

Exact effective action for [(1 + 1)-dimensional] fermions in an Abelian background at finite temperature

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In an effort to further understand the structure of effective actions for fermions in an external gauge background at finite temperature, we study the example of (1 + 1)-dimensional fermions interacting with an arbitrary Abelian gauge field. We evaluate the effective action exactly at finite temperature. This effective action is non-analytic as is expected at finite temperature. However, contrary to the structure at zero temperature and contrary to naive expectations, the effective action at finite temperature has interactions to all (even) orders (which, however, do not lead to any quantum corrections). The covariant structure thus obtained may prove useful in studying (2 + 1)-dimensional models in arbitrary backgrounds. We also comment briefly on the solubility of various (1 + 1)-dimensional models at finite temperature. [S0556-2821(99)03410-4]

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I. INTRODUCTION

Finite temperature introduces various new features [1] into quantum field theories that we are not used to at zero temperature. Thus, for example, it is known that various amplitudes as well as the effective actions can become non-analytic at finite temperature [1–3] (beyond 0 + 1 dimensions) which is connected with the existence of additional channels of reactions possible in a thermal medium. There are also various subtleties that arise, such as the modified Feynman combination formula [1,4,5], because the propagators do not have simple analytic behavior at finite temperature. More recently, it is also found that the effective action at finite temperature can be non-extensive [6] unlike at zero temperature. Thus, for the (0 + 1)-dimensional fermions interacting with an Abelian gauge field, the effective action at finite temperature becomes a non-polynomial function of $(\int dt A)$ where A represents the external, Abelian gauge field. This new structure of the effective action has led to a successful understanding of the question of large gauge invariance, at finite temperature, in this model. This model has properties similar to that of the (2 + 1)-dimensional fermions interacting with an arbitrary external gauge field in the sense that the radiative corrections induce a Chern-Simons term whose coefficient is a continuous function of temperature [7,8] and, therefore, incompatible with the quantization condition necessary for large gauge invariance to hold [9]. The study of the (0 + 1)-dimensional model suggests a way for the understanding of the question of large gauge invariance in the (2 + 1)-dimensional model at finite temperature and there have been several attempts to generalize the results of the (0 + 1)-dimensional model to the case of the (2 + 1)-dimensional model [10,11,12]. However, these attempts, typically, deal with very specific gauge backgrounds and a systematic study of the effective action for the (2 + 1)-dimensional fermions interacting with an arbitrary gauge background is still lacking.

While various properties of the (0 + 1)-dimensional

model, at finite temperature, are quite well understood [13,14] it is not at all obvious how the structure should be generalized to higher dimensions. For one thing, in 0 + 1 dimensions, there is only one component for the gauge field and, consequently, it is not clear what would be the appropriate covariant structure that would generalize to higher dimensions. Second, as pointed out earlier, at finite temperature, the effective action can become non-analytic beyond 0 + 1 dimensions and this makes any generalization of the results of the (0 + 1)-dimensional model (where there is no problem of non-analyticity) to higher dimensions additionally tricky. For these reasons, we have chosen to study, in this paper, a model of intermediate complexity, namely, the (1 + 1)-dimensional fermions interacting with an arbitrary external Abelian gauge field with the hope that it would shed light on some of the issues raised.

We consider massless fermions interacting with an external Abelian gauge field which, of course, can be exactly solved at zero temperature (leads to only quadratic terms in the effective action) and is associated with the solubility of various two dimensional models [15,16,17,18]. This model, of course, is not directly related to the question of large gauge invariance, but it is the structure of the effective action at finite temperature that we are interested in. It is well known that the chiral anomaly (which is responsible for the solubility of the model) of this model is not changed in the presence of temperature [1,19,20]. In a gauge invariant regularization (which is what we will use, but let us emphasize that the finite temperature calculations are all finite and it is the zero temperature calculation that needs a regularization), therefore, it would seem, *a priori*, that there would be no temperature dependent corrections to the effective action. Namely, if there is a temperature dependent correction, $\Gamma^{(\beta)}$, it must satisfy

$$\partial_\mu \frac{\delta \Gamma^{(\beta)}}{\delta A_\mu} = 0 = \epsilon_{\mu\nu} \partial^\mu \frac{\delta \Gamma^{(\beta)}}{\delta A_\nu}. \quad (1)$$

With the usual assumptions of locality, then, it would follow

that $\Gamma^{(\beta)}=0$. However, as we have learnt from the study of the $(0+1)$ -dimensional model [6], the effective action at finite temperature can be non-extensive in which case, it is not necessary for $\Gamma^{(\beta)}$ to vanish. In fact, taking from the results of the $(0+1)$ -dimensional model [6,13], we note that a simple, non-extensive quadratic term in the effective action of the form (c is a constant)

$$\Gamma_q = c \left(\int d^2x A_\mu(x) \right) \left(\int d^2y A^\mu(y) \right) \quad (2)$$

can give rise to a current that has vanishing divergence and curl. Thus, we would like to systematically study the structure of the effective action for this $(1+1)$ -dimensional model at finite temperature.

The paper is organized as follows. In Sec. II, we recapitulate briefly the structure of the quadratic effective action at zero temperature. We, then, point out that the generalization of a theorem of zero temperature [21] which shows that the effective action would continue to be quadratic even at finite temperature may be too naive. In Sec. III, we evaluate the two point function to show that the quadratic term does have a temperature dependent correction which does not alter the current conservation and the anomaly of the theory. The non-analytic structure of this correction is pointed out and the quadratic temperature dependent term of the effective action is expressed in a manifestly covariant fashion. In Sec. IV, we calculate explicitly the 3-point and the 4-point functions. While the 3-point function vanishes (in fact, all the odd point functions must vanish because of charge conjugation invariance), the 4-point function is nontrivial at finite temperature and correspondingly, there is a quartic term in the temperature dependent effective action. We also obtain the general form for the $2n$ -point function which is nontrivial and, thereby, determine the complete effective action at finite temperature. We show that, in a dynamical gauge theory, these additional interactions, however, do not generate any quantum mechanical correction which is yet a new feature at finite temperature. In Sec. V, we make some brief comments about the solubility of various two dimensional models at finite temperature and present some brief conclusions in Sec. VI.

II. GENERAL THEOREM

In this section, we will recapitulate briefly the structure of the effective action at zero temperature following from a general theorem [21] and make some observations concerning the structure of the effective action at finite temperature. Let us note that we are interested in the $(1+1)$ -dimensional model described by the Lagrangian density

$$\mathcal{L} = \bar{\psi} \gamma^\mu (i \partial_\mu - e A_\mu) \psi. \quad (3)$$

We use $\eta^{\mu\nu} = (+, -)$ with $\mu, \nu = 0, 1$. Although not necessary, a representation for the Dirac matrices can be chosen to be $\gamma^0 = \sigma_2$, $\gamma^1 = i\sigma_1$. In $1+1$ dimensions, the gamma matrices further satisfy the identity

$$\gamma^\mu \gamma^\nu = \eta^{\mu\nu} + \epsilon^{\mu\nu} \gamma_5 \quad (4)$$

where $\gamma_5 = \gamma^0 \gamma^1$ and $\epsilon^{\mu\nu}$ is the anti-symmetric Levi-Civita tensor with $\epsilon^{01} = 1$. Integrating out the fermions in the path integral leads to the effective action

$$\Gamma[A] = -i \text{Tr} \ln(1 - eS(i\partial)\mathbb{A}). \quad (5)$$

Here, we have normalized the effective action so that it vanishes for vanishing external field. The ‘‘Tr’’ in Eq. (5) stands for the trace in a complete basis as well as a Dirac trace and $S(p)$ is the propagator for the fermion.

We can expand the logarithm in Eq. (5) which leads to a power series representation for the effective action

$$\Gamma[A] = i \text{Tr} \left((eS(i\partial)\mathbb{A}) + \frac{(e^2 S(i\partial)\mathbb{A} S(i\partial)\mathbb{A})}{2} + \dots \right). \quad (6)$$

Here $S(i\partial)$ and $A(x)$ are supposed to be non-commuting operators and the effective action, in general, contains an infinite number of terms. However, let us note that, in $1+1$ dimensions, we can decompose the vector field as

$$A_\mu = \frac{1}{e} (\partial_\mu \sigma + \epsilon_{\mu\nu} \partial^\nu \phi) \quad (7)$$

and that at zero temperature, the fermion propagator has the form ($i\epsilon$ prescription is understood)

$$S(p) = \frac{1}{\not{p}}. \quad (8)$$

It is, therefore, clear that at zero temperature, we can write [using Eq. (4)]

$$\begin{aligned} eS(i\partial)\mathbb{A} &= (i\partial)^{-1} (\partial\sigma - \gamma_5 \partial\phi) \\ &= (i\partial)^{-1} [\partial, \sigma] + \gamma_5 (i\partial)^{-1} [\partial, \phi] \\ &= -i\sigma + i(i\partial)^{-1} \sigma \partial - i\gamma_5 \phi \\ &\quad + i\gamma_5 (i\partial)^{-1} \phi \partial. \end{aligned} \quad (9)$$

It now follows from this that

$$(eS(i\partial)\mathbb{A})^2 = -i[\sigma + \gamma_5 \phi, (eS(i\partial)\mathbb{A})] \quad (10)$$

which gives

$$\text{Tr}[(eS(i\partial)\mathbb{A})^{n+1}] = -\frac{i}{n} \text{Tr}[\sigma + \gamma_5 \phi, (eS(i\partial)\mathbb{A})^n]. \quad (11)$$

For $n > 1$, these integrals are convergent and hence one can use the cyclicity of the trace to conclude that all the terms in the effective action in Eq. (6) which are higher order than the quadratic vanish at zero temperature. Even the linear term in Eq. (6) vanishes because of the odd nature of the integrand. The quadratic term in the effective action can be evaluated in a straightforward manner and a gauge invariant regularization gives the complete effective action at zero temperature to be (although there is a one parameter freedom of regularization, we will use a gauge invariant regularization for simplicity)

$$\Gamma^{(0)}[A] = \frac{e^2}{2\pi} \int d^2x A_\mu \left(\eta^{\mu\nu} - \frac{\partial^\mu \partial^\nu}{\partial^2} \right) A_\nu. \quad (12)$$

We can now try to generalize the above proof to finite temperature. Let us recall that, at finite temperature, the fermion propagator has the form [1]. [At finite temperature, in the real time formalism (which is what we will use), there is a doubling of the field degrees of freedom. Consequently, the propagator acquires a 2×2 matrix structure. For a complete calculation of the effective action in the real time formalism, one needs all the components of the propagator. However, we are interested only in that part of the effective action which depends only on the original (not the doubled) degree of freedom. For such a calculation, at one loop, only the first component (the $++$ component in the terminology of closed time path formalism) is needed which we give below]

$$S(p) = \frac{1}{p} + 2i\pi \not{p} n(|p^0|) \delta(p^2) \quad (13)$$

where $n(|p^0|)$ represents the fermion distribution function ($\beta = 1/kT$, $k = \text{Boltzmann constant}$)

$$n(|p^0|) = \frac{1}{e^{\beta|p^0|} + 1}. \quad (14)$$

We note that, with the parametrization in Eq. (7), we can proceed again as before and show that the only non-vanishing term in the effective action would be quadratic in the field variables suggesting that the only modification that temperature might induce is at most to change the two point function. However, such a conclusion is too naive because at finite temperature, there are more tensor structures present and the decomposition in Eq. (7) is not the most general. Consequently, one must examine the structure of the effective action explicitly which we do next.

III. TWO POINT FUNCTION

To study the structure of the effective action at finite temperature, we, therefore, study the n -point amplitudes of the theory systematically. As mentioned earlier, we follow the real time formalism (in fact, the closed time path formalism) where one needs to double the degrees of freedom. The original field variables are denoted by a subscript $+$ while the doubled field variables are conventionally denoted by a subscript $-$. In such a calculation of the effective action, therefore, there arise amplitudes with all $+$ thermal index, all $-$ thermal index as well as terms with mixed indices. In what follows, we calculate that part of the effective action which contains only the original field variables, namely, the ones with only $+$ thermal index. For such a calculation, at one loop, (as mentioned before) one only needs the $++$ component of the propagator in the internal lines. The calculation of the two point function is not really very difficult. There is only one Feynman diagram to evaluate which has the form ($i\epsilon$ prescription is understood)

$$\begin{aligned} i\Gamma_{++}^{\mu\nu}(p^0, p^1) &= -e^2 \int \frac{d^2k}{(2\pi)^2} \{k_+^\mu(k+p)_+^\nu + k_-^\mu(k+p)_-^\nu\} \\ &\times \left(\frac{1}{k^2} + 2i\pi n(|k^0|) \delta(k^2) \right) \\ &\times \left(\frac{1}{(k+p)^2} + 2i\pi n(|k^0+p^0|) \delta((k+p)^2) \right) \end{aligned} \quad (15)$$

where we have defined

$$k_\pm^\mu = (\eta^{\mu\nu} \pm \epsilon^{\mu\nu}) k_\nu. \quad (16)$$

The zero temperature part, of course, can be read out from Eq. (12) and, therefore, we would concern ourselves, in this section, only with possible temperature dependent corrections to the two point function. We note from the definition in Eq. (16) that only two independent tensor structures arise from Eq. (15). The evaluation is straightforward and we have

$$\begin{aligned} i\Gamma_{++}^{00(\beta)}(p^0, p^1) &= i\Gamma_{++}^{11(\beta)}(p^0, p^1) \\ &= (\delta(p_-) + \delta(p_+)) I_2 \\ i\Gamma_{++}^{01(\beta)}(p^0, p^1) &= i\Gamma_{++}^{10(\beta)}(p^0, p^1) \\ &= (\delta(p_-) - \delta(p_+)) I_2. \end{aligned} \quad (17)$$

Here, we have defined

$$p_\pm = p^0 \pm p^1 \quad (18)$$

and

$$\begin{aligned} I_2 &= \frac{(2ie\pi)^2}{2} \int \frac{dk^1}{(2\pi)^2} [\epsilon(k^1) \epsilon(k^1+p^1) \\ &\times \{n(|k^1|) + n(|k^1+p^1|) \\ &- 2n(|k^1|)n(|k^1+p^1|)\}]. \end{aligned} \quad (19)$$

Here $\epsilon(x)$ stands for the alternating step function.

Thus, we see that there is indeed a temperature dependent correction to the two point function. Furthermore, there are several things to note from the structure of the temperature dependent part in Eq. (17). First, it is easy to verify from Eq. (17) that

$$p_\mu \Gamma_{++}^{\mu\nu(\beta)}(p^0, p^1) = 0 = p_\nu \Gamma_{++}^{\mu\nu(\beta)}(p^0, p^1) \quad (20)$$

so that this additional correction is transverse as gauge invariance would require. Furthermore, it is also equally straightforward to check that

$$\epsilon_{\mu\nu} p^\mu \Gamma_{++}^{\nu\lambda(\beta)}(p^0, p^1) = 0 = \epsilon_{\mu\lambda} p^\mu \Gamma_{++}^{\nu\lambda(\beta)}(p^0, p^1). \quad (21)$$

In other words, this temperature dependent correction would lead to a modification in the current which has vanishing divergence as well as curl (and yet is not trivial).

The two point function in Eq. (17) is clearly non-analytic at the origin in the (p^0, p^1) plane which is best seen by writing

$$p^0 = \alpha p^1$$

and noting that in the limit $p^1 \rightarrow 0$ the amplitude depends on the parameter α . This is the well known non-analyticity in the two point function that is expected at finite temperature. However, it is interesting to note that the non-analyticity, in this case, manifests essentially in the structure of the delta functions which is also quite crucial for the current conservation as well as the vanishing of the anomaly [which is clear from Eqs. (20),(21)]. We suspect that this is a structure that may generalize to higher dimensions in a calculation with an arbitrary gauge background. [We would like to point out here that this dependence on the delta function is a particular generalization of the $(0+1)$ -dimensional result [13,14] where the two point function has only one component and is proportional to $\delta(p)$.]

The two point function, of course, can be expressed in a more covariant form. The standard way to do this is to introduce a velocity for the heat bath, u^μ , such that [22]

$$u^\mu u_\mu = 1.$$

Without going into too much detail, let us note that every four-vector can now be decomposed along parallel and perpendicular directions to u^μ as [23]

$$\begin{aligned} k^\mu &= \Omega u^\mu - \epsilon^{\mu\nu} u_\nu \bar{k} \\ p^\mu &= \omega u^\mu - \epsilon^{\mu\nu} u_\nu \bar{p} \end{aligned} \quad (22)$$

where the Lorentz invariant quantities Ω , ω , \bar{k} and \bar{p} are defined by

$$\begin{aligned} \Omega &= k^\mu u_\mu; \quad \bar{k} = \epsilon^{\mu\nu} k_\mu u_\nu \\ \omega &= p^\mu u_\mu; \quad \bar{p} = \epsilon^{\mu\nu} p_\mu u_\nu. \end{aligned} \quad (23)$$

We can also define the component of the velocity four-vector perpendicular to p^μ as

$$\bar{u}^\mu(p) = u^\mu - \frac{\omega}{\bar{p}} \epsilon^{\mu\nu} u_\nu. \quad (24)$$

The calculation of the two point function can be easily carried out in terms of these variables and the temperature dependent correction has the form

$$i\Gamma_{++}^{\mu\nu(\beta)}(\omega, \bar{p}) = (\delta(\omega - \bar{p}) + \delta(\omega + \bar{p})) \bar{u}^\mu(p) \bar{u}^\nu(-p) \bar{I}_2 \quad (25)$$

where

$$\begin{aligned} \bar{I}_2 &= \frac{(2ie\pi)^2}{2} \int \frac{d\bar{k}}{(2\pi)^2} \epsilon(\bar{k}) \epsilon(\bar{k} + \bar{p}) [n(|\bar{k}|) \\ &+ n(|\bar{k} + \bar{p}|) - 2n(|\bar{k}|)n(|\bar{k} + \bar{p}|)]. \end{aligned} \quad (26)$$

This single tensor structure indeed generates the two independent structures noted in Eq. (17) and, in fact, reduces to them in the rest frame of the heat bath for which $u^\mu = (1,0)$. We also note here that the transversality of the two point function follows trivially from the fact that $\bar{u}^\mu(p)$ is transverse to p^μ . The vanishing of the curl, however, does not follow from the transversality of \bar{u}^μ ; rather, it is a consequence of the delta function structure of the two point function. (Note also that, in addition to the delta functions being non-analytic, \bar{u}^μ also depends on the direction along which we approach the origin.)

Once we have a covariant expression for the temperature dependent correction to the two point function, we can easily write the additional quadratic term that would be generated at finite temperature in the effective action (here, for simplicity, we are identifying $A_+^\mu = A^\mu$, namely, the original fields),

$$\begin{aligned} \Gamma_2^{(\beta)}[A] &= \frac{1}{2!} \int \frac{d\omega d\bar{p}}{(2\pi)^2} A_\mu(p) (i\Gamma_{++}^{\mu\nu(\beta)}) A_\nu(-p) \\ &= \frac{1}{2!} \int \frac{d\omega d\bar{p}}{(2\pi)^2} (\bar{u} \cdot A)(p) (\bar{u} \cdot A)(-p) \\ &\quad \times \bar{I}_2 (\delta(\omega - \bar{p}) + \delta(\omega + \bar{p})). \end{aligned} \quad (27)$$

As is obvious, this action is highly nonlocal although it does not have the non-extensive structure found in the $(0+1)$ -dimensional model. It is not obvious to us, at this point, whether the non-extensive structure is a special feature in odd space-time dimensions or simply an accidental feature of the $(0+1)$ -dimensional model. We would also like to comment here that the structure of the quadratic term in Eq. (27) is manifestly gauge invariant because of the transversality of \bar{u}^μ . While it is not obvious, it can be easily checked that the same structure is also invariant for non-Abelian gauge fields which may be an interesting thing to note for generalizations to non-Abelian theories in higher dimensions.

IV. HIGHER POINT FUNCTIONS

The general proof of Sec. II would seem to suggest that the quadratic term is all the correction that temperature would induce in the effective action. However, as we have pointed out earlier, such a conclusion is too naive. Consequently, in what follows, we would like to calculate explicitly the 3-point and the 4-point functions for this theory at finite temperature.

The calculation of the 3-point function is only slightly more difficult than the two point function. In this case, the amplitude involves evaluating two Feynman diagrams. If we denote the two independent external momenta by p and q , then, the two independent diagrams would correspond to exchanging $p \leftrightarrow q$ (of course, with the appropriate interchange of the tensor indices). The 3-point function has the structure

$$\begin{aligned}
i\Gamma_{+++}^{\mu\nu\lambda}(p,q) = & -e^3 \int \frac{d^2k}{(2\pi)^2} \left[(k_+^\mu(k+p)_+^\nu(k+p+q)_+^\lambda + k_-^\mu(k+p)_-^\nu(k+p+q)_-^\lambda) \right. \\
& \times \left(\frac{1}{k^2} + 2i\pi n(|k^0|)\delta(k^2) \right) \left(\frac{1}{(k+p)^2} + 2i\pi n(|k^0+p^0|)\delta((k+p)^2) \right) \\
& \left. \times \left(\frac{1}{(k+p+q)^2} + 2i\pi n(|k^0+p^0+q^0|)\delta((k+p+q)^2) \right) + (p,\nu \leftrightarrow q,\lambda) \right] \quad (28)
\end{aligned}$$

where we have used the definitions in Eq. (16). Once again, we can easily see that the temperature dependent corrections have only two independent structures, the ones with an even number of space-like indices (they are equal) and the ones with an odd number of space-like indices (which are also equal). The two independent structures can be evaluated to have the simple forms

$$\begin{aligned}
i\Gamma_{+++}^{000(\beta)}(p,q) &= (\delta(p_-)\delta(q_-) + \delta(p_+)\delta(q_+))I_3 \\
i\Gamma_{+++}^{001(\beta)}(p,q) &= (\delta(p_-)\delta(q_-) - \delta(p_+)\delta(q_+))I_3 \quad (29)
\end{aligned}$$

where

$$\begin{aligned}
I_3(p,q) = & \frac{(2ie\pi)^3}{2^2} \int \frac{dk^1}{(2\pi)^2} [\epsilon(k^1)\epsilon(k^1+p^1)\epsilon(k^1+p^1+q^1)\{-n(|k^1|) + n(|k^1+p^1|) + n(|k^1+p^1+q^1|)\} \\
& + 2(n(|k^1|)n(|k^1+p^1|) + n(|k^1|)n(|k^1+p^1+q^1|) + n(|k^1+p^1|)n(|k^1+p^1+q^1|)) \\
& - 4n(|k^1|)n(|k^1+p^1|)n(|k^1+p^1+q^1|)] + (p \leftrightarrow q). \quad (30)
\end{aligned}$$

With an appropriate change of variables, it is easy to see that I_3 vanishes because of the anti-symmetry of an odd number of alternating step functions. (Namely, the two diagrams exactly cancel each other.) In fact, one can show, in general, that all the odd-point functions vanish because of charge conjugation invariance in the theory. Namely, the Lagrangian density in Eq. (3) is invariant under

$$\psi \rightarrow \eta C \bar{\psi}^T; \quad A_\mu \rightarrow -A_\mu$$

where η is a phase and C represents the charge conjugation matrix. This invariance requires that the effective action can only depend on an even number of A_μ fields. However, in spite of this general result, we went through the explicit calculation to show the generalization of the structure of the two point function to the case of the three point function. [Had the three point function not vanished, transversality as well as vanishing anomaly would have required the structure to be a generalization of the two point function in the form in Eq. (29).]

Let us next turn to the 4-point function. This is much more involved than the 3-point function. Furthermore, there are now six diagrams to be evaluated. However, each of them has the generic form (p, q and r are the independent external momenta)

$$\begin{aligned}
= & -e^4 \int \frac{d^2k}{(2\pi)^2} \left[\{k_+^\mu(k+p)_+^\nu(k+p+q)_+^\lambda(k+p+q+r)_+^\rho + k_-^\mu(k+p)_-^\nu(k+p+q)_-^\lambda(k+p+q+r)_-^\rho\} \right. \\
& \times \left(\frac{1}{k^2} + 2i\pi n(|k^0|)\delta(k^2) \right) \left(\frac{1}{(k+p)^2} + 2i\pi n(|k^0+p^0|)\delta((k+p)^2) \right) \\
& \times \left(\frac{1}{(k+p+q)^2} + 2i\pi n(|k^0+p^0+q^0|)\delta((k+p+q)^2) \right) \\
& \left. \times \left(\frac{1}{(k+p+q+r)^2} + 2i\pi n(|k^0+p^0+q^0+r^0|)\delta((k+p+q+r)^2) \right) \right].
\end{aligned}$$

The calculation for the temperature dependent part is exactly similar to the two and the three point functions, but much more tedious. We have evaluated these explicitly and adding all the six diagrams, we find, again, that there are only two independent structures that arise. There are the ones with an even number of space-like indices (and they are all equal) and the other kind is for the ones with an odd number of space-like indices (which are again all equal) with the forms given by [with appropriate definitions given in (18)]

$$i\Gamma_{++++}^{0000(\beta)}(p,q,r) = (\delta(p_-)\delta(q_-)\delta(r_-) + \delta(p_+)\delta(q_+)\delta(r_+))I_4 \quad (31)$$

$$i\Gamma_{++++}^{0001(\beta)}(p, q, r) = (\delta(p_-) \delta(q_-) \delta(r_-) - \delta(p_+) \delta(q_+) \delta(r_+)) I_4$$

where

$$\begin{aligned} I_4(p, q, r) = & \frac{(2ie\pi)^4}{2^3} \int \frac{dk^1}{(2\pi)^2} [\epsilon(k^1) \epsilon(k^1+p^1) \epsilon(k^1+p^1+q^1) \epsilon(k^1+p^1+q^1+r^1) \{n(|k^1|) + n(|k^1+p^1|) + n(k^1+p^1+q^1|) \\ & + n(|k^1+p^1+q^1+r^1|) - 2(n(|k^1|)n(|k^1+p^1|) + \text{all quadratic permutations}) \\ & + 4(n(|k^1|)n(|k^1+p^1|)n(|k^1+p^1+q^1|) + \text{all cubic permutations}) \\ & - 8n(|k^1|)n(|k^1+p^1|)n(|k^1+p^1+q^1|)n(|k^1+p^1+q^1+r^1|)\} + \text{all permutations of } (p, q, r)]. \end{aligned} \quad (32)$$

There are several things to note here. First of all, the 4-point function involves an even number of alternating step functions and, therefore, does not vanish unlike the three point function. However, from the structure in Eq. (31), it is clear that it is divergence free and does not contribute to the anomaly either. In fact, the structure is a generalization of the two point and the three point functions in Eqs. (17) and (29) respectively. The non-analyticity continues to be present in the structure of the 4-point function. Furthermore, we can also write the 4-point function in a manifestly covariant form as in the case of the two point function. Let us identify

$$p^\mu = p_1^\mu; \quad q^\mu = p_2^\mu; \quad r^\mu = p_3^\mu \quad (33)$$

and define, as in Eqs. (22),(23),

$$p_i^\mu = \omega_i u^\mu - \epsilon^{\mu\nu} u_\nu \bar{p}_i, \quad i = 1, 2, 3. \quad (34)$$

One can calculate the 4-point function with these variables and it takes the covariant form

$$\begin{aligned} i\Gamma_{++++}^{\mu\nu\lambda\rho(\beta)}(p_1, p_2, p_3) = & (\delta(\omega_1 - \bar{p}_1) \delta(\omega_2 - \bar{p}_2) \delta(\omega_3 - \bar{p}_3) \\ & + \delta(\omega_1 + \bar{p}_1) \delta(\omega_2 + \bar{p}_2) \delta(\omega_3 + \bar{p}_3)) \bar{u}^\mu(p_1) \bar{u}^\nu(p_2) \bar{u}^\lambda(p_3) \bar{u}^\rho(- (p_1 + p_2 + p_3)) \bar{I}_4 \end{aligned} \quad (35)$$

with

$$\begin{aligned} \bar{I}_4 = & \frac{(2ie\pi)^4}{2^3} \int \frac{d\bar{k}}{(2\pi)^2} [\epsilon(\bar{k}) \epsilon(\bar{k} + \bar{p}_1) \epsilon(\bar{k} + \bar{p}_1 + \bar{p}_2) \epsilon(\bar{k} + \bar{p}_1 + \bar{p}_2 + \bar{p}_3) \{n(|\bar{k}|) + n(|\bar{k} + \bar{p}_1|) \\ & + \dots - 2(n(|\bar{k}|)n(|\bar{k} + \bar{p}_1|) + \text{all quadratic permutations}) + 4(n(|\bar{k}|)n(|\bar{k} + \bar{p}_1|)n(|\bar{k} + \bar{p}_1 + \bar{p}_2|) \\ & + \text{all cubic permutations}) - 8n(|\bar{k}|)n(|\bar{k} + \bar{p}_1|)n(|\bar{k} + \bar{p}_1 + \bar{p}_2|)n(|\bar{k} + \bar{p}_1 + \bar{p}_2 + \bar{p}_3|)\} \\ & + \text{all permutations of } (\bar{p}_1, \bar{p}_2, \bar{p}_3)]. \end{aligned} \quad (36)$$

It is clear, therefore, that unlike at zero temperature, the higher order terms do not vanish at finite temperature. This is very much like the behavior of the (0+1)-dimensional theory [6,13,14] where, at zero temperature, the effective action is only linear in the external field while, in the presence of a heat bath, interactions to all orders are generated. Here, however, we have the simplification that only even amplitudes are nonvanishing. Furthermore, the dependence on the delta functions is a very particular generalization of the (0+1)-dimensional result [13,14] where the n -point function is proportional to $\delta(p_1) \delta(p_2) \dots \delta(p_{n-1})$ and is a simple consequence of gauge invariance alone. From the calculations presented so far (as well as from the requirement of vanishing divergence and curl), the structure of the temperature dependent corrections to the higher point functions is quite clear. The covariant form of the $2n$ -point function can be written as follows. Let $p_1, p_2, \dots, p_{2n-1}$ denote the independent external momenta. Then, with the generalization of the decomposition given in Eq. (34), we can write the $2n$ -point function as

$$\begin{aligned} i\Gamma_{++++}^{\mu_1 \dots \mu_{2n}}(p_1, \dots, p_{2n-1}) = & \{(\delta(\omega_1 - \bar{p}_1) \dots \delta(\omega_{2n-1} - \bar{p}_{2n-1}) \\ & + \delta(\omega_1 + \bar{p}_1) \dots \delta(\omega_{2n-1} + \bar{p}_{2n-1}))\} \bar{u}^{\mu_1}(p_1) \dots \bar{u}^{\mu_{2n}}(- (p_1 + \dots + p_{2n-1})) \bar{I}_{2n} \end{aligned} \quad (37)$$

where

$$\begin{aligned}
\bar{I}_{2n} = & \frac{(2ie\pi)^{2n}}{2^{2n-1}} \int \frac{d\bar{k}}{(2\pi)^2} [\epsilon(\bar{k})\epsilon(\bar{k}+\bar{p}_1)\cdots\epsilon(\bar{k}+\cdots+\bar{p}_{2n-1})\{n(|\bar{k}|)+\cdots+n(|\bar{k}+\cdots+\bar{p}_{2n-1}|) \\
& -2(n(|\bar{k}|)n(|\bar{k}+\bar{p}_1|)+\text{all quadratic permutations})+4(n(|\bar{k}|)n(|\bar{k}+\bar{p}_1|)n(|\bar{k}+\bar{p}_1+\bar{p}_2|)+\text{all cubic permutations}) \\
& +\cdots-2^{2n-1}n(|\bar{k}|)\cdots n(|\bar{k}+\cdots+\bar{p}_{2n-1}|)\}] + \text{all permutations of } (\bar{p}_1, \dots, \bar{p}_{2n-1}). \tag{38}
\end{aligned}$$

Thus, collecting all terms, we can write the full effective action at finite temperature to be

$$\Gamma[A] = \Gamma^{(0)}[A] + \sum_{n=1}^{\infty} \Gamma_{2n}^{(\beta)}[A] \tag{39}$$

where (once again, we have identified $A_+^\mu = A^\mu$)

$$\begin{aligned}
\Gamma_{2n}^{(\beta)}[A] = & \frac{1}{2n!} \int \frac{d\omega_1 d\bar{p}_1}{(2\pi)^2} \cdots \frac{d\omega_{2n-1} d\bar{p}_{2n-1}}{(2\pi)^2} (\bar{u} \cdot A)(p_1) \cdots (\bar{u} \cdot A)(-p_1 + \cdots + p_{2n-1}) \\
& \times \bar{I}_{2n}(\delta(\omega_1 - \bar{p}_1) \cdots \delta(\omega_{2n-1} - \bar{p}_{2n-1}) + \delta(\omega_1 + \bar{p}_1) \cdots \delta(\omega_{2n-1} + \bar{p}_{2n-1})) \tag{40}
\end{aligned}$$

and $\Gamma^{(0)}[A]$ is the effective action at zero temperature given in Eq. (12).

This shows that, at finite temperature, the effective action has interactions to all even orders unlike the case at zero temperature. The structures of all these additional, temperature dependent terms are what we were interested in and we note that they are such that they do not change the current conservation as well as the anomaly of the theory. [We would like to add here that we have calculated that part of the effective action which depends only on the original field variables of the theory. The part of the effective action that contains only the doubled degrees of freedom can be easily derived from this since the corresponding amplitudes are related to the ones we have calculated by mere complex conjugation. The part of the effective action containing mixed degrees of freedom, on the other hand, cannot be obtained from what we have calculated, but the evaluation of these amplitudes is similar to what we have carried out and even relatively simpler since the off-diagonal matrix elements of

the propagator are on-shell. The actual value (coefficient) of these terms is not as relevant for our subsequent discussion as the structure of these terms which can be easily seen to be of the form we have already calculated (namely, the dependence on the delta functions) which would not modify current conservation as well as anomaly.]

The presence of these temperature dependent terms raises the interesting possibility that, unlike at zero temperature, there may be finite temperature effects giving rise to two loop and higher loop contributions in the fundamental theory. In fact, such contributions will correspond to diagrams where two (or more) gauge fields are contracted in the effective action (of course, we are assuming here that the gauge fields are dynamical and that one must take into consideration the full effective action containing A_+^μ and A_-^μ and the full 2×2 matrix structure for the photon propagator). But, a little analysis would show that every such contraction would involve a factor of the form [remembering the transversality of $\bar{u}^\mu(p)$]

$$\begin{aligned}
\bar{u}^\mu(p) D_{\mu\nu}(p) \bar{u}^\nu(-p) \delta(\omega \mp \bar{p}) &= - \left(\frac{1}{p^2 - m^2} - 2i\pi n_B(u \cdot p) \delta(p^2 - m^2) \right) \bar{u}^\mu(p) \bar{u}_\mu(-p) \delta(\omega \mp \bar{p}) \\
&= - \left(\frac{1}{p^2 - m^2} - 2i\pi n_B(u \cdot p) \delta(p^2 - m^2) \right) \left(\frac{p^2}{\bar{p}^2} \right) \delta(\omega \mp \bar{p}) = 0. \tag{41}
\end{aligned}$$

Here we have used for $D_{\mu\nu}(p)$ the $++$ component of the temperature dependent propagator for the gauge field, $m^2 = e^2/\pi$, n_B is the bosonic distribution function and we have used Eq. (24) in the final step. However, it is also clear that

the same conclusion holds for any other component of the propagator as well and follows from the very special structure of the temperature dependent vertex functions in this theory (namely, the delta functions). [As a side remark, we-

would like to point out that in 1 + 1 dimensions,¹ $\bar{u}^\mu \bar{u}^\nu$ can be expressed in terms of the usual transverse projection operator as follows:

$$\bar{u}^\mu(p) \bar{u}^\nu(-p) = -\frac{p^2}{\bar{p}^2} \left(\eta^{\mu\nu} - \frac{p^\mu p^\nu}{p^2} \right). \quad (42)$$

This observation makes the derivation of the propagator rather simple.] This is quite interesting, for it says that, even though there are higher point functions present in the effective theory, all the radiative corrections in the fundamental theory are of order one loop. The consequence of this is that the temperature dependent, effective theory obtained in Eqs. (39),(40) is purely classical—it cannot generate any quantum correction. This is interesting and is indeed quite unusual and, as is clear from Eq. (41), is a direct consequence of the specific dependence on delta functions which is also necessary to maintain the current conservation as well as the anomaly of the theory. This is indeed yet a new feature that finite temperature field theories can have. We also note from Eq. (41) that, for $m=0$, such a contraction will not vanish. Consequently, in perturbation theory (where the photon does not have a mass), the higher loop contributions will not vanish individually. They would, in fact, be highly infrared singular. However, if the perturbation is summed to all orders, all such contributions would add up to zero as is clear from Eq. (41).

V. SOLUBLE MODELS

As is well known [15,21] once the effective action for the fermion field in an external Abelian gauge background is known, various soluble models can be directly studied. Therefore, we will be rather brief in this section. First, let us recall that the Schwinger model [24] is defined by the Lagrangian density

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi} \gamma^\mu (i\partial_\mu - eA_\mu) \psi. \quad (43)$$

It is clear, therefore, that integrating out the fermions would lead to an effective action which is the sum of the kinetic term for the photons and the effective action derived in Eq. (39). Thus, the effective action, in addition to containing the mass term for the photon also contains now the additional interactions whose properties we have already discussed.

The general model, in 1 + 1 dimensions, [16,17,18] is described by the Lagrangian density

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi} \gamma^\mu (i\partial_\mu - e(1+r\gamma_5)A_\mu) \psi. \quad (44)$$

Here r is an arbitrary parameter and this model is known to reduce to various soluble models under different limits and reductions. We note that if we define a new gauge field as

$$B^\mu = (\eta^{\mu\nu} + r\epsilon^{\mu\nu}) A_\nu, \quad (45)$$

then, using the two dimensional identities in Eq. (4), it is easy to show that the fermion part of the Lagrangian in Eq. (44) becomes identical to that for the Schwinger model, but in terms of the B_μ field. We have already evaluated the effective action for this and so, expressing everything back in terms of the A_μ field, we would have the effective action for the general model which then, would give the effective action for various soluble models at finite temperature under different limits and reductions [25]. We would simply like to note here that for the gradient coupling model,

$$A_\mu = \partial_\mu \phi$$

namely, the gauge field can be identified with the gradient of a scalar. In such a case, however, it is clear that

$$\bar{u}(p) \cdot A(p) = 0. \quad (46)$$

Consequently, all the temperature dependent corrections to this model identically vanish and the zero temperature effective action is the full action (independent of the regularization used).

VI. CONCLUSION

In this paper, we have studied, systematically, the effective action for (1 + 1)-dimensional, massless fermions interacting with an external Abelian gauge field at finite temperature. While the naive expectation would be that only the two point function is corrected by temperature, we have calculated and shown that the effective action, in fact, contains interaction terms to all (even) orders. The exact form of the effective action is obtained and it is shown that these additional temperature dependent terms do not change the anomaly or the current conservation. The non-analytic structure of the effective action at finite temperature is pointed out. It is also pointed out that these temperature dependent terms in the effective action have a very specific structure which prevents them from generating any quantum mechanical correction. To the best of our knowledge, this is a new feature of field theories at finite temperature. The solubility of various two dimensional models is also briefly discussed. We hope that some of the features found here will help in the understanding of the structure of the effective action for a fermion interacting with an arbitrary gauge field in 2 + 1 dimensions. As a final comment, we would like to add that we have also calculated the retarded Green's function [1] in this model. All the temperature dependent parts vanish which is consistent with our conclusion that these new terms in the finite temperature effective action cannot lead to any quantum correction [26].

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- [1] A. Das, *Finite Temperature Field Theory* (World Scientific, Singapore, 1997).
- [2] H. A. Weldon, Phys. Rev. D **28**, 2007 (1983).
- [3] A. Das and M. Hott, Phys. Rev. D **50**, 6655 (1994).
- [4] H. A. Weldon, Phys. Rev. D **47**, 594 (1993).
- [5] P. F. Bedaque and A. Das, Phys. Rev. D **47**, 601 (1993).
- [6] G. Dunne, K. Lee, and C. Lu, Phys. Rev. Lett. **78**, 3434 (1997).
- [7] K. S. Babu, A. Das, and P. Panigrahi, Phys. Rev. D **36**, 3725 (1987); A. Das and S. Panda, J. Phys. A **25**, L245 (1992).
- [8] I. Aitchison, C. S. Fosco, and J. Zuk, Phys. Rev. D **48**, 5895 (1993); I. Aitchison and J. Zuk, Ann. Phys. (N.Y.) **242**, 77 (1995).
- [9] S. Deser, R. Jackiw, and S. Templeton, Phys. Rev. Lett. **48**, 975 (1982); S. Deser, R. Jackiw, and S. Templeton, Ann. Phys. (N.Y.) **140**, 372 (1982).
- [10] S. Deser, L. Griguolo, and D. Seminara, Phys. Rev. Lett. **79**, 1976 (1997); S. Deser, L. Griguolo, and D. Seminara, Phys. Rev. D **57**, 7444 (1998).
- [11] C. Fosco, G. Rossini, and F. Schaposnik, Phys. Rev. Lett. **79**, 1980 (1997); C. Fosco, G. Rossini, and F. Schaposnik, Phys. Rev. D **56**, 6547 (1997).
- [12] I. Aitchison and C. Fosco, Phys. Rev. D **57**, 1171 (1998).
- [13] A. Das and G. Dunne, Phys. Rev. D **57**, 5023 (1998).
- [14] J. Barcelos-Neto and A. Das, Phys. Rev. D **58**, 085022 (1998).
- [15] W. Thirring, Ann. Phys. (N.Y.) **3**, 91 (1958); V. Glaser, Nuovo Cimento **9**, 990 (1958); K. Johnson, *ibid.* **20**, 773 (1961); C. Sommerfield, Ann. Phys. (N.Y.) **26**, 1 (1964); L. S. Brown, Nuovo Cimento **29**, 617 (1963); C. R. Hagen, Nuovo Cimento B **51**, 169 (1967); C. R. Hagen, Nuovo Cimento A **51**, 1033 (1967); J. H. Lowenstein and A. Swieca, Ann. Phys. (N.Y.) **68**, 172 (1971); A. Das and C. R. Hagen, Phys. Rev. D **32**, 2024 (1985); A. Das and V. S. Mathur, *ibid.* **33**, 489 (1986); R. Jackiw and R. Rajaraman, Phys. Rev. Lett. **54**, 1219 (1985).
- [16] A. Bassetto, L. Griguolo, and P. Zanca, Phys. Rev. D **50**, 1077 (1994).
- [17] A. Das and M. Hott, Z. Phys. C **67**, 707 (1995).
- [18] C. R. Hagen, Phys. Rev. D **55**, 1021 (1997).
- [19] L. Dolan and R. Jackiw, Phys. Rev. D **9**, 3320 (1974).
- [20] A. Das and A. Karev, Phys. Rev. D **36**, 623 (1987).
- [21] A. Das and A. Karev, Phys. Rev. D **36**, 2591 (1987).
- [22] H. A. Weldon, Phys. Rev. D **26**, 1394 (1982).
- [23] A. Das and M. Hott, Mod. Phys. Lett. A **9**, 3383 (1994). We take this opportunity to point out that the second of Eq. (24) in this paper is in error and, consequently, the second conclusion following in Eq. (25) as well because of the singular delta function structure.
- [24] J. Schwinger, Phys. Rev. **128**, 2425 (1962).
- [25] It is interesting to compare our results with those in S. T. Love, Phys. Rev. D **23**, 420 (1981); F. Ruiz Ruiz and R. F. Estrada, *ibid.* **35**, 3161 (1987); H. Yokota, Prog. Theor. Phys. **77**, 1450 (1987).
- [26] F. Brandt, A. Das, J. Frenkel, and A. J. da Silva, Phys. Rev. D **59**, 065004 (1999).