Effective dynamics of hot, soft non-Abelian gauge fields: Color conductivity and $\log(1/\alpha)$ effects

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Bödeker has recently argued that non-perturbative processes in very high temperature non-Abelian plasmas (such as electroweak baryon number violation in the very hot early Universe) are logarithmically enhanced over previous estimates and take place at a rate per unit volume of order $\alpha^5 T^4 \ln(1/\alpha)$ for small coupling. We give a simple physical interpretation of Bödeker's qualitative and quantitative results in terms of Lenz's Law—the fact that conducting media resist changes in the magnetic field—and earlier authors' calculations of the color conductivity of such plasmas. In the process, we resolve some confusion in the literature about the value of the color conductivity and present an independent calculation. We also discuss the issue of whether the classical effective theory proposed by Bödeker has a good continuum limit. [S0556-2821(99)05008-0]

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I. INTRODUCTION

Standard electroweak theory violates baryon number via non-perturbative processes involving the electroweak anomaly.[1] Such processes are exponentially suppressed under normal conditions, but are unsuppressed at very high temperatures in the early Universe. Non-perturbative baryon number violation is a key ingredient in scenarios for electroweak baryogenesis, which attempt to explain the matter anti-matter asymmetry of the Universe in terms of the physics of the electroweak phase transition. Such scenarios typically depend on (among other things) the equilibrium rate of baryon number violation in the hot, symmetry-restored phase of electroweak theory.[2] The rate of baryon number violation—and more generally the rate of any generic non-perturbative process in high-temperature non-Abelian plasmas—has long been a source of theoretical confusion. In fact, it is only recently becoming clear how the rate scales with the fine structure constant α of the relevant gauge interactions in the arbitrarily weak coupling limit.

For non-Abelian plasmas at ultra-relativistic temperatures, non-perturbative fluctuations of the gauge field are associated with magnetic fluctuations over distance scales $R \sim 1/g^2T$ (to be reviewed momentarily). For a long time in the literature, it was assumed that the time scale t of non-perturbative processes was also of order $1/g^2T$, and that the rate Γ per unit volume was therefore of order $1/R^3t \sim \alpha^4 T^4$. Two years ago, we argued [3] that damping effects in the plasma slow the time scale down to $t \sim 1/g^4T$, giving a rate of $\Gamma \sim 1/R^3t \sim \alpha^5T^4$. (See also Refs. [4,5].) More recently, Bödeker [6] has claimed that there is an additional logarithmic suppression of the time scale, so that

$$t \sim \frac{1}{g^4 T \ln(1/g)}, \quad \text{and} \quad \Gamma \sim \alpha^5 T^4 \ln(1/\alpha). \tag{1.1}$$

Bödeker has also proposed an effective theory for the relevant distance and time scales in the form of simple stochastic dynamics for the gauge fields. Numerical simulation of this effective theory would give the non-perturbative numerical coefficient c of the logarithm:

$$\Gamma \simeq c\, \alpha^5 T^4 \ln(b/\alpha) \tag{1.2}$$

for small α. (No one has yet proposed an explicit way to calculate the constant b under the log, and one should expect there to be sub-leading corrections suppressed only by powers of $1/\ln \alpha$.) The goal of the present work is to show that Bödeker's results can be reproduced and interpreted through a simple argument based on the fact that plasmas are conductive.

Before presenting the essential argument, let us take a moment to review the physical origin of the length scale $1/g^2T$ associated with non-perturbative fluctuations. (For more formal arguments, see [7].) Imagine a fluctuation of the gauge field of spatial size R and amplitude A. Non-perturbative means that, for example, gA is not a perturbation in the covariant derivative $\mathbf{D} = \boldsymbol{\nabla} - ig\mathbf{A}$. So non-

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[1]For some reviews of electroweak baryon number violation and electroweak baryogenesis, see Ref. [1].

[2]We use the term "symmetric phase" loosely since, depending on the details of the Higgs sector, there may not be any sharp transition between the symmetric and "symmetry-broken" phases of the theory [2]. A sharp transition is in fact required for electroweak baryogenesis. The analysis of this paper applies directly whenever the temperature is sufficiently high that the infrared dynamics of the Higgs is irrelevant at lengths of $O(1/g^2T)$, which is the case either (a) far above the electroweak phase transition or "crossover," or (b) in the symmetric phase at the transition in cases where there is a first-order transition and the transition is not exceedingly weak.

perturbative means $A \gtrsim O(1/gR)$, and hence the energy E of this fluctuation is $\gtrsim O(1/g^2R)$. The probability of such a fluctuation in energy is exponentially suppressed by the Maxwell-Boltzmann factor $\exp(-\beta E) \sim \exp[-1/(g^2RT)]$ unless $R \gtrsim O(1/g^2T)$. Because of entropy effects, non-perturbative processes will be dominated by the smallest size scale for which the probability is unsuppressed (since there are more small-wavelength degrees of freedom than large-wavelength ones), and so the characteristic length scale of non-perturbative physics is $R \sim 1/g^2T$. Static electric fields are screened by the Debye effect on smaller distance scales, of order $1/gT$. For this reason, non-perturbative physics in the hot plasma is essentially magnetic. More technically, it is only the transverse degrees of freedom of the gauge field which are important.

In the next section, we present the simple relationship between the color conductivity and non-perturbative dynamics at leading-log order, and reproduce Bödeker's effective theory for the non-perturbative dynamics. In Sec. III, we review the somewhat confusing literature on color conductivity and present our own calculation based on the Boltzmann equation with a collision term. Finally, in Sec. IV, we argue that Bödeker's effective theory is ultraviolet insensitive—a crucial property for numerical simulations.

II. THE ESSENTIAL ARGUMENT

We now turn to the essence of the argument, which is quite short. It is based on realizing that the dynamics of magnetic fluctuations in plasmas is slowed down by Lenz's law: conducting media resist changes in magnetic field. In the context of high temperature baryon number violation, this qualitative explanation of the slow time scale for non-perturbative processes is due to Moore [8]. Let's make it quantitative. This derivation will be a little fast and loose, but its advantage is that the physics is very simple.

Imagine splitting the gauge field into soft degrees of freedom—those associated with momenta of order g^2T, and hard degrees of freedom—those associated with much higher momenta such as T. The details of exactly how this split is made will not be relevant at the order we shall consider.[3] The amplitude of fluctuations is non-perturbative for the soft modes but perturbative for the hard ones. As is well known [10], the soft modes are also effectively classical—there are a large number of quanta in each mode because of Bose statistics. Now, treating the soft modes classically, start with the Maxwell equation

$$\mathbf{D}\times\mathbf{B} = D_t\,\mathbf{E} + \mathbf{J}_{\text{hard}} \tag{2.1}$$

for the soft degrees of freedom, where $\mathbf{B} = \mathbf{D}\times\mathbf{A}$ and where all covariant derivatives are to be understood as only involving the soft gauge field degrees of freedom. $\mathbf{J}_{\text{hard}}$ is the color current[4] due to the hard degrees of freedom, which we shall later see is dominated by excitations with momenta of order T. It is important to distinguish between the hard momenta of the particles which contribute to $\mathbf{J}_{\text{hard}}$ and the momentum components of $\mathbf{J}_{\text{hard}}$ itself (which is bilinear in the fundamental fields). It is the soft momentum components of $\mathbf{J}_{\text{hard}}$ which are relevant in the context of Eq. (2.1).

Plasmas are conductors. Hence, for sufficiently small momentum and frequency (exactly how small will be discussed later), we have

$$\mathbf{J}_{\text{hard}} = \sigma\mathbf{E}, \tag{2.2}$$

where σ is the color analog of conductivity. The Maxwell equation then becomes

$$\mathbf{D}\times\mathbf{B} = D_t\,\mathbf{E} + \sigma\mathbf{E}. \tag{2.3}$$

Let us assume that non-perturbative processes will be slow enough (which we will verify *a posteriori*) that we can neglect the time derivative term. Then the Maxwell equation becomes simply

$$\mathbf{D}\times\mathbf{B} = \sigma\mathbf{E}. \tag{2.4}$$

In $A_0 = 0$ gauge, this is a simple first-order equation of motion:

$$\sigma\frac{d}{dt}\mathbf{A} = -\mathbf{D}\times\mathbf{B}. \tag{2.5}$$

This equation is dissipative and describes the relaxation of fluctuations of the soft fields away from equilibrium. The dissipation results from interactions of the soft modes with the hard degrees of freedom, which are accelerated by and steal energy from the soft fields. Interactions with the hard modes, however, not only provide dissipation for the soft modes; they also serve as a source of thermal noise. In the above analysis, the noise has been implicitly disregarded, and we will need to put it in if we wish to describe equilibrium fluctuations. Fortunately, this is simple to do after the fact because noise and dissipation are intimately related by the fluctuation-dissipation theorem. In the language of an effective theory of the soft modes, equilibrium requires a delicate balance between the soft modes' excitation from thermal noise and their dissipative decay.

To be more specific, note that Eq. (2.5) has the general form

$$\sigma\frac{d}{dt}\mathbf{q} = -\boldsymbol{\nabla}_{\mathbf{q}}V(\mathbf{q}), \tag{2.6}$$

where $V(\mathbf{q})$ is the potential energy of the degrees of freedom $\mathbf{q}$ (which in our case is the non-Abelian magnetic energy $\frac{1}{2}\int_{\mathbf{x}}B^2$). Such systems are common in physics, and a simple way to incorporate thermal noise is to include a random force $\boldsymbol{\zeta}$:

[3]That is fortunate, because trying to make such a split explicit creates a host of difficulties. See Ref. [9].

[4]We are using ''color'' as a descriptive name for some non-Abelian gauge field. It should be emphasized that all discussion of ''color'' is applicable to the dynamics of, in particular, the SU(2) electroweak gauge field.

$$\sigma \frac{d}{dt}\mathbf{q} = -\nabla_{\mathbf{q}} V(\mathbf{q}) + \boldsymbol{\zeta}. \tag{2.7}$$

This is a typical example of a Langevin equation. The simplest possible choice of thermal noise, Gaussian white noise, reproduces the correct equilibrium distribution $\exp(-\beta V)$ if the noise variance is suitably scaled with the amount of dissipation,

$$\langle \zeta_i(t)\zeta_j(t')\rangle = 2\sigma T\, \delta_{ij}\, \delta(t-t'). \tag{2.8}$$

This well-known result can be verified by converting the Langevin equation (2.7) into a Fokker-Planck equation for the probability distribution. (See, for example, Chap. 4 of Ref. [11].)

Why should one believe the noise distribution is so simple? First, the noise can be treated as Gaussian if the soft dynamics of interest has a time scale large compared to the decorrelation time of the noise, which is caused by fluctuations of the hard modes. Averaging the noise over time scales small compared to the soft dynamics scale but large compared to the noise decorrelation time, the central limit theorem implies that the resulting distribution will approach a Gaussian shape. We will see later (Sec. III A) that in our case the relevant decorrelation time for hard fluctuations is $1/(g^2 T \ln g^{-1})$ whereas the time scale for soft dynamics is the longer scale $1/(g^4 T \ln g^{-1})$ asserted earlier. Second, if the theory were linearized, then the fact that the spectrum of this Gaussian noise is white noise would follow rigorously from the fluctuation-dissipation theorem. More generally, any noise spectrum $f(\omega)$ may be regarded as frequency-independent (i.e., white noise) at sufficiently small frequency ω provided $f(0)$ is finite and non-zero. So effective theories for long time scales can generally be expected to have Gaussian white noise. Finally, one might wonder why there could not be some non-linear coupling to the noise, in the form of a function $e(\mathbf{q})$ multiplying the noise term $\boldsymbol{\zeta}$ in Eq. (2.7). Generically, the introduction of such a $\mathbf{q}$-dependence would change the equilibrium distribution produced by Eq. (2.7) so that it would not correctly reproduce $\exp(-\beta V)$.

Based on the above discussion, let us introduce noise as in Eq. (2.7). Translating back to our particular system (2.5), we obtain the following effective theory for the soft modes:

$$\sigma \frac{d}{dt}\mathbf{A} = -\mathbf{D}\times\mathbf{B} + \boldsymbol{\zeta}, \tag{2.9a}$$

$$\langle \zeta_i^a(t,\mathbf{x})\, \zeta_j^b(t',\mathbf{x}')\rangle = 2\sigma T\, \delta^{ab}\delta_{ij}\, \delta(t-t')\, \delta(\mathbf{x}-\mathbf{x}'), \tag{2.9b}$$

where i,j and a,b are spatial vector and adjoint color indices, respectively. Those readers interested in a more technical derivation of the noise term starting somewhat closer to first principles should consult Bödeker [6].

Astute readers may notice a peculiarity of Eq. (2.9): it introduces noise for the longitudinal as well as transverse modes of $\mathbf{A}$, whereas the effective theory is only meant to describe the transverse modes. (The longitudinal modes are the pieces of $\mathbf{E}$ which contribute to $\mathbf{D}\cdot\mathbf{E}$ and perturbatively correspond to polarizations parallel to the spatial momentum $\mathbf{k}$.) We discuss the noise-driven longitudinal dynamics generated by Eq. (2.9) in great detail in Ref. [12], but the matter is not directly relevant to the present discussion.

The effective equation (2.9) turns out to have the wonderful property that it is insensitive to how the soft modes are cut off at large momentum. (We will discuss this in greater depth in Sec. IV.) It means that one can ignore the soft/hard separation that was necessary to write Eq. (2.1) but which was never specified in detail. It means that Eq. (2.9) will be insensitive to short-distance lattice cut-offs used in numerical simulations. Finally, it also means that such simulations will not be plagued by lattice artifacts, such as loss of rotational invariance, that were thought to arise in other approaches [5].

From Eq. (2.9a) and $\mathbf{B}=\mathbf{D}\times\mathbf{A}$, one can immediately see that the time scale of non-perturbative dynamics is given by

$$\sigma t^{-1} A \sim R^{-2} A, \tag{2.10}$$

so that

$$t \sim R^2 \sigma \sim \frac{\sigma}{g^4 T^2}. \tag{2.11}$$

Thus, one need only know the color conductivity σ. There has been some confusion in the literature (described later) about this quantity, but the correct value was first presented by Selikhov and Gyulassy [13]. The color conductivity is of order

$$\sigma \sim \frac{T}{\ln(1/g)}. \tag{2.12}$$

We will review later how to understand this physically. Inserting Eq. (2.12) into Eq. (2.11) then gives the time scale

$$t \sim \frac{1}{g^4 T \ln(1/g)}, \tag{2.13}$$

and so $\Gamma \sim \alpha^5 T^4 \ln(1/\alpha)$, which has the logarithmic enhancement claimed by Bödeker. Later, we will see that earlier estimates [3] of the time scale as $t \sim 1/(g^4 T)$ correspond to ignoring the effects of collisions on the conductivity. Note that ignoring the time derivative term in Eq. (2.3) was justified since the characteristic time scale (2.13) is much greater than the inverse conductivity σ^{-1} determined by (2.12).

On a more quantitative level, the color conductivity is [13][5]

$$\sigma \approx \frac{m_{\rm pl}^2}{\gamma_{\rm g}}, \tag{2.14}$$

where $m_{\rm pl}$ is the plasma frequency and

$$\gamma_{\rm g} \approx \alpha\, C_{\rm A} T \ln(1/g) \tag{2.15}$$

[5]The reader of Ref. [13] should beware the final equation of that paper, Eq. (47). In that equation, the authors replace their result by something rough and approximate.

is the damping rate for hard thermal gauge bosons [14].[6] Here, C_A is the adjoint Casimir, conventionally normalized as $C_A = N$ for the gauge group SU(N), and ''$\approx$'' means equality up to relative corrections suppressed by powers of $\ln(1/g)$. That is, no claim is made about discriminating $\ln(1/g)$ from $\ln(2/g)$.[7] The only place where the matter content of the theory enters is in the value of the plasma frequency. For hot electroweak theory with a single Higgs doublet, it is given by

$$m_{\rm pl}^2 = \frac{(5+2n_{\rm f})}{18} g^2 T^2 [1+O(g)], \tag{2.16}$$

where $n_{\rm f}$ is the number of fermion families. The Langevin equation (2.9) with the value (2.14) of σ precisely reproduces the effective theory derived by Bödeker [6].

It is interesting to note that, if the time is rescaled, the Langevin equation (2.9) is equivalent to the stochastic quantization of three-dimensional Euclidean gauge theory.[8] In that context, the time t is usually considered a fictitious additional variable, corresponding in simulations to Monte Carlo time. Amusingly, the present application provides an instance where Monte Carlo time for gauge theories is actually real time, up to a calculable rescaling.

III. COLOR CONDUCTIVITY

A. Qualitative description

We now review why the color conductivity depends on coupling as in Eq. (2.12), and show how earlier estimates [3–5] of the time scale for non-perturbative processes as $t \sim 1/(g^4 T)$ correspond to ignoring collision effects. Begin by considering the current response to an external electric field in a *collisionless* ultra-relativistic plasma. For simplicity of notation, consider a QED plasma for the moment rather than a non-Abelian one. If the external field were static and homogeneous, particles in the plasma with charge g would respond to the field by a change in momentum

$$\Delta \mathbf{p} = g \mathbf{E}\, \Delta t \tag{3.1}$$

over a time Δt. For small deviations, the change in velocity of a typical particle whose energy is order T would then be

$$\Delta \mathbf{v} \sim \frac{\Delta \mathbf{p}}{p^0} \sim \frac{g \mathbf{E}\, \Delta t}{T}, \tag{3.2}$$

and the resulting current would be

$$\mathbf{J} \sim n\, g\, \Delta \mathbf{v} \sim (g^2 T^2 \Delta t)\, \mathbf{E}, \tag{3.3}$$

where $n \sim T^3$ is the density of hard particles. The current is dominated by the most prevalent particles in the plasma: those with momentum of order T. The current (3.3) grows indefinitely with the length of time the electric field is applied. There are two things which can cut off this growth of the current: (a) collisions, and (b) temporal or spatial oscillation of the electric field. Stick with the collisionless plasma for a moment and consider oscillations of $\mathbf{E}$. As we have discussed, the time scale for non-perturbative processes turns out to be slow. So suppose, for example, that the electric field varies in the z direction as $\mathbf{E} \sim \mathbf{E}_0 \cos(kz)$ but not significantly in time. Then current carriers, which have an rms z velocity of $1/\sqrt{3}$, will move from regions of positive E_z to regions of negative E_z in a time of order

$$\Delta t \sim k^{-1}. \tag{3.4}$$

This change in direction of the electric field felt by the charge carriers then limits the average current response to a magnitude

$$J \sim \frac{g^2 T^2}{k} E. \tag{3.5}$$

If we identify the (k-dependent) conductivity as

$$\sigma(k) \sim \frac{g^2 T^2}{k} \quad \text{(collisionless)}, \tag{3.6}$$

and take k to be the inverse spatial scale $g^2 T$ for non-perturbative physics, then the time scale t for non-perturbative physics (2.11) would be

$$t \sim \frac{1}{g^2 k} \sim \frac{1}{g^4 T}, \tag{3.7}$$

provided we could indeed ignore the effects of collisions on the conductivity. This is the qualitative physics behind the more formal and quantitative discussions of Refs. [3–5].[9]

The divergence of the conductivity (3.6) as $k \to 0$ is cut off in real physical systems by the effects of collisions, as pointed out by Drude in 1900. Let us continue to focus on a QED plasma for the moment. A charge accelerated by the electric field eventually experiences a collision with other particles in the plasma which changes the charged particle's direction by a large angle. Such collisions randomize the

[6] In the literature, the hard thermal ''damping rate'' is defined (in one-loop perturbation theory) as the imaginary part of the pole energy for a propagating gauge boson. In particular, it is defined so that the *amplitudes* of plasma waves decay as $\exp(-\gamma t)$. This is in contrast to the standard usage of the ''width'' Γ of a resonance (for example, of the Z boson at zero temperature), which is defined so that the probability (or equivalently the intensity or particle number) associated with the resonance decays as $\exp(-\Gamma t)$. The relation is simply $\Gamma = 2\gamma$.

[7] It is not clear whether the color conductivity even has meaning except as an approximate concept valid at the level of leading logarithms. We do not know, for instance, of any directly measurable (gauge-invariant, non-perturbative) definition of the color conductivity.

[8] See, for example, Chap. 17 of Ref. [11].

[9] $\sigma(k)$ corresponds exactly to the damping coefficient γ introduced in Ref. [5].

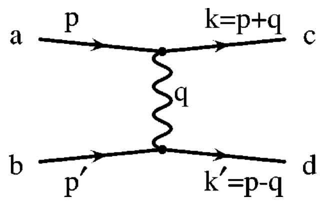

FIG. 1. The dominant scattering process: t-channel gauge boson exchange. The solid lines represent any sort of hard particles, including gauge bosons themselves. The labels a,b,c,d show our convention for naming color indices of the various lines.

direction of the particle and so randomize its contribution to the current. So the relevant time Δt determining the conductivity (3.2) becomes the mean collision time $\tau_{\rm large}$ for large angle scatterings:

$$\tau_{\rm large}^{-1} \sim g^4 T \ln\left(\frac{T}{m_{\rm pl}}\right). \tag{3.8}$$

The g^4 above just comes from the square of the scattering matrix element. The logarithm arises because the randomization of the velocity can occur either through a single large-angle scattering or through the cumulative effect of many (individually more probable) small-angle scatterings.[10] If $\tau_{\rm large}$ were the relevant mean free time in the non-Abelian case, then the effects of collisions on the conductivity $\sigma(k)$ could safely be ignored when investigating non-perturbative fluctuations. That is because $\tau_{\rm large} \gg 1/k \sim 1/g^2 T$, and so it would be the collisionless time scale $1/k$ instead of $\tau_{\rm large}$ that determines Δt and hence $\sigma(k)$.

However, Selikhov and Gyulassy [13] have pointed out that $\tau_{\rm large}$ is not the relevant mean free time in the non-Abelian case. In the non-Abelian case, even arbitrarily small angle scatterings can randomize the current, not by randomizing the velocity of the current carriers, but by randomizing their color charge. The crucial difference with QED is that an exchanged non-Abelian gauge boson, no matter how soft, carries color and so changes the color charge of the scatterers, whereas an exchanged photon is neutral. The relevant time scale for the non-Abelian case is then the mean free time $\tau_{\rm small}$ for *any*-angle scattering, which is much shorter than the mean free time for large-angle scattering. Specifically, t-channel gauge boson exchange, shown in Fig. 1, gives a cross section $\hat{\sigma}$ such that

$$\tau_{\rm small}^{-1} \sim n\hat{\sigma} \sim n g^4 \int \frac{dt_{\rm M}}{t_{\rm M}^2}, \tag{3.9}$$

where $n \sim T^3$ is the density of particles and $t_{\rm M} = -Q^2$ is the virtuality of the exchanged gauge boson. $\tau_{\rm small}^{-1}$ is also known as the thermal damping rate of the hard particle carrying the current [14,17,18]. For $t_{\rm M}$ below $m_{\rm pl}^2 \sim (gT)^2$, screening effects in the plasma turn out to reduce the linear $t_{\rm M} \to 0$ divergence in Eq. (3.9) to a logarithmic one. The result is then that[11]

$$\tau_{\rm small}^{-1} \sim \frac{n g^4}{m_{\rm pl}^2} \ln\left(\frac{m_{\rm pl}}{g^2 T}\right) \sim g^2 T \ln\left(\frac{1}{g}\right), \tag{3.10}$$

where the scale $k \sim g^2 T$ of non-perturbative physics has been used as an infrared cut-off. Using Eq. (3.2) and comparing $\tau_{\rm small}$ to the collisionless time scale $1/k$, the zero-frequency conductivity $\sigma \sim g^2 T^2 \Delta t$ is then

$$\sigma(k) \sim \begin{cases} g^2 T/k, & k \gtrsim \tau_{\rm small}^{-1}, \\ g^2 T\, \tau_{\rm small}, & k \lesssim \tau_{\rm small}^{-1}. \end{cases} \tag{3.11}$$

$\tau_{\rm small}$ wins by a logarithm for $k \sim g^2 T$. This means that, in the small coupling, large logarithm limit, the $k \to 0$ value of the conductivity, namely $\sigma \sim T/\ln(1/g)$, is what is relevant to non-perturbative physics in non-Abelian plasmas.

Some readers may want to know what Feynman diagrams, in the underlying, fundamental quantum field theory, correspond to the color conductivity discussed above. In the next section, we formulate a leading-log calculation of the conductivity in terms of the Boltzmann equation. Based on (a) the analogy of QCD with scalar $\phi^3 + \phi^4$ theory (both have 3- and 4-point interactions), (b) the diagrammatic analysis of transport coefficients for the latter theory and its equivalence to the Boltzmann equation as explained in Refs. [19,20], and (c) the fact that, in the gauge theory case, only t-channel scattering processes are relevant at the order of interest, we believe that the relevant series of Feynman diagrams are the ladder diagrams shown in Fig. 2. This is similar to the class of diagrams considered in Ref. [17] for QED. Diagrammatic perturbation theory in this form is awkward and cumbersome, however, and we shall avoid it.

B. Quantitative description

The original calculation of the color conductivity by Selikhov and Gyulassy [13] was clever but not absolutely convincing. For one thing, it was based on an approximation to the evolution of color distribution functions which assumes that there is no coupling between the different velocity components of a fluctuation. (We shall explain more clearly what this means below.) The approximation is incorrect in general but, as we shall see, does not affect the calculation of the color conductivity in particular. Subsequently, Heiselberg [21] analyzed the quark contribution to the conductivity by starting with a Boltzmann equation with an appropriate collision term. He obtained the same dependence on coupling g as Selikhov and Gyulassy but a different numerical coefficient. As we shall later explain, this difference was primarily due to the use of a plausible but inadequate variational ansatz. Selikhov and Gyulassy [22] subsequently published an alternative derivation of the color conductivity that also

[10] For a slightly more detailed but still qualitative summary see, for example, Sec. III of Ref. [15]. $\tau_{\rm large}$ is also known as the ''momentum relaxation'' time (see, for example, Ref. [16]).

[11] Again, for more qualitative detail, see the review in Sec. III of Ref. [15]. For the original work, see Ref. [14].

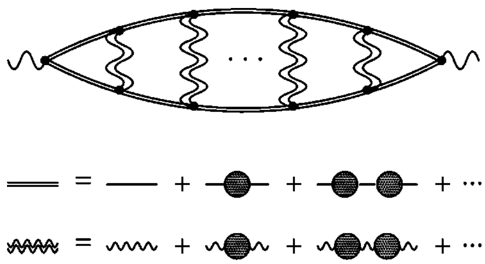

FIG. 2. The Feynman diagrams (assuming finite-temperature Feynman rules) that produce the conductivity due to hard excitations at leading-log order. Specifically, the ladder diagrams are for the self-energy of the soft fields, whose imaginary part is proportional to the conductivity at low frequency (it's $\omega\sigma$ in $A_0=0$ gauge). The external lines have soft momentum (g^2T) and softer frequencies, the solid lines correspond to any type of colored particle with hard momentum (T), and the rungs have semi-hard momentum ($g^2T\ln g^{-1}\ll q\ll gT$). The double lines indicate that the dominant one-loop contributions to the self-energies have been included in the propagators. The analog of the two-loop chain diagram of ϕ^3 theory [19,20] is not included because we only integrate out hard and semi-hard, but not soft, fields to obtain the effective theory of interest. Other diagrams relevant to ϕ^3 theory (e.g., non-pinching boxes and chain diagrams) have been dropped because they do not correspond to t-channel scattering and so should be sub-leading in the gauge-theory case.

started from the Boltzmann equation with a collision term [23]. Unfortunately, the collision term they used did not account for quantum statistics of the hard particles and, as a result, they were unable to obtain a final answer without making some very rough approximations along the way.[12] Indirectly, therefore, Bödeker's [6] results seem to be the first complete (albeit terse) quantitative analysis of the color conductivity, even though that is not the language he uses. We present here our own direct derivation of the conductivity, based on the collision term approach, which will be more familiar to some readers (and so perhaps more comforting) than Bödeker's methods.

There are three scales relevant to understanding the conductivity at leading-log order. Following Bödeker, we will label them as (a) the hard scale, corresponding to momentum T, and characteristic of the charges which carry the current, (b) the soft scale, corresponding to momentum g^2T, characteristic of the non-perturbative electric fields that the hard charge carriers respond to, and (c) the *semi-hard* scale, corresponding to momenta q in the range $g^2T\ll q\lesssim gT$, which (at leading-log order) is the momentum scale of the t-channel gauge bosons that mediate color randomization of the hard charge carriers. Remember that the logarithm in the conductivity is a logarithm of the plasma frequency scale gT over the soft scale g^2T. If we tried to go beyond leading-log order, then the distinction between semi-hard and soft would blur, because soft gauge bosons can also mediate color-randomizing processes.

[12]Such as the approximation made in and just above Eq. (15) of Ref. [22].

We will describe the hard, perturbative modes of the theory by a Boltzmann equation. This is well known to reproduce exactly (at leading order in coupling) a large variety of thermal results obtained by a more fundamental analysis of diagrammatic perturbation theory.[13] This kinetic description is valid whenever the mean free time is long enough that the hard particles can be treated as propagating classically (i.e., on-shell) between collisions—a condition to be discussed momentarily. We will couple the hard particles to a soft electric field, and we will incorporate the semi-hard mediated scattering processes into the collision term.

The requirement that particles propagate classically between collisions means that the de Broglie wavelength and the collision times must be small compared to the mean free path and mean free time, respectively. The relevant mean free time (and path) here is $\tau_{\rm small}\sim(g^2T\ln)^{-1}$, where here, and henceforth, we use ‘‘ln’’ as shorthand for $\ln g^{-1}$. The de Broglie wavelength of the hard particles is order $1/T \ll\tau_{\rm small}$. The duration of collisions[14] mediated by semi-hard gauge bosons is order $1/|q\pm q_0|\sim 1/q$ and is small compared to $\tau_{\rm small}$ when $g^2T\ln\ll q$. This requirement means that we can only properly account for scatterings with semi-hard momentum transfer q having $g^2T\ln\ll q\lesssim gT$ rather than $g^2T \ll q\lesssim gT$. This will not, however, affect results at leading-log order, which does not distinguish between $\ln(gT/g^2T)$ and $\ln(gT/g^2T\ln)$.

1. The Boltzmann (Waldmann-Snider) equation

To introduce the Boltzmann equation we will use, let's start by ignoring the details of color and the non-Abelian nature of the problem. (We will return to the non-Abelian case momentarily.) Pretend, for the moment, that we were interested in the Boltzmann equation for hard particles in a QED plasma.[15] Schematically, the Boltzmann equation for the local distribution $n(\mathbf{x},\mathbf{p},t)$ of hard particles is of the form

$$\frac{dn}{dt}=-C[n], \tag{3.12}$$

where $C[n]$ is a collision term describing the net loss, due to semi-hard scattering, of hard particles with momentum $\mathbf{p}$.

[13]For an explicit discussion of the relationship between diagrammatic perturbation theory and the Boltzmann equation in scalar theory, see Ref. [20]. For kinetic theory descriptions of gauge theories, including the extraction of ‘‘hard thermal loops’’ from collisionless kinetic theory, see Refs. [24,25].

[14]One way to estimate the scattering duration is to consider specific time orderings of Fig. 1 and to estimate the energy difference ΔE between the initial and intermediate states. The duration is then order $1/\Delta E$. Alternatively, one may consider a Feynman diagram representing two successive collisions of a particle and verify the requirement that a typical particle will be sufficiently on-shell between collisions that no error (at the desired order) is made by treating the collisions separately.

[15]Kinetic theory for QED plasmas has, of course, a long history. See for example Ref. [26].

The total time derivative can be rewritten in terms of a convective derivative and the force exerted by the soft fields as

$$\frac{dn}{dt} \equiv (\partial_t + \dot{\mathbf{x}}\cdot\boldsymbol{\nabla}_{\mathbf{x}} + \dot{\mathbf{p}}\cdot\boldsymbol{\nabla}_{\mathbf{p}})\, n = \partial_t n + \mathbf{v}\cdot\boldsymbol{\nabla}_{\mathbf{x}}\, n + g(\mathbf{E}+\mathbf{v}\times\mathbf{B})\cdot\boldsymbol{\nabla}_{\mathbf{p}}\, n, \tag{3.13}$$

where **E** and **B** are to be understood as soft. [The difference between this and Bödeker's approach [6] is that Bödeker starts with a collisionless Boltzmann equation ($dn/dt=0$) but includes coupling to dynamical semi-hard fields in Eq. (3.13).]

The collision term is dominated by $2\rightarrow 2$ collisions and has the form[16]

$$C[n] = \int_{\mathbf{p}'\mathbf{k}\mathbf{k}'} |M_{\mathbf{p}\mathbf{p}'\rightarrow\mathbf{k}\mathbf{k}'}|^2 [n_{\mathbf{p}} n_{\mathbf{p}'}(1\pm n_{\mathbf{k}})(1\pm n_{\mathbf{k}'}) - n_{\mathbf{k}} n_{\mathbf{k}'}(1\pm n_{\mathbf{p}})(1\pm n_{\mathbf{p}'})] \tag{3.14a}$$

$$\equiv n_{\mathbf{p}}\,\mathcal{I}_- - (1\pm n_{\mathbf{p}})\,\mathcal{I}_+ , \tag{3.14b}$$

where $M_{\mathbf{p}\mathbf{p}'\rightarrow\mathbf{k}\mathbf{k}'}$ is the matrix element for the collision, and the $1\pm n$ are final-state Bose enhancement or Fermi blocking factors, depending on whether the hard particles are bosons $(+)$ or fermions $(-)$. The first term of $C[n]$ in Eq. (3.14) is a loss term, representing scattering out of momentum state **p**, and the second term is a gain term, representing scattering into state **p**. Coefficients of the loss and gain terms, $\mathcal{I}_\mp$, have been introduced for later convenience. In equilibrium, $C[n]$ vanishes.

We have not included the coupling of the soft electromagnetic fields to the spin of the hard particles in Eq. (3.13). There are a number of independent reasons for this: (a) the small-angle scatterings that determine the conductivity are insensitive to the spins of the colliding particles, (b) such terms vanish when one linearizes the Boltzmann equation [25], as we shall eventually do, and (c) for hard massless quarks, at least, the spin dynamics is made trivial by conservation of helicity. See Refs. [25,27,28] for a discussion of including spin effects in the Boltzmann equation.

We must now face the one subtlety in this derivation, which is how to incorporate color into the collision term. It is easy to put flavor indices into a collision term if all distribution functions n are diagonal in flavor: one must simply use the specific matrix elements for flavors a,b to collide and produce flavors c,d and then sum appropriately over flavor indices. The problem is more subtle for color, however, since, as we will review, the distribution functions that describe color fluctuations are not diagonal in color space. The need to deal with a non-diagonal distribution n is a problem that has arisen previously in applications involving massive particles with spin: if one quantizes spin in the z direction, there is then no way to describe spin 1/2 particles with spin, say, in the x direction in terms of definite numbers $n_\pm$ of particles with spin in the $\pm z$ direction. In fact, the problem goes back to the physics of dilute gases of molecules with spins that, between collisions, precess in an external magnetic field; the generalization of the Boltzmann equation which solved that problem is known as the Waldmann-Snider equation.[17] We need the appropriate generalization to the problem at hand.

In preparation, let us review the incorporation of color into the collisionless part (3.13) of the Boltzmann equation, which is relatively well known [31]. We will give a quick summary, rather than starting from first principles. The first thing to note is that number operators for particles are of the form $a_{\mathbf{p}}^\dagger a_{\mathbf{p}}$ in terms of creation and annihilation operators.[18] Since both a and $a^\dagger$ carry color indices, we see that n is a matrix and transforms under color as $\mathcal{R}\times\overline{\mathcal{R}}$ if the hard particles are in the representation $\mathcal{R}$. To generalize the convective derivative in Eq. (3.13) to the non-Abelian case, gauge-invariance then requires the derivatives ∂_t and $\boldsymbol{\nabla}_{\mathbf{x}}$ to be replaced by gauge-covariant derivatives $\mathcal{D}_t$ and $\mathcal{D}_{\mathbf{x}}$ acting in the $R\times\overline{R}$ representation. That is,

$$\partial_\mu n \rightarrow \mathcal{D}_\mu n = \partial_\mu n - ig\,[A_\mu , n], \tag{3.15}$$

where A_μ is the soft gauge field expressed in terms of generators of the representation $\mathcal{R}$ of the hard particles. The third term in Eq. (3.13)—the electromagnetic force term—could be color contracted as either $[E,\nabla_{\mathbf{p}} n]$ or $\{E,\nabla_{\mathbf{p}} n\}$, with E expressed in terms of generators of $\mathcal{R}$. The fact that n, and so also dn/dt, are Hermitian rules out the commutator. So the non-Abelian Boltzmann equation is

$$(\mathcal{D}_t + \mathbf{v}\cdot\mathcal{D}_{\mathbf{x}})\, n + \tfrac{1}{2} g\,\{(\mathbf{E}+\mathbf{v}\times\mathbf{B})_i , \boldsymbol{\nabla}_{p_i} n\} = -C[n], \tag{3.16}$$

where we have yet to specify the collision term $C[n]$. In the Appendix, we discuss how to generalize the collision term (3.14). (The result is substantively equivalent to one derived by Botermans and Malfliet [32] in the context of one-boson exchange processes in nuclear matter.[19]) Here, we will just try to make the result plausible. It is fairly easy to guess how to contract all the color indices in Eq. (3.14) *other* than the one whose net loss is being described (that is, other than the one associated with **p**). Refer to Fig. 1:[20]

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[16]Our notation is $\int_{\mathbf{p}} \equiv \int [d^3p/(2\pi)^3]$ for momentum integrals and $\int_{\mathbf{x}} \equiv \int d^3x$ for position integrals. With this convention, transition matrix elements M should be understood to have non-relativistic rather than covariant normalization.

[17]For a review, see Ref. [30].

[18]More technically, the localized number densities $n(\mathbf{x},\mathbf{p})$ correspond to expectations of the Wigner operators $a^\dagger_{\mathbf{p}+\mathbf{k}/2} a_{\mathbf{p}-\mathbf{k}/2}$ where **k** is the Fourier transform variable conjugate to **x** and should be regarded as small compared to **p**.

[19]Botermans and Malfliet, however, absorb the $\mathrm{Re}\,\Sigma$ term in Eq. (3.19) by redefining their flavor states to diagonalize the effective Hamiltonian.

[20]We make no particular distinction between upper and lower color indices.

$$\mathcal{I}_-^{a\bar{a}} = \int_{\mathbf{p}'\mathbf{k}\mathbf{k}'} M^*_{\overline{abcd}} M_{abcd} n_{\mathbf{p}'}^{\bar{b}b} (1 \pm n_{\mathbf{k}})^{c\bar{c}} (1 \pm n_{k'})^{d\bar{d}}, \tag{3.17}$$

$$\mathcal{I}_+^{a\bar{a}} = \int_{\mathbf{p}'\mathbf{k}\mathbf{k}'} M^*_{\overline{abcd}} M_{abcd} n_{\mathbf{k}}^{c\bar{c}} n_{\mathbf{k}'}^{d\bar{d}} (1 \pm n_{\mathbf{p}'})^{\bar{b}b}. \tag{3.18}$$

Here and henceforth, there will always be an implied summation over the types and spins of the particles associated with $\mathbf{p}'$ (quarks, anti-quarks, gauge bosons, Higgs, etc.). In terms of $\mathcal{I}_\pm$, the correct collision term then turns out to be

$$C[n] = \tfrac{1}{2}\{n_{\mathbf{p}}, \mathcal{I}_-\} - \tfrac{1}{2}\{1 \pm n_{\mathbf{p}}, \mathcal{I}_+\} - i\,[\mathrm{Re}\,\overline{\Sigma}, n_{\mathbf{p}}], \tag{3.19}$$

where all commutators are in color space and $\overline{\Sigma}$ is the self-energy of the hard particles (non-relativistically normalized). Equation (3.19) assumes that $\mathrm{Re}\,\overline{\Sigma}$ can be treated as small compared to tree-level energies, which is indeed true for hard excitations.

The appearance of the self-energy term is easy to understand, although it will disappear when we linearize the Boltzmann equation. Time evolution of observables, ignoring dissipation, is given by

$$\frac{dA}{dt} = i\,[H_{\mathrm{eff}}, A], \tag{3.20}$$

and the real part of the self-energy contributes to the effective Hamiltonian. The loss and gain terms are related to the imaginary part of the self-energy, and a simple mnemonic (although hardly a real derivation) for the appearance of anti-commutators in those terms is to consider the time evolution of an observable with a non-Hermitian effective Hamiltonian:

$$A(t) = U^\dagger(t)\, A(0)\, U, \quad U(t) = e^{-iH_{\mathrm{eff}}t}, \text{ and } H_{\mathrm{eff}} = R + iI, \tag{3.21}$$

so that

$$\frac{dA}{dt} = \{I, A\} + i\,[R, A]. \tag{3.22}$$

This is the same sort of structure that appears in the collision term. As already mentioned, the real argument for Eq. (3.19) is given in the Appendix.

If the final-state statistical factors are ignored, so that $1 \pm n \to 1$, then Eq. (3.19) has exactly the form of the relativistic collision term presented in Ref. [27][21] for spin (as opposed to color) degrees of freedom.

[21]Specifically, Eq. (26) of Sec. B IV 3 of Ref. [27].

2. The linearized Boltzmann equation

We will now linearize the Boltzmann equation, since the fluctuations in the hard particles induced by soft fields are small (as parameterized by powers of the coupling). Write

$$n^{a\bar{a}} = n_{\mathrm{eq}}\, \delta^{a\bar{a}} + \delta n^{a\bar{a}}, \tag{3.23}$$

where n_{eq} is the equilibrium distribution and is colorless. The linearization of the Boltzmann equation given by Eq. (3.16) is

$$(\mathcal{D}_t + \mathbf{v}\cdot\mathcal{D}_{\mathbf{x}})\, \delta n + g\, \mathbf{E}\cdot\mathbf{v} \frac{dn_{\mathrm{eq}}}{dp} = -\delta C[\delta n], \tag{3.24}$$

where δC is the linearization of Eq. (3.19). The equilibrium self-energy must be colorless (proportional to $\delta^{a\bar{a}}$), and so the linearization of the self-energy term in δC vanishes:

$$\delta[\mathrm{Re}\,\Sigma, n] = [\delta(\mathrm{Re}\,\Sigma), n_{\mathrm{eq}}] + [\mathrm{Re}\,\Sigma_{\mathrm{eq}}, \delta n] = 0. \tag{3.25}$$

The linearization of the loss and gain ($\mathcal{I}_\pm$) pieces of the collision integral may be simplified by recalling that small-angle collisions will dominate the physics. The dominant momentum transfer q lies between g^2T and gT, and is small compared to the momenta of the colliding hard particles. So, to leading order in coupling, we can replaces $n_{\mathbf{k}} = n_{\mathbf{p}+\mathbf{q}}$ and $n_{\mathbf{k}'} = n_{\mathbf{p}'-\mathbf{q}}$ by $n_{\mathbf{p}}$ and $n_{\mathbf{p}'}$. The result of linearizing the collision term (3.19) in this small momentum-transfer approximation is then[22,23]

$$\delta C[\delta n] = \tfrac{1}{2} \int_{\mathbf{p}'\mathbf{q}} |\mathcal{M}|^2 \{ t_{\mathcal{R}'} [T^a_{\mathcal{R}}, [T^a_{\mathcal{R}}, \delta n_{\mathbf{p}}]]\, n_{\mathbf{p}'} (1 \pm n_{\mathbf{p}'}) - C_{\mathrm{A}} T^c_{\mathcal{R}}\, \mathrm{tr}(T^c_{\mathcal{R}'}\, \delta n_{\mathbf{p}'})\, n_{\mathbf{p}} (1 \pm n_{\mathbf{p}})\}, \tag{3.26}$$

where here (and henceforth) we have dropped the subscript "eq" from the equilibrium distribution n_{eq}. The matrices $\{T^a_{\mathcal{R}}\}$ are color generators for the representation $\mathcal{R}$; $\mathcal{M}$ is the t-channel matrix element of Fig. 1 stripped of color generators, $M_{abcd} = \mathcal{M}\, T^{ac}_{\mathcal{R}} T^{bd}_{\mathcal{R}'}$; and $t_{\mathcal{R}}$ is the normalization constant defined by

$$\mathrm{tr}(T^a_{\mathcal{R}} T^b_{\mathcal{R}}) = t_{\mathcal{R}}\, \delta^{ab}, \tag{3.27}$$

which is C_{A} for the adjoint representation and, with conventional normalization, $\frac{1}{2}$ for the fundamental.

[22]The collision term in Eq. (3.26) is the same as the ΔC_2 given by Selikhov and Gyulassy in Eq. (6) of Ref. [22] (originally derived by Selikhov [23]) except for the statistical factors of $1 \pm n$.

[23]We have swept under the rug the fact that the matrix element depends on the self-energy Π of the exchanged gauge boson, which in turn depends on the distribution functions n. One should consider fluctuations of these distribution functions as well, but, at linear order in δn, these variations do not contribute to δC because the loss and gain terms cancel.

Note that the expression (3.26) for δC vanishes for color neutral fluctuations, i.e., when δn is proportional to the identity. To treat such fluctuations, one must expand $n_{\mathbf{k}}=n_{\mathbf{p}+\mathbf{q}}$ and $n_{\mathbf{k}'}=n_{\mathbf{p}-\mathbf{q}}$ to higher order in $\mathbf{q}$ than we have done, which leads to suppression by more powers of g. (An example is the difference between the inverse momentum relaxation time $\tau_{\rm large}^{-1}$ and the color relaxation time $\tau_{\rm small}^{-1}$ discussed earlier.)

For comparison to δC (3.26), note that the hard thermal damping rate defined from the imaginary part of the self-energy in equilibrium is, to leading order in coupling,[24]

$$\gamma_{\mathcal{R}}=\frac{1}{2}\,\frac{d}{dn_{\mathbf{p}}}\biggl\{\int_{\mathbf{p}'\mathbf{q}}|\mathcal{M}|^2\,C_{\mathcal{R}}\,t_{\mathcal{R}'}\,[n_{\mathbf{p}}\,n_{\mathbf{p}'}\,(1\pm n_{\mathbf{k}})\,(1\pm n_{\mathbf{k}'})$$
$$-n_{\mathbf{k}}\,n_{\mathbf{k}'}\,(1\pm n_{\mathbf{p}})\,(1\pm n_{\mathbf{p}'})]\biggr\}$$
$$=\frac{1}{2}\int_{\mathbf{p}'\mathbf{q}}|\mathcal{M}|^2\,C_{\mathcal{R}}\,t_{\mathcal{R}'}\,[n_{\mathbf{p}'}\,(1\pm n_{\mathbf{k}})\,(1\pm n_{\mathbf{k}'})$$
$$\mp n_{\mathbf{k}}\,n_{\mathbf{k}'}\,(1\pm n_{\mathbf{p}'})]$$
$$\simeq\frac{1}{2}\int_{\mathbf{p}'\mathbf{q}}|\mathcal{M}|^2\,C_{\mathcal{R}}\,t_{\mathcal{R}'}\,n_{\mathbf{p}'}\,(1\pm n_{\mathbf{p}'}),\tag{3.28}$$

where in the last step we have used the small momentum-transfer limit (valid at leading order in coupling). As always, there is an implicit summation over the particle type and spin associated with $\mathbf{p}'$ in Eq. (3.28). Note that the coefficient of δn in the first term of Eq. (3.26) for δC is, up to color factors, just the thermal damping rate $\gamma_{\mathcal{R}}$.

The representation $\mathcal{R}\times\overline{\mathcal{R}}$ we have been ascribing to fluctuations δn is reducible and a bit over-general for our needs. There is only one irreducible component of $\mathcal{R}\times\overline{\mathcal{R}}$ which contributes to the conductivity—the adjoint representation. There are a number of ways to see this. First, the color current $\mathbf{J}$ is given by

$$\mathbf{J}^a=g\int_{\mathbf{p}}\mathrm{tr}(T^a_{\mathcal{R}}\,\delta n_{\mathbf{p}})\,\mathbf{v}_{\mathbf{p}}\tag{3.29}$$

(with implicit summation over particle type and spin), and $\mathbf{J}$ only receives contributions from the pieces of δn proportional to the generators T^a. Alternatively, on the left-hand side of the Boltzmann equation (3.24), the driving term $\mathbf{E}\cdot\mathbf{v}\,(dn/dp)$ is in the adjoint representation. We may thus specialize δn to fluctuations of the form

$$\delta n_{\mathcal{R}}=T^a_{\mathcal{R}}\,\delta N^a.\tag{3.30}$$

[24] The overall factor of 1/2 arises because the damping rate is defined in the literature as the decay rate for the quantum-mechanical amplitude of an excitation rather than the decay rate for the number density of an excitation. See footnote 6.

The linearized Boltzmann equation given by Eqs. (3.24) and (3.26) then becomes

$$[(D_t+\mathbf{v}\cdot\mathbf{D}_{\mathbf{x}})\,\delta N]^a+g\,\mathbf{E}^a\cdot\mathbf{v}\frac{dn}{dp}$$
$$=\tfrac{1}{2}\,C_{\rm A}\,t_{\mathcal{R}'}\int_{\mathbf{p}'\mathbf{q}}|\mathcal{M}|^2\{\delta N^a_{\mathbf{p}}\,n_{\mathbf{p}'}(1\pm n_{\mathbf{p}'})$$
$$-\delta N^a_{\mathbf{p}'}\,n_{\mathbf{p}}(1\pm n_{\mathbf{p}})\},\tag{3.31}$$

where the covariant derivatives D_μ now act in the adjoint representation. Comparison with Eq. (3.28) shows that the first term above is just $\gamma_{\rm g}\,\delta N^a_{\mathbf{p}}$, where $\gamma_{\rm g}$ is the thermal damping rate of hard gauge bosons. The color current resulting from δN^a is

$$\mathbf{J}^a=g\,t_{\mathcal{R}}\int_{\mathbf{p}}\delta N^a\,\mathbf{v}.\tag{3.32}$$

Now let us finally turn to the matrix element $\mathcal{M}$. At small momentum transfers, the classic Coulomb scattering amplitude may be written in the form

$$\int_{\mathbf{q}}|\mathcal{M}|^2=g^4\int\frac{d^4Q}{(2\pi)^4}\left|V_\mu\frac{\delta^{\mu\nu}}{Q^2}V'_\nu\right|^2$$
$$\times 2\pi\delta(Q\cdot V)\,2\pi\delta(Q'\cdot V)\quad(\text{no screening}),\tag{3.33}$$

where $Q=(q_0,\mathbf{q})$ and $V=(1,\mathbf{v})$. However, as discussed earlier, the conductivity is dominated by momentum transfers q small enough that plasma screening effects are important. In particular, the momentum range of relevance at leading-log order is $g^2T\ll q\ll gT$. In this regime, longitudinal forces are Debye screened, and hard particles only interact through transverse (magnetic) forces. The above $\delta^{\mu\nu}$ should therefore be replaced by the transverse projection operator $\delta^{ij}-q^iq^j$, and the transverse self-energy $\Pi_{\rm T}$ should be resummed into the propagator of the exchanged gauge boson:

$$\int_{\mathbf{q}}|\mathcal{M}|^2\approx g^4\int\frac{d^4Q}{(2\pi)^4}\left|v_i\frac{(\delta_{ij}-q_iq_j)}{Q^2+\Pi_{\rm T}(Q)}v_j\right|^2$$
$$\times 2\pi\delta(q_0-\mathbf{q}\cdot\mathbf{v})\,2\pi\delta(q_0-\mathbf{q}\cdot\mathbf{v}'),\tag{3.34}$$

where the $\approx$ sign indicates we have now made approximations valid only at the leading-log level. The full one-loop result for $\Pi_{\rm T}(Q)$ is well known [33,24], but we shall see in a moment that we need only its $q_0\ll\mathbf{q}$ limit. In that domain, it is simply

$$\Pi_{\rm T}(Q)\approx i\,\sigma_0(q)\,\omega\tag{3.35}$$

where

$$\sigma_0(q)\equiv\frac{3\pi m_{\rm pl}^2}{4q}\tag{3.36}$$

is the collisionless conductivity discussed earlier. As also discussed earlier, the collisionless approximation is in fact only valid for $g^2T\ln\ll q$ rather than $g^2T\ll q$. (Diagrammatically, the breakdown for $g^2T\lesssim q\lesssim g^2T\ln$ appears as a failure of the one-loop approximation to Π_T.) So our current approximations are really only valid for $g^2T\ln\ll q\ll gT$. As we shall see shortly, this will not affect the result for Eq. (3.34) at leading-log order.

Given these approximations, the q integration in Eq. (3.34) is dominated by $q_0\sim q^2/\sigma_0(q)\ll q$, justifying the small q_0 approximation. Performing the q_0 and angular integrations first, one obtains

$$\int_{\mathbf{q}}|\mathcal{M}|^2\approx\frac{32\alpha^2}{3m_{\rm pl}^2}\;\frac{(\mathbf{v}\cdot\mathbf{v}')^2}{\sqrt{1-(\mathbf{v}\cdot\mathbf{v}')^2}}\int\frac{dq}{q}. \qquad (3.37)$$

The logarithmic integral is cut off by gT on one side (above which the un-approximated integrand starts to fall more rapidly) and the soft scale g^2T or the inverse collision time $g^2T\ln$ on the other side—it does not matter which. The result at leading-log order is

$$\int_{\mathbf{q}}|\mathcal{M}|^2\approx\frac{32\alpha^2}{3m_{\rm pl}^2}\ln(g^{-1})\,\frac{(\mathbf{v}\cdot\mathbf{v}')^2}{\sqrt{1-(\mathbf{v}\cdot\mathbf{v}')^2}}. \qquad (3.38)$$

At this point, we have all the elements we need. To proceed, it is convenient to follow Bödeker [6] and others [34] and combine the different color distribution functions δN^a for different particles and different $|\mathbf{p}|$ by noticing that the current $\mathbf{J}$ depends only on the combination

$$W^a(\mathbf{x},\mathbf{v},t)\equiv\frac{g}{3m_{\rm pl}^2}\sum_{\substack{\rm type\\ \rm spin}}\int\frac{4\pi|\mathbf{p}|^2d|\mathbf{p}|}{(2\pi)^3}\,t_{\mathcal{R}}\,\delta N^a(\mathbf{x},\mathbf{p},t), \qquad (3.39)$$

where we have integrated over $|\mathbf{p}|$ but not $\mathbf{v}\equiv\hat{\mathbf{p}}$. Integrate and sum both sides of the Boltzmann equation (3.31) similarly. Then, making use of the value

$$m_{\rm pl}^2=\frac{g^2}{3T}\sum_{\substack{\rm type\\ \rm spin}}\int_{\mathbf{p}}t_{\mathcal{R}}\,n_{\mathbf{p}}(1\pm n_{\mathbf{p}}) \qquad (3.40)$$

and of the fact that the result (3.38) for $\int_{\mathbf{q}}|\mathcal{M}|^2$ depends only on angles, and comparing to the adjoint representation expression $\gamma_{\rm g}$ of the hard thermal damping rate (3.28), one obtains

$$(D_t+\mathbf{v}\cdot\mathbf{D}_{\mathbf{x}})\,W-\mathbf{E}\cdot\mathbf{v}=-\,\delta C[W], \qquad (3.41a)$$

$$\delta C[W](\mathbf{v})=\gamma_{\rm g}\left[W(\mathbf{v})-\frac{4}{\pi}\left\langle\frac{(\mathbf{v}\cdot\mathbf{v}')^2}{\sqrt{1-(\mathbf{v}\cdot\mathbf{v}')^2}}\,W(\mathbf{v}')\right\rangle_{\mathbf{v}'}\right], \qquad (3.41b)$$

and

$$\mathbf{J}=3m_{\rm pl}^2\langle\mathbf{v}W(\mathbf{v})\rangle_{\mathbf{v}}, \qquad (3.42)$$

where $\langle\cdots\rangle_{\mathbf{v}}$ indicates angular averaging over $\mathbf{v}$. This precisely reproduces the result that Bödeker derived by another method [6].

It is the second term in Eq. (3.41b) that was dropped in the original analysis of Selikhov and Gyulassy [13]. It is relevant to some aspects of color dynamics, a prime example (noted by Bödeker) being the conservation $D_\mu J^\mu=0$ required of the hard current $J^\mu=3m_{\rm pl}^2\langle V^\mu W(\mathbf{v})\rangle_{\mathbf{v}}$ by the effective Maxwell equation $D_\mu F^{\mu\nu}=J^\nu$ for the soft fields. From Eq. (3.41a), this conservation requires $\langle\delta C[W]\rangle_{\mathbf{v}}=0$, which is indeed satisfied by Eq. (3.41b).

The fact $\langle\delta C[W]\rangle_{\mathbf{v}}=0$ can be rephrased to say that the symmetric operator δC has zero modes: it annihilates anything that is independent of $\mathbf{v}$. (This can be rephrased in bra-ket notation in $\mathbf{v}$-space as $\langle{\rm const}|\delta C|W\rangle=\langle W|\delta C|{\rm const}\rangle=0$ for any W.)

3. The conductivity

To solve the linearized Boltzmann equation (3.41) for the W at leading-log order, Bödeker [6] argues that the covariant derivative terms are together order $g^2T\,W$ and so can be ignored compared to the collision term, which is order $\gamma_{\rm g}W\sim(g^2T\ln g^{-1})\,W$. This approximation is actually flawed because of the zero mode of δC. We analyze this flaw in the approximation in Ref. [12] and show that it does not affect the transverse dynamics. Here, we shall instead simply continue with the naive approximation. Dropping the covariant derivative terms from the Boltzmann equation gives simply

$$\mathbf{E}\cdot\mathbf{v}\approx\delta C[W]. \qquad (3.43)$$

Next note that δC maps even (odd) functions of $\mathbf{v}$ into even (odd) functions of $\mathbf{v}$. (In contrast, the $\mathbf{v}\cdot\mathbf{D}_{\mathbf{x}}$ operator that we dropped does not.) Since $\mathbf{E}\cdot\mathbf{v}$ is odd in $\mathbf{v}$, the solution W to Eq. (3.43) must be odd as well. But for odd functions of $\mathbf{v}$, the form (3.41b) of δC simplifies to $\delta C[W]=\gamma_{\rm g}W$.[25] The solution is then

$$W\approx\frac{\mathbf{E}\cdot\mathbf{v}}{\gamma_{\rm g}}, \qquad (3.44)$$

which inserted into Eq. (3.42) generates a current

$$\mathbf{J}\approx\frac{m_{\rm pl}^2}{\gamma_{\rm g}}\mathbf{E}. \qquad (3.45)$$

This is Selikhov and Gyulassy's leading-log result (2.14) for the conductivity. [We show in Ref. [12] that a more careful analysis of the Boltzmann equation (3.41) reveals that the $\mathbf{E}$ above should really be the transverse projection of $\mathbf{E}$.]

[25] This is gratifyingly simpler than the leading-log collision terms one obtains for most transport phenomena, where δC reduces to a linear differential operator [35] and the linearized Boltzmann equation must be solved either numerically or approximately.

4. Variational methods

We are now in a position to understand the problem with the estimate of the color conductivity made by Heiselberg in Ref. [21]. Heiselberg uses a variational method to approximately solve the Boltzmann equation. The variational ansatz he uses is one that works stunningly well for the diffusion of global or Abelian charges. Using the imagery of QED, one imagines the linear response of the system as a simple boost of equilibrium distributed positive charges moving in one direction and of negative charges moving in the other, with the boost velocities depending on the charges of the particles:

$$n_i=\frac{1}{e^{\beta(\epsilon_{\mathbf{p}}-\mathbf{u}_i\cdot\mathbf{p})}\mp 1}\simeq n_i^{\rm eq}-\frac{dn}{d\epsilon_{\mathbf{p}}}\mathbf{u}_i\cdot\mathbf{p}, \qquad (3.46)$$

where i is a flavor index. Equation (3.46) is Heiselberg's ansatz, with the velocity $\mathbf{u}_i$ to be determined variationally.

When deriving the conductivity, we found it convenient to combine all particles together and work with W instead of δn. If one instead follows through the argument of Sec. III B 3 with the original Boltzmann equation (3.31) for δn, one finds that

$$\delta n \propto \frac{dn}{d\epsilon_p}\mathbf{E}\cdot\mathbf{v} \qquad (3.47)$$

at leading-log order. For color diffusion, Heiselberg's ansatz (3.46) misses the mark by a factor of $|\mathbf{p}|$. The actual linear response (3.47) cannot be interpreted as simple boosts of fluids corresponding to different charges.

IV. ULTRAVIOLET INSENSITIVITY

We will now elaborate on our earlier claim that Bödeker's effective theory (2.9) of the soft modes has the wonderful property that it is insensitive to how the soft modes are cut off in the ultraviolet. Equivalently, but more technically, the effective theory does not require any ultraviolet renormalization—it is ultraviolet finite.

As preparation, let us ignore the dynamics for a moment and remember that the equilibrium properties of the classical theory are described by the partition function

$$\mathcal{Z}=\int[\mathcal{D}A]\,e^{-\beta\mathcal{V}} \qquad (4.1)$$

where

$$\mathcal{V}=\tfrac{1}{2}\int_{\mathbf{x}}B^2=\tfrac{1}{4}\int_{\mathbf{x}}F^a_{ij}F^a_{ij} \qquad (4.2)$$

is the potential energy associated with the magnetic field. This is nothing other than the partition function for three-dimensional Euclidean gauge theory, and three-dimensional gauge theory is ultraviolet finite. One way to see this is to rescale the fields $\mathbf{A}$ and coupling g (hiding in the definition of the field strength) to absorb $\beta=T^{-1}$ by

$$\mathbf{A}\to T^{1/2}\mathbf{A},\quad g\to T^{-1/2}g. \qquad (4.3)$$

The partition function is then

$$\int[\mathcal{D}A]\exp\left(-\frac{1}{4}\int_{\mathbf{x}}F^a_{ij}F^a_{ij}\right). \qquad (4.4)$$

From Eq. (4.3) or (4.4), the gauge field can now be seen to have scaling dimension of $[A]=\frac{1}{2}$, and the coupling constant has dimension $[g]=\frac{1}{2}$. There are no other relevant terms (in the sense of mass dimension) that could be added to the action which are gauge and parity invariant. So the dimension-$\frac{1}{2}$ coupling g is the *only* relevant parameter of this theory. Now suppose we modify or integrate out (in a gauge-invariant manner) ultraviolet degrees of freedom associated with an arbitrarily large momentum scale Λ. We then potentially need to modify Eq. (4.4) to

$$\int[\mathcal{D}A]\exp\left(-\frac{1}{4}Z\int_x F^a_{ij}F^a_{ij}\right), \qquad (4.5)$$

where Z is a renormalization constant. However, it follows by dimensional analysis that the perturbative expansion of Z must be in powers of g^2/Λ, which vanishes for $\Lambda\to\infty$.

Now let us turn to Bödeker's effective theory (2.9), which we write in the generic form

$$\sigma\frac{d}{dt}\mathbf{A}=-\frac{\delta}{\delta\mathbf{A}}\mathcal{V}+\boldsymbol{\zeta}, \qquad (4.6a)$$

$$\langle\zeta(t,\mathbf{x})\zeta(t',\mathbf{x}')\rangle=2\sigma T\,\delta(t-t')\,\delta(\mathbf{x}-\mathbf{x}'), \qquad (4.6b)$$

suppressing color and vector indices. By rescaling fields and coupling as before, and also rescaling time by

$$t\to\sigma t, \qquad (4.7)$$

one can put this in the form

$$\frac{d}{dt}\mathbf{A}=-\frac{\delta}{\delta\mathbf{A}}\mathcal{V}+\boldsymbol{\zeta}, \qquad (4.8a)$$

$$\langle\zeta(t,\mathbf{x})\zeta(t',\mathbf{x}')\rangle=2\,\delta(t-t')\,\delta(\mathbf{x}-\mathbf{x}'). \qquad (4.8b)$$

Once again, the theory appears to depend on only one parameter: the dimension-$\frac{1}{2}$ coupling g. The essential point to understand is that no other relevant terms can be added to this equation—that is, more complicated time dependence (which survives when the cutoff scale $\Lambda\to\infty$) cannot be generated for the long-distance modes when one modifies or integrates out short-distance physics. This has been proven in a general analysis of the renormalizability of purely dissipative stochastic field equations by Zinn-Justin and Zwanziger

[36].[26] They analyze the problem by first finding a path-integral representation of the stochastic equation, and then using dimensional analysis and various Becchi-Rouet-Stora (BRS) symmetries to determine the allowed relevant terms. When translated back into a stochastic equation, the result is that no more-complicated time dependence can be generated and that a renormalized version of Bödeker's effective theory will take the form

$$Z_t \frac{d}{dt}\mathbf{A} = -\frac{\delta}{\delta \mathbf{A}} \mathcal{V}_{\rm R} + \boldsymbol{\zeta}, \tag{4.9a}$$

$$\langle \zeta(t,\mathbf{x}) \zeta(t',\mathbf{x}') \rangle = 2 Z_t \, \delta(t-t') \, \delta(\mathbf{x}-\mathbf{x}'), \tag{4.9b}$$

where $\mathcal{V}_{\rm R}$ is the renormalized potential and Z_t is a new renormalization constant. We already know that the potential is not renormalized. And, just as before, Z_t must have a perturbative expansion in g^2/Λ and so generates no relevant correction to the equation. This demonstrates why Bödeker's equation is insensitive to the ultraviolet.

If one imposes a gauge-invariant lattice cutoff on Bödeker's effective theory, in order to perform numerical simulations, this ultraviolet insensitivity implies that physical quantities such as the topological transition rate will have a finite continuum limit as the lattice spacing is sent to zero.

ACKNOWLEDGMENTS

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APPENDIX: THE COLLISION TERM

In this appendix we review the derivation of the Boltzmann equation from first principles. We will work up from the simplest case of a single-component ϕ^4 theory, to multi-component scalar theories, and then to gauge theories. We will discuss scalar QED first and finally derive the Boltzmann equation for the case of primary interest, non-Abelian gauge theory. While the treatment of a simple ϕ^4 theory, or QED, may be easily found in the literature (see for example [26]), the appropriate generalizations for multi-component or non-Abelian theories are much less well-known.

1. Single-component ϕ^4 theory

Our starting point is the Schwinger–Keldysh closed-time-path (CTP) formalism [37,38]. Since both the Schwinger-Keldysh CTP formalism and the derivation of the Boltzmann equation from it can be found in the literature, the exposition here will be rather concise. For more details, see Ref. [26].

[26]For a review, see Chap. 16 and 17, and especially Sec. 17.5.2, of Ref. [11].

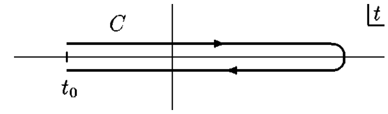


FIG. 3. The Schwinger-Keldysh closed-time-path contour.

In the Schwinger–Keldysh formalism, one considers time evolution along a time contour, denoted C in Fig. 3, running from some initial time $t=t_0$ to $t=+\infty$ and then returning to t_0. Correspondingly, the propagator is defined as $i\,G_C(x,y) = \langle \mathcal{T}_C(\phi(x)\phi(y))\rangle$, where $\mathcal{T}_C$ denotes contour ordering. Since both x and y may lie on either the upper or the lower portion of the contour, the propagator G_C may be separated into 4 different components,

$$i\,G_{11}(x,y) = \langle \mathcal{T}(\phi(x)\phi(y))\rangle, \quad i\,G_{12}(x,y) = \langle \phi(y)\phi(x)\rangle,$$
$$i\,G_{21}(x,y) = \langle \phi(x)\phi(y)\rangle, \qquad i\,G_{22}(x,y) = \langle \overline{\mathcal{T}}(\phi(x)\phi(y))\rangle, \tag{A1}$$

where $\overline{\mathcal{T}}$ denotes anti-time-ordering. From Eq. (A1) it is apparent that the four components of G_C satisfy the relation

$$G_{11} + G_{22} = G_{12} + G_{21}. \tag{A2}$$

It is also useful to introduce retarded and advanced propagators,

$$i\,G_R(x,y) = \theta(x_0-y_0)\langle[\phi(x),\phi(y)]\rangle,$$
$$i\,G_A(x,y) = -\theta(y_0-x_0)\langle[\phi(x),\phi(y)]\rangle, \tag{A3}$$

which are related to the components of G_C in the following ways:

$$G_R = G_{11} - G_{12} = G_{21} - G_{22},$$
$$G_A = G_{12} - G_{22} = G_{11} - G_{21}. \tag{A4}$$

The retarded and advanced propagators are the boundary values of the Euclidean time-ordered propagator when the imaginary (Matsubara) frequency $i\omega_n$ is analytically continued to just above, or just below, the real frequency axis.

For a free scalar field with Lagrangian $L = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2} m^2 \phi^2$, the Fourier transforms of the propagator components are

$$\tilde{G}_{11}(p) = \frac{1}{p_0^2 - \omega_{\mathbf{p}}^2 + i\epsilon} - \frac{i\pi}{\omega_{\mathbf{p}}}[n_{\mathbf{p}}\,\delta(p_0-\omega_{\mathbf{p}}) + n_{-\mathbf{p}}\,\delta(p_0+\omega_{\mathbf{p}})], \tag{A5a}$$

$$\tilde{G}_{22}(p) = \frac{-1}{p_0^2 - \omega_{\mathbf{p}}^2 - i\epsilon} - \frac{i\pi}{\omega_{\mathbf{p}}}[n_{\mathbf{p}}\,\delta(p_0-\omega_{\mathbf{p}}) + n_{-\mathbf{p}}\,\delta(p_0+\omega_{\mathbf{p}})], \tag{A5b}$$

$$\tilde{G}_{12}(p)=-\frac{i\pi}{\omega_{\mathbf{p}}}[n_{\mathbf{p}}\,\delta(p_0-\omega_{-\mathbf{p}})+(1+n_{-\mathbf{p}})\,\delta(p_0+\omega_{\mathbf{p}})], \tag{A5c}$$

$$\tilde{G}_{21}(p)=-\frac{i\pi}{\omega_{\mathbf{p}}}[(1+n_{\mathbf{p}})\,\delta(p_0-\omega_{-\mathbf{p}})+n_{-\mathbf{p}}\,\delta(p_0+\omega_{\mathbf{p}})], \tag{A5d}$$

where $n_{\mathbf{p}}$ is the occupation number. The retarded and advanced propagators are

$$\tilde{G}_R(p)=\frac{1}{(p_0+i\epsilon)^2-\omega_{\mathbf{p}}^2},\quad \tilde{G}_A(p)=\frac{1}{(p_0-i\epsilon)^2-\omega_{\mathbf{p}}^2}. \tag{A6}$$

To compute the propagator for an interacting scalar theory, it is useful to introduce the self-energy $\Sigma(x,y)$ which is related to the propagator by the equations

$$(-\partial_x^2-m^2)\,G_C(x,y)=\eta_x\,\delta(x-y)+\int_C dz\,\Sigma_C(x,z)\,G_C(z,y), \tag{A7}$$

$$(-\partial_y^2-m^2)\,G_C(x,y)=\eta_y\,\delta(x-y)+\int_C dz\,G_C(x,z)\,\Sigma_C(z,y), \tag{A8}$$

where η_x equals $+1$ if x is on the upper part of the contour, and -1 when x is on the lower part. As with the propagator, the self-energy may be decomposed into 4 components, Σ_{11}, Σ_{12}, Σ_{21}, and Σ_{22}. If one forms 2×2 matrices from the components of G_C and Σ_C,

$$G(x,y)\equiv\begin{pmatrix} G_{11}(x,y) & G_{12}(x,y)\\ G_{21}(x,y) & G_{22}(x,y)\end{pmatrix},$$

$$\Sigma(x,y)\equiv\begin{pmatrix} \Sigma_{11}(x,y) & \Sigma_{12}(x,y)\\ \Sigma_{21}(x,y) & \Sigma_{22}(x,y)\end{pmatrix}, \tag{A9}$$

then Eqs. (A7) and (A8) become

$$(-\partial_x^2-m^2)\,G(x,y)=\sigma_3\,\delta(x-y)+\int dz\,\Sigma(x,z)\,\sigma_3\,G(z,y), \tag{A10}$$

$$(-\partial_y^2-m^2)\,G(x,y)=\sigma_3\,\delta(x-y)+\int dz\,G(x,z)\,\sigma_3\,\Sigma(z,y), \tag{A11}$$

where $\int dz$ now means ordinary spacetime integration and σ_3 is the usual Pauli matrix (but has nothing to do with spin here). Using Eq. (A10) [or (A11)] one can show that the identity (A2) for the propagator implies a corresponding identity for the self-energy,

$$\Sigma_{11}+\Sigma_{22}=\Sigma_{12}+\Sigma_{21}. \tag{A12}$$

One can also introduce retarded and advanced self-energies in a manner similar to Eq. (A4),

$$\Sigma_R=\Sigma_{11}-\Sigma_{12}=\Sigma_{21}-\Sigma_{22}$$

$$\Sigma_A=\Sigma_{12}-\Sigma_{22}=\Sigma_{11}-\Sigma_{21}. \tag{A13}$$

It can be easily shown that $G_{R,A}$ are related to $\Sigma_{R,A}$ by $G_{R,A}=(G_0^{-1}-\Sigma_{R,A})^{-1}$, where G_0 is the corresponding free retarded or advanced propagator.

Let us now turn to the derivation of the Boltzmann equation. Subtracting Eq. (A7) from Eq. (A8), one obtains

$$(\partial_x^2-\partial_y^2)\,G_C(x,y)=\int_C dz\,[G_C(x,z)\,\Sigma_C(z,y)-\Sigma_C(x,z)\,G_C(z,y)]. \tag{A14}$$

Up to this point, we have not made any approximation. Now we will assume that the overall evolution of the system occurs on a time scale much larger than the typical wavelength of a particle. In terms of the propagator $G(x,y)$, this means that it varies much more slowly as a function of the average position $(x+y)/2$ than with the separation $x-y$. Having this in mind, let us change variables in Eq. (A14) from x, y, z to new variables X, s and s', where

$$x=X+\frac{s}{2},\quad y=X-\frac{s}{2},\quad z=X+\frac{s}{2}-s'. \tag{A15}$$

Equation (A14) becomes

$$2\frac{\partial}{\partial X^\mu}\frac{\partial}{\partial s_\mu}G_C(X,s)=\int_C ds'\left[G_C\left(X+\frac{s-s'}{2},s'\right)\Sigma_C\left(X-\frac{s'}{2},s-s'\right)-\Sigma_C\left(X+\frac{s-s'}{2},s'\right)G_C\left(X-\frac{s'}{2},s-s'\right)\right]. \tag{A16}$$

Since G and Σ vary slowly as a function of X, one can replace the first argument in G and Σ on the right-hand side of Eq. (A16) by X. The (12) component of Eq. (A16) then reads

$$2\frac{\partial}{\partial X^\mu}\frac{\partial}{\partial s_\mu}G_{12}(X,s)=\int ds'[G_{11}(X,s')\,\Sigma_{12}(X,s-s')-G_{12}(X,s')\,\Sigma_{22}(X,s-s')-\Sigma_{11}(X,s')\,G_{12}(X,s-s')+\Sigma_{12}(X,s')\,G_{22}(X,s-s')]. \tag{A17}$$

Fourier transforming with respect to the relative separation, $\tilde{G}(X,p)\equiv\int ds\,e^{-ips}\,G(X,s)$, etc., converts Eq. (A17) to

$$-2ip^{\mu}\partial_{\mu}\tilde{G}_{12}=\tilde{\Sigma}_{12}(\tilde{G}_{11}+\tilde{G}_{22})-\tilde{G}_{12}(\tilde{\Sigma}_{11}+\tilde{\Sigma}_{22}), \tag{A18}$$

where, for the simplicity of notation, we have omitted the arguments of $\tilde{G}$ and $\tilde{\Sigma}$ which are now always (X,p). Making use of Eqs. (A2) and (A12) allows this result to be written in the form

$$-2ip^{\mu}\partial_{\mu}\tilde{G}_{12}=\tilde{\Sigma}_{12}\tilde{G}_{21}-\tilde{G}_{12}\tilde{\Sigma}_{21}. \tag{A19}$$

Finally, to obtain the Boltzmann equation from Eq. (A19), we make the following ansatz for the propagators G_{12} and G_{21}: [27]

$$\tilde{G}_{12}(X,p)=-\frac{i\pi}{\omega_{\mathbf{p}}}[\,\delta(p_0-\omega_{\mathbf{p}})\,n_{\mathbf{p}}(X)+\delta(p_0+\omega_{\mathbf{p}})\,(1+n_{-\mathbf{p}}(X))], \tag{A20}$$

$$\tilde{G}_{21}(X,p)=-\frac{i\pi}{\omega_{\mathbf{p}}}[\,\delta(p_0-\omega_{\mathbf{p}})\,(1+n_{\mathbf{p}}(X))+\delta(p_0+\omega_{\mathbf{p}})\,n_{-\mathbf{p}}(X)]. \tag{A21}$$

In other words, one assumes that $\tilde{G}$ has the same form as the free propagator, except that the distribution function is now an arbitrary function of both X and $\mathbf{p}$. By comparing the coefficient of $\delta(p_0-\omega_{\mathbf{p}})$, one derives from Eq. (A19) that

$$(\partial_t+\mathbf{v}\cdot\partial_{\mathbf{x}})\,n_{\mathbf{p}}=\frac{i}{2\omega_{\mathbf{p}}}[\tilde{\Sigma}_{12}(\omega_{\mathbf{p}},\mathbf{p})\,(1+n_{\mathbf{p}})-\tilde{\Sigma}_{21}(\omega_{\mathbf{p}},\mathbf{p})\,n_{\mathbf{p}}]. \tag{A22}$$

One can see the Boltzmann equation emerging. Indeed, the term $(i/2\omega_{\mathbf{p}})\tilde{\Sigma}_{12}(1+n_{\mathbf{p}})$ is the ''gain'' term and $-(i/2\omega_{\mathbf{p}})\tilde{\Sigma}_{21}n_{\mathbf{p}}$ is the ''loss'' term in the collision integral. To produce the conventional form of the Boltzmann equation, we need to compute the leading-order contribution to the self-energy. There is no one-loop contribution to $\tilde{\Sigma}_{12}$ or $\tilde{\Sigma}_{21}$. The first non-zero contribution comes from the two-loop diagram (Fig. 4). Using the explicit propagators (A5), one finds,

[27] We are assuming that the dispersion relation of quasi-particle excitations is adequately approximated by the zero-temperature dispersion relation. In a weakly-coupled high-temperature theory, this is always the case for typical (hard) excitations which are relativistic. If $\lambda n^{2/3}$ is not small compared to the zero-temperature physical mass, then soft (non-relativistic) excitations will have significant corrections to their dispersion relation. Nevertheless, the error in the description of these excitations does not affect the leading behavior of many quantities, such as the conductivity, which are dominantly sensitive to hard excitations. (An improved treatment, which correctly describes soft as well as hard excitations, is needed for the bulk viscosity. See [20] for a discussion of the construction of such an improved ''effective'' kinetic theory.)

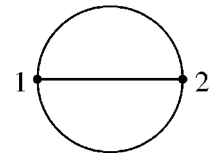


FIG. 4. The leading contribution to Σ_{12}.

$$\tilde{\Sigma}_{12}(\omega_{\mathbf{p}},\mathbf{p})=-\frac{i\lambda^2}{2}\int\frac{d\mathbf{p}'\,d\mathbf{k}}{(2\pi)^6\,2\omega_{\mathbf{p}'}2\omega_{\mathbf{k}}2\omega_{\mathbf{k}'}}n_{\mathbf{p}'}\,n_{\mathbf{k}}\times(1+n_{\mathbf{k}'})\,2\pi\delta(\omega_{\mathbf{p}}+\omega_{\mathbf{p}'}-\omega_{\mathbf{k}}-\omega_{\mathbf{k}'}),$$

$$\tilde{\Sigma}_{21}(\omega_{\mathbf{p}},\mathbf{p})=-\frac{i\lambda^2}{2}\int\frac{d\mathbf{p}'\,d\mathbf{k}}{(2\pi)^6\,2\omega_{\mathbf{p}'}2\omega_{\mathbf{k}}2\omega_{\mathbf{k}'}}(1+n_{\mathbf{p}'})\times(1+n_{\mathbf{k}})\,n_{\mathbf{k}'}\,2\pi\delta(\omega_{\mathbf{p}}+\omega_{\mathbf{p}'}-\omega_{\mathbf{k}}-\omega_{\mathbf{k}'}), \tag{A23}$$

where $\mathbf{k}'\equiv\mathbf{p}+\mathbf{p}'-\mathbf{k}$. Substituting Eq. (A23) into Eq. (A22), one obtains the Boltzmann equation

$$(\partial_t+\mathbf{v}\cdot\partial_{\mathbf{x}})\,n_{\mathbf{p}}=\frac{\lambda^2}{2}\int\frac{d\mathbf{p}'\,d\mathbf{k}}{(2\pi)^6\,2\omega_{\mathbf{p}}2\omega_{\mathbf{p}'}2\omega_{\mathbf{k}}2\omega_{k'}}\times2\pi\delta(\omega_{\mathbf{p}}+\omega_{\mathbf{p}'}-\omega_{\mathbf{k}}-\omega_{\mathbf{k}'})\times[n_{\mathbf{p}_1}n_{\mathbf{p}_2}(1+n_{\mathbf{p}_3})(1+n_{\mathbf{p}})-(1+n_{\mathbf{p}_1})(1+n_{\mathbf{p}_2})\,n_{\mathbf{p}_3}n_{\mathbf{p}}], \tag{A24}$$

which coincides with the result one would derive naively from kinetic theory.

2. The multi-component case

The extension of the derivation in the previous subsection to the case of a multi-component scalar field is straightforward. Let the field be ϕ^a, where a is some isospin index. The propagator and self-energy components become matrices, $G_{12}=||G_{12}^{ab}||$, etc., but everything in the preceding discussion up to (and including) Eq. (A17) remains valid. However, Eq. (A18) is not correct since the components of G and Σ no longer commute. Instead, one rewrites Eq. (A17) as

$$(\partial_x^2-\partial_y^2)\,G_{12}=\tfrac{1}{2}(\{G_{11}+G_{22},\Sigma_{12}\}+[G_{11}-G_{22},\Sigma_{12}]-\{\Sigma_{11}+\Sigma_{22},G_{12}\}-[\Sigma_{11}-\Sigma_{22},G_{12}]). \tag{A25}$$

The anti-commutator terms can be simplified by using Eqs. (A2) and (A12) to produce

$$(\partial_x^2-\partial_y^2)\,G_{12}=\tfrac{1}{2}(\{G_{21},\Sigma_{12}\}-\{\Sigma_{21},G_{12}\}+[G_{11}-G_{22},\Sigma_{12}]-[\Sigma_{11}-\Sigma_{22},G_{12}]). \tag{A26}$$

To progress further, we make the following ansatz for the propagator:

$$G_{12}^{ab}(X,p) = -\frac{i\pi}{\omega_{\mathbf{p}}}[\,\delta(p_0-\omega_{\mathbf{p}})\,n_{\mathbf{p}}^{ab}(X) + \delta(p_0+\omega_{\mathbf{p}})\,(\delta^{ab}+n_{-\mathbf{p}}^{ba}(X))],$$

$$G_{21}^{ab}(X,p) = -\frac{i\pi}{\omega_{\mathbf{p}}}[\,\delta(p_0-\omega_{\mathbf{p}})\,(\delta^{ab}+n_{\mathbf{p}}^{ab}(X)) + \delta(p_0+\omega_{\mathbf{p}})\,n_{-\mathbf{p}}^{ba}(X)]. \quad \text{(A27)}$$

Physically, $n_{\mathbf{p}}^{ab}(X)$ is the density matrix (in isospin space) of the momentum $\mathbf{p}$ excitations. The ansatz (A27) immediately implies that $G_{12}-G_{21}$ is proportional to δ^{ab}, and this in turn implies the same for $G_{11}-G_{22}$. Therefore, the first commutator term in Eq. (A26) vanishes and the kinetic equation becomes

$$(\partial_t+\mathbf{v}\cdot\partial_{\mathbf{x}})\,n_{\mathbf{p}} = \frac{i}{4\omega_{\mathbf{p}}}(\{\Sigma_{12}(\omega_{\mathbf{p}},\mathbf{p}),(1+n_{\mathbf{p}})\} - \{\Sigma_{21}(\omega_{\mathbf{p}},\mathbf{p}),n_{\mathbf{p}}\} - [\Sigma_{11}(\omega_{\mathbf{p}},\mathbf{p})-\Sigma_{22}(\omega_{\mathbf{p}},\mathbf{p}),n_{\mathbf{p}}]). \quad \text{(A28)}$$

This has the same form as Eq. (3.19) in the main text if one identifies

$$\mathcal{I}_- = \frac{i}{2\omega_{\mathbf{p}}}\Sigma_{12}(\omega_{\mathbf{p}},\mathbf{p}), \quad \mathcal{I}_+ = \frac{i}{2\omega_{\mathbf{p}}}\Sigma_{21}(\omega_{\mathbf{p}},\mathbf{p}), \quad \text{(A29)}$$

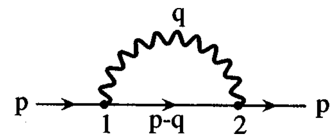
and

$$\mathrm{Re}\,\bar{\Sigma} = \frac{1}{4\omega_{\mathbf{p}}}(\Sigma_{11}-\Sigma_{22}) = \frac{1}{4\omega_{\mathbf{p}}}(\Sigma_R+\Sigma_A) = \frac{1}{2\omega_{\mathbf{p}}}\mathrm{Re}\,\Sigma_R(\omega_{\mathbf{p}},\mathbf{p}). \quad \text{(A30)}$$

The quantity $\bar{\Sigma}\equiv(1/2\omega_{\mathbf{p}})\Sigma_R$ has a simple physical meaning: it is the correction to the energy of an excitation with momentum $\mathbf{p}$ (in other words, it is the self-energy in the nonrelativistic normalization). It is straightforward to find the explicit form of $\mathcal{I}_\pm$ at the leading (two-loop) level, and show that one obtains Eqs. (3.17) and (3.18) discussed in the main text. Let us, however, move on to the case of gauge theories.

3. Gauge theories

The approach of the previous subsection can be carried over to the gauge theory case without substantial modification. We consider scalar QED first, and will find the Boltzmann equation describing the kinetics of the hard scalar particles. The basic equation remains Eq. (A22). However, the computation of the scalar self-energy Σ is slightly more complicated than in the ϕ^4 case. For scalar QED, the leading contribution to the self-energy Σ_{12} comes from the one-loop diagram:



This gives

$$\tilde{\Sigma}_{12}(p) = -ie^2\int\frac{d^4q}{(2\pi)^4}(2p-q)_\mu(2p-q)_\nu \times D_{12}^{\mu\nu}(q)\,\tilde{G}_{12}(p-q) \quad \text{(A31)}$$

where $D^{\mu\nu}$ is the photon propagator. Typical scatterings between bosons have small momentum exchange, so we will assume that the internal photon momentum q is small. The soft photon propagator $D(q)$ is obtained by summing the bubble diagrams: or equivalently, introducing the photon self-energy Π. Π has four components and is related to the photon propagator through the equations [see Eq. (A10) with $\Sigma\to\Pi$ and $m=0$]

$$q^2D_{12} = \Pi_{11}D_{12}-\Pi_{12}D_{22},$$

$$q^2D_{22} = \Pi_{21}D_{12}-\Pi_{22}D_{22}-1. \quad \text{(A32)}$$

We have suppressed the Lorentz indices here for notational simplicity in Eq. (A32). [In fact, the longitudinal and transverse parts of D and Π satisfy Eqs. (A32) separately.] Solving for D_{12}, one finds

$$D_{12}(q) = \frac{\Pi_{12}}{(q^2-\Pi_{11})(q^2+\Pi_{22})+\Pi_{12}\Pi_{21}}. \quad \text{(A33)}$$

As in the scalar case, the photon self-energy Π satisfies the identity (A12) and the retarded and advanced self-energies $\Pi^{R,A}$ can be introduced in a manner similar to Eq. (A13). It is easy to show that $(q^2-\Pi_{11})(q^2+\Pi_{22})+\Pi_{12}\Pi_{21} = (q^2-\Pi_R)(q^2-\Pi_A)$, and therefore

$$D_{12}(q) = \frac{\Pi_{12}}{(q^2-\Pi_R)(q^2-\Pi_A)} = D_R\,\Pi_{12}\,D_A = |D_R|^2\,\Pi_{12}. \quad \text{(A34)}$$

The photon self-energy Π_{12} is determined by the scalar one-loop diagram, which gives

$$\Pi_{12}^{\mu\nu}(q) = -ie^2\int\frac{dp'}{(2\pi)^4}(2p'+q)^\mu(2p'+q)^\nu \times G_{12}(p')\,G_{21}(p'+q). \quad \text{(A35)}$$

We now insert the ansatz [analogous to Eq. (A21)] for the propagator of the complex scalar ϕ,[28]

[28]The comments of footnote 27 equally apply to this gauge theory case. Since we are interested in a kinetic description for hard excitations in the plasma, neglecting background-field dependent corrections to the dispersion relations of excitations is sufficient for our purposes. See Refs. [28,29] for discussions of more general cases.

$$\tilde{G}_{12}(p) = -\frac{i\pi}{\omega_{\mathbf{p}}}[n_{\mathbf{p}}\,\delta(p_0-\omega_{\mathbf{p}})+(1+\bar{n}_{-\mathbf{p}})\,\delta(p_0+\omega_{\mathbf{p}})]. \tag{A36}$$

Here, $n_{\mathbf{p}}$ and $\bar{n}_{\mathbf{p}}$ are distribution functions of particles and anti-particles, respectively. Substituting this into Eq. (A35), one finds

$$\Pi_{12}^{\mu\nu}(q) = ie^2\int\frac{d\mathbf{p}'}{(2\pi)^3}(2p'+q)^{\mu}(2p'+q)^{\nu} \times[n_{\mathbf{p}'+\mathbf{q}}(1+n_{\mathbf{p}'})+\bar{n}_{\mathbf{p}'+\mathbf{q}}(1+\bar{n}_{\mathbf{p}'})]. \tag{A37}$$

Inserting Eqs. (A34), (A36), and (A37) in the scalar self-energy (A31) yields

$$\Sigma_{12}(\omega_{\mathbf{p}},\mathbf{p}) = -ie^4\int\frac{d\mathbf{p}'\,d\mathbf{q}}{(2\pi)^6\,2\omega_{\mathbf{p}'}2\omega_{\mathbf{p}-\mathbf{q}}2\omega_{\mathbf{p}'+\mathbf{q}}} \times(2\pi)\,\delta(\omega_{\mathbf{p}}+\omega_{\mathbf{p}'}-\omega_{\mathbf{p}-\mathbf{q}}-\omega_{\mathbf{p}'+\mathbf{q}}) \times|(2p-q)_{\mu}(2p'+q)_{\nu}D_R^{\mu\nu}(q)|^2 \times[(1+n_{\mathbf{p}'})\,n_{\mathbf{p}-\mathbf{q}}n_{\mathbf{p}'+\mathbf{q}} +(1+\bar{n}_{\mathbf{p}'})\,n_{\mathbf{p}-\mathbf{q}}\bar{n}_{\mathbf{p}'+\mathbf{q}}]. \tag{A38}$$

The other off-diagonal self-energy component, Σ_{21}, may be computed completely analogously. Inserting the expressions for Σ_{12} and Σ_{21} into Eq. (A22) yields the scalar QED Boltzmann equation,

$$(\partial_t+\mathbf{v}\cdot\partial_{\mathbf{x}})\,n_{\mathbf{p}} = \int\frac{d\mathbf{p}'\,d\mathbf{k}}{(2\pi)^6\,2\omega_p 2\omega_{\mathbf{p}'}2\omega_{\mathbf{k}}2\omega_{\mathbf{k}}}|M_{\mathbf{pp}'\to\mathbf{kk}'}|^2 \times(2\pi)\,\delta(\omega_{\mathbf{p}}+\omega_{\mathbf{p}'}-\omega_{\mathbf{k}}-\omega_{\mathbf{k}'}) \times\{(1+n_{\mathbf{p}})\,n_{\mathbf{k}}[(1+n_{\mathbf{p}'})\,n_{\mathbf{k}'} +(1+\bar{n}_{\mathbf{p}'})\,\bar{n}_{\mathbf{k}'}]-n_{\mathbf{p}}(1+n_{\mathbf{k}}) \times[n_{\mathbf{p}'}(1+n_{\mathbf{k}'})+\bar{n}_{\mathbf{p}'}(1+\bar{n}_{\mathbf{k}'})]\}, \tag{A39}$$

where we have introduced $k\equiv p-q$, $k'\equiv p'+q$, and the scattering amplitude $M_{\mathbf{pp}'\to\mathbf{kk}'}\equiv e^2(p+p')_{\mu}(k+k')_{\nu}D_R^{\mu\nu}(q)$. This has the same form as the naive Boltzmann equation, with the exception that the scattering amplitude is to be computed using the resummed propagator $D_R(q)$ for the exchanged photon instead of the bare photon propagator.

Finally, combining our treatment of multi-component scalar theory with that of QED, one may write down the Boltzmann equation for a non-Abelian gauge theory. The distribution functions become matrices $n_{\mathbf{k}}^{\overline{ab}}$ with respect to group indices.[29] The Boltzmann equation has the form shown in Eq. (A28). The loss and gain terms $\mathcal{I}_-$ and $\mathcal{I}_+$, which come from one-loop contributions to the self-energy (computed with a soft resummed gauge boson propagator), are trivial generalizations of Eq. (A38). The final results are given in Eqs. (3.16)–(3.19) of the main text.

[29] To simplify the discussion, we assume that distribution functions are trivial with respect to polarization.

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