve the problem of Landau levels in the presence of the cosmic string, in the framework of Kaluza-Klein theory, finding the energy spectrum and wave functions for this problem. In this case, the uniform magnetic field is described by the fifth dimension.

II. THE KALUZA-KLEIN MAGNETIC COSMIC STRING

In this section we study the quantum dynamics of a scalar particle in the five-dimensional space-time of a magnetic cosmic string. The line element corresponding to the magnetic flux string in five dimensions is given by [10]

$$dS^2=dt^2-dz^2-d\rho^2-\alpha^2\rho^2 d\varphi^2-\left(dx^5+\frac{\Phi}{2\pi}d\varphi\right)^2, \qquad (1)$$

with $\alpha=1-4\mu$, where μ is the linear mass density of the string. By using the standard Kaluza-Klein decomposition of the five-dimensional metric, it can be shown that this metric represents the magnetic flux string with a longitudinal field

$$B^z=\kappa^{-1}\Phi\,\delta^{(2)}(\vec{r}), \qquad (2)$$

where κ is the Kaluza constant. The azimuthal vector potential $A_\varphi=\kappa^{-1}(\Phi/2\pi)$ is pure gauge, except for a singularity on the axis. Notice that in Eq. (1) the parameter α characterizes the cosmic string. Klein-Gordon equation in the metric (1) reduces to

$$\left\{\partial_t^2-\frac{1}{\rho}\partial_\rho(\rho\partial_\rho)-\frac{1}{\alpha^2\rho^2}\left(\partial_\varphi-\frac{\Phi}{2\pi}\partial_x\right)^2-\partial_x^2-\partial_z^2+M^2\right\}\psi$$
$$=0, \qquad (3)$$

where we made $x^5=x$. The solution of this equation can be obtained using the ansatz

$$\psi(t,\rho,\varphi,z,x)=\exp[-iEt+il\varphi+iKz+iQx]R(\rho), \quad (4)$$

where E,l,K and Q are constants. We consider now the coordinate transformation that changes the metric (1) into a flat metric. The transformation is

$$x=X-\frac{\Phi}{2\pi}\varphi, \qquad (5)$$

$$\varphi=\frac{\theta}{\alpha}. \qquad (6)$$

In the flat coordinates (t,ρ,θ,z,X), Eq. (4) becomes

$$\psi(t,\rho,\theta,z,X)=\exp[-iEt+il_{eff}\theta+iKz+iQX]R(\rho), \qquad (7)$$

where

$$l_{eff}=\frac{l-\dfrac{\Phi Q}{2\pi}}{\alpha} \tag{8}$$

is an effective angular momentum due both to the boundary condition, which states that the total angle around the string is $2\pi\alpha$, and to the coupling with the electromagnetic field. And, of course, $l=0,\pm 1,\pm 2,\ldots$.

Formula (8) suggests that, when the particle circles the string, the wave function changes according to

$$\psi\rightarrow\psi'=e^{2\pi i l_{eff}}\psi=\exp\left\{\frac{2\pi i}{\alpha}\left[l-\frac{\Phi Q}{2\pi}\right]\right\}\psi. \tag{9}$$

This shift in the angular momentum is analogous to the shift $l\rightarrow l-Q\Phi/2\pi$ in the eletromagnetic Aharonov-Bohm effect [13] combined with its gravitational counterpart [14], which corrects the angular momentum by a factor of $1/\alpha$. An immediate consequence of Eq. (8) is that the angular momentum operator may be redefined as

$$\hat{l}_{eff}=-\frac{i}{\alpha}\left(\partial_{\varphi}+\frac{\Phi}{2\pi}\partial_x\right), \tag{10}$$

where the additional contribution, $(-i/\alpha)(\Phi/2\pi)\partial_x$, takes into account the electromagnetic flux.

Now, using the ansatz given by Eq. (4), Eq. (3) reduces to

$$\left\{\rho\partial_{\rho}(\rho\partial_{\rho})-\frac{\left(l-\dfrac{\Phi}{2\pi}Q\right)^2}{\alpha^2}+[E^2-M^2-K^2-Q^2]\rho^2\right\}R(\rho)=0. \tag{11}$$

This is a Bessel differential equation whose regular solution is given by

$$R^{Reg}_{lQK}(\lambda\rho)\propto(\pm 1)^l J_{|l-\Phi Q'/2\pi|/\alpha}(\lambda\rho), \tag{12}$$

where $\lambda^2=E^2-(K^2+M^2+Q^2)$ and $\mu^2=(l-\Phi Q/2\pi)/\alpha^2$. The ''plus sign'' corresponds to the case $l\geqslant-[\Phi Q/(2\pi)]$, and the ''minus sign'' corresponds to the case $l<-[\Phi Q/(2\pi)]$, where $[x]$ means the largest integer less than or equal to x.

The associated phase shifts

$$\delta_l=\pm\frac{\pi}{2}\left[l(1-\alpha^{-1})+\frac{\Phi Q}{2\pi\alpha}\right] \tag{13}$$

follow immediately from Hankel's asymptotic expansion [15]. From expression (13) we can compute the scattering amplitude $f(\theta)$ which is defined by [16]

$$f(\varphi)=\frac{1}{\sqrt{2\pi K}}\sum_l (e^{2i\delta_l}-1)e^{il\varphi}. \tag{14}$$

Therefore, using Eqs. (13) and (14) and doing the appropriate regularizations [16], we get the following result for the scattering amplitude:

$$f(\varphi)=-\frac{\pi}{\sqrt{2\pi K}}\sum_l [e^{-i(\Phi Q/2\alpha)}\delta(\varphi+\omega-2\pi l)+e^{i(\Phi Q/2\alpha)}\delta(\varphi-\omega-2\pi l)-2\delta(\varphi-2\pi l)]+f^{(0)}(\varphi), \tag{15}$$

where

$$f^{(0)}(\varphi)=\frac{i}{\sqrt{2\pi K}}e^{-i(\Phi Q/2\alpha)}A(\varphi,\omega)-e^{i(\Phi Q/2\alpha)}A(\varphi,-\omega), \tag{16}$$

$$A(\varphi,\pm\omega)=e^{\pm i[\Phi Q/2\pi](\varphi\pm\omega)}/(i-e^{i(\varphi\pm\omega)}), \tag{17}$$

and $\omega=-\pi(1-\alpha^{-1})$. From this we conclude that the phase shift and the scattering amplitude depend on the parameters α and Φ. In summary, in this section we have shown a generalization of the Aharonov-Bohm effect due to a magnetic cosmic string in the Kaluza-Klein framework: the angular momentum of a scalar particle is corrected not only by Φ, in the usual Aharonov-Bohm way, but also by α giving a gravitational analogue to the Aharonov-Bohm effect. The scattering states are also affected by the topology of spacetime as can be seen in the phase shift and scattering amplitude expressions, which depend on α.

III. THE KALUZA-KLEIN MAGNETIC CHIRAL COSMIC STRING

In this section we extend the problem studied in Sec. II to a more general spacetime, that of a chiral string, or cosmic dislocation. We write the line element as

$$dS^2=(dt+J_0d\varphi)^2-d\rho^2-\alpha^2\rho^2d\varphi^2-(dz+J_zd\varphi)^2-\left(dx+\frac{\Phi}{2\pi}d\varphi\right)^2. \tag{18}$$

The parameters J_0 and J_z are related to angular momentum and torsion, respectively, of the string. With $J_0=0$, $J_z=0$ and $\Phi=0$, Eq. (18) represents the metric of the cosmic string in five dimensions; with $J_z=0$ only, it represents a spinning string [17,18]; and with only $J_0=0$, $\Phi=0$, it represents a space-time generated by a cosmic dislocation [11,19]. For $\Phi=0$ it represents a space-time generated by a chiral cone [14].

The Klein-Gordon equation in the metric (18) is given by

$$\left\{\partial_t^2-\partial_z^2-\frac{1}{\rho}\partial_{\rho}(\rho\partial_{\rho})-\partial_x^2-\frac{1}{\alpha\rho^2}\left[\partial_{\varphi}-J_0\partial_t-J_z\partial_z-\frac{\Phi}{2\pi}\partial_x\right]^2\right\}\psi=0. \tag{19}$$

The metric (18) is actually a locally flat one as we can see by applying the coordinate transformation

$$t=T-J_0\varphi, \tag{20}$$

$$\varphi=\frac{\theta}{\alpha}, \tag{21}$$

$$z=Z-J_z\varphi,$$

$$x=X-\frac{\Phi}{2\pi}\varphi. \tag{22}$$

Using a reasoning similar to the one used in Sec. II we get

$$l_{eff}=\frac{l+J_0E-\left(J_zK+\dfrac{\Phi Q}{2\pi}\right)}{\alpha}, \tag{23}$$

with $l=0,\pm1,\pm2,\ldots$, as before. Therefore we have an effective angular momentum for the particle due to the coupling with the electromagnetic field, to the angular momentum of the string, and to the torsion, which is also corrected by the angular factor α.

From the result given by Eq. (23) we see that the wave function must be changed according to

$$\psi\rightarrow\psi'=e^{2\pi il_{eff}}=\exp\left\{\frac{2\pi i}{\alpha}(l+J_0E)-\left(J_zK+\frac{\Phi Q}{2\pi}\right)\right\}\psi, \tag{24}$$

when the particle circles the string once. Again, this shift in the angular momentum due to the flux Φ, angular momentum J_0 and torsion J_z, corresponds to a generalized Aharonov-Bohm effect [20].

It follows from Eq. (24) that the angular momentum operator may be redefined as

$$\hat{l}_{eff}=-\frac{i}{\alpha}\left(\partial_\varphi-J_0\partial_t+J_z\partial_z+\frac{\Phi}{2\pi}\partial_x\right). \tag{25}$$

Compared to the corresponding $\hat{l}_{eff}$ for a spinning cosmic string, given by Eq. (10), there is an additional contribution $-(i/\alpha)(-J_0\partial_t-J_z\partial_z)$ that takes into account the rotation of the string and the torsion.

Using the ansatz given by Eq. (4), the Klein-Gordon equation (19) reduces to

$$\left\{\rho\partial_\rho(\rho\partial_\rho)+\rho^2[E^2-K^2+Q^2+M^2]-\frac{1}{\alpha^2}\times\left(l+J_0E-J_zK-\frac{\Phi Q}{2\pi}\right)^2\right\}R(\rho)=0, \tag{26}$$

which is again a Bessel equation whose regular solution is given by

$$R^{Reg}_{lQK}(\lambda\rho)\propto(\pm1)^lJ_{\frac{|l+J_0E-J_zK-\Phi Q/2\pi|}{\alpha}}(\lambda\rho), \tag{27}$$

where again $\lambda^2=E^2-K^2+M^2+Q^2$, but now $\mu^2=(l+J_0E-J_zK-\Phi Q/2\pi)^2/\alpha^2$. The "plus sign" corresponds to the case $l\geqslant-[J_0E-J_zK-\Phi Q/2\pi]$ and the "minus sign" corresponds to the case $l<-[J_0E-J_zK-\Phi Q/2\pi]$, the square brackets meaning "the largest integer less than or equal to," as before. The associated phase shift is

$$\delta_l=\pm\frac{\pi}{2}\left[l(1-\alpha^{-1})-J_0E+J_zK+\frac{\Phi Q}{2\pi}\right], \tag{28}$$

which leads to the following scattering amplitude:

$$f(\varphi)=\frac{i\pi}{\sqrt{2\pi K}}\sum_l\left[e^{-i\pi[(\Phi/2\pi\alpha)Q+J_zK-J_0E]}\delta(\varphi+\omega-2\pi l)+e^{i\pi[\Phi Q/2\pi\alpha+J_zK-J_0E]}\delta(\varphi-\omega-2\pi l)-2\delta(\varphi-2\pi l)\right]+f^{(0)}(\varphi), \tag{29}$$

where

$$f^{(0)}(\varphi)=\frac{i}{\sqrt{2\pi K}}e^{-(i\pi/\alpha)[\Phi Q/2\pi+J_zK-J_0E]}A(\varphi,\omega)-e^{(i\pi/\alpha)(\Phi Q/2\pi+J_zK-J_0E)}A(\varphi,-\omega), \tag{30}$$

with

$$A(\varphi,\pm\omega)=e^{\pm i[\Phi Q/2\pi+J_zK-J_0E](\varphi\pm\omega)}/(1-e^{i(\varphi\pm\omega)}) \tag{31}$$

and $\omega=-\pi(1-\alpha^{-1})$. It is clear that, now, the phase shift and the scattering amplitude depend on the parameters α,Φ,J_z and J_0.

In summary, in this section we further generalize the Aharonov-Bohm effect to be caused not only by the magnetic flux Φ, but also by the conical topology (represented by α), by the rotation (represented by J_0), and by the torsion (represented by J_z) of the spacetime.

IV. LANDAU LEVELS IN A FIVE-DIMENSIONAL SPACETIME

In this section we consider the relativistic and nonrelativistic problems of Landau levels in the presence of defects. In this new approach the uniform magnetic field is introduced geometrically by use of Kaluza-Klein theory. The five-dimensional metric that corresponds to a uniform magnetic field is

$$dS^2=dt^2-d\rho^2-dz^2-\alpha^2\rho^2d\varphi^2-\left(dx-\frac{B_0\rho^2}{2}d\varphi\right)^2, \tag{32}$$

where $A_\varphi=B_o\rho/(2\alpha)$ is the vector potencial and the magnetic field is $B^z=B_0$. The Klein-Gordon equation for one particle in this background is

$$\left\{\partial_t^2-\partial_z^2-\partial_x^2-\frac{1}{\rho}\partial_\rho(\rho\partial_\rho)-\frac{1}{\alpha^2\rho^2}(\partial_\varphi-B_o\rho^2\partial_x)^2+M^2\right\}\varphi=0. \tag{33}$$

This equation has as solution

$$\psi(t,\rho,\varphi,z,x)=e^{-iEt+iKz+il\phi+iQx}F\left(-n,\frac{|l|}{\alpha}+1,\frac{\rho^2}{2}\right), \quad (34)$$

where $F(a,b,x)$ is the hypergeometric function and $n=0,\ 1,\ 2,\ldots$. For this problem we obtain the energy levels as

$$E=\sqrt{\frac{B_0Q}{\alpha}\left(2n+\frac{|l|}{\alpha}-\frac{l}{\alpha}+1\right)+M^2+K^2+Q^2}. \quad (35)$$

Notice that these results depend on the parameter α and therefore the introduction of defects breaks the degeneracy of the Landau levels.

Now, we consider the nonrelativistic case. To do this let us consider the covariant Schrödinger equation in the metric (32), which is given by

$$-\frac{1}{2m}\partial_z^2+\frac{1}{\rho}\partial_\rho(\rho\partial\rho)+\partial_x^2+\frac{1}{\alpha^2\rho^2}\left(\partial_\varphi-\frac{B_0\rho^2}{2}\partial_x\right)^2\psi=i\frac{\partial\psi}{\partial t}. \quad (36)$$

The wave function is now given by

$$\psi(t,\rho,\varphi,z,x)=C_{nl}e^{-iEt+iKz+iQx+il\varphi}\times e^{-B_0Q\rho^2/4\alpha}\rho^{|l|}F\left(-n,\frac{|l|}{\alpha}+1,\frac{\rho^2}{2}\right) \quad (37)$$

and eigenvalues

$$E=\frac{B_0Q}{2m\alpha}\left(n+\frac{|l|}{2\alpha}-\frac{l}{2\alpha}+\frac{1}{2}\right)+\frac{K^2}{2m}+\frac{Q^2}{2m}. \quad (38)$$

This result is in perfect agreement with our earlier work [21] where the magnetic field is introduced by minimal coupling. This also agrees with Eq. (35) in the nonrelativistic limit as can easily be demonstrated by taking M to be very large.

V. CONCLUDING REMARKS

In this work we used Kaluza-Klein theory to investigate global effects of the special boundary conditions that cosmic strings impose on a quantum scalar particle. We considered the situations involving a magnetic flux string and a chiral magnetic cosmic string. The magnetic flux of the string gives rise to the Aharonov-Bohm effect. In addition, the topological factor α, that characterizes the string, and the parameters J_0 and J_z, that characterize the angular momentum and torsion, respectively, of the string spacetime, correct both the angular momentum of the particle and its energy, giving rise to a generalized Aharonov-Bohm effect (a recent publication [22], by one of us and a coworker, deals with this generalized A-B effect in the context of condensed matter physics). We have also studied, within the Kaluza-Klein framework, the problem of Landau levels in the presence of a cosmic string, both relativistic and not. The result, as shown in a previous publication, is a break of degeneracy of the levels. To conclude, we wish to emphasize the straightforwardness and elegance of the Kaluza-Klein approach to deal with the quantum dynamics of a scalar particle in the background spacetime of a cosmic string.

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