

## Holographic probes of anti-de Sitter spacetimes

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We describe probes of anti-de Sitter spacetimes in terms of conformal field theories on the AdS boundary. Our basic tool is a formula that relates bulk and boundary states—classical bulk field configurations are dual to expectation values of operators on the boundary. At the quantum level we relate the operator expansions of bulk and boundary fields. Using our methods, we discuss the CFT description of local bulk probes including normalizable wave packets, fundamental and D-strings, and D-instantons. Radial motions of probes in the bulk spacetime are related to motions in scale on the boundary, demonstrating a scale-radius duality. We discuss the implications of these results for the holographic description of black hole horizons in the boundary field theory. [S0556-2821(99)03808-4]

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### I. INTRODUCTION

A recurring theme of recent work is that gravitational theories can sometimes be formulated as gauge theories in fewer dimensions. This point of view has had some encouraging successes, but we still do not understand how the famous problems of quantum gravity—for example, information loss in black hole evaporation—are solved. All of our intuitions about gravity and spacetime physics is based on a classical, geometric picture valid when  $\hbar$  is small and the field configurations macroscopic. In this regime the spacetime physics displays at least approximate locality and causality, and a well-defined geometry in which free particles follow geodesics. These properties seem obscure in the gauge theory formulation. Once we understand their origin, we can investigate precisely when and how they break down.

An important avenue for understanding these issues is the most recent manifestation of the gravity-gauge theory connection: the conjecture [1] that string theory on an anti-de Sitter (AdS) background is dual to a conformal field theory residing on the spacetime boundary. So far, the proposal has been checked by comparing spectra and low-order correlation functions of the dual theories. Such checks are based on a remarkably compact and powerful statement of the equality between certain path integrals in the dual theories [2,3]. We would like to use this equality to learn how the classical geometric description of the bulk emerges and ultimately breaks down in the holographic boundary representation.

This article begins such a study by describing a variety of spacetime probes from the boundary perspective. Our basic tool is a compact formula relating bulk and boundary states; specifically, the asymptotic behavior of fluctuating, classical bulk fields is related to expectation values of the dual boundary operators in excited boundary states. At the operator level we relate the quantized mode expansion of bulk fields

to a mode expansion of boundary operators; so bulk quanta are dual to conformal field theory (CFT) states created by modes of boundary operators acting on the vacuum. We develop this formalism in Sec. II and, more carefully, in Appendix A. This development continues the work in [4] which identified fluctuating supergravity modes as dual to boundary states and nonfluctuating modes implementing boundary conditions as dual to boundary sources. Here, we will be interested mainly in the classical limit of the fluctuating states and in bulk configurations generated by brane sources.

Using the methods of Sec. II, we discuss the boundary description of three kinds of probes: D-instantons, F- and D-strings, and dilation wave packets. In all cases the characteristic radial position of the bulk probe is mapped to the characteristic scale of the boundary configuration, as understood on general grounds by comparing the action of bulk isometries with conformal transformations on the boundary [1]. This scale-radius duality gives rise to pleasantly physical interpretations of bulk dynamics. For instance, strings or particles move in AdS spacetime to reduce gravitational potential energy; this is dual in the boundary theory to the spreading of localized field distributions to reduce gradient energy. In Euclidean  $AdS_5 \times S^5$  it has been conjectured that a D-instanton at radial position  $z$  is dual to an instanton in  $d = 4$  super Yang-Mills (SYM) theory with scale size  $z$  [5–8]. As an application of our methods we *derive* this correspondence from the fundamental formulation of the AdS-CFT conjecture given in [2,3]. We conclude by discussing the meaning of the classical limit on both sides of the AdS-CFT duality and by discussing the implications of our results for the holographic representation of black hole horizons in the boundary gauge theory.

### II. RELATING BULK AND BOUNDARY STATES

In order to study how bulk geometry is encoded in the boundary description, we wish to introduce probes into the AdS spacetime. In this section we develop methods that identify boundary configurations corresponding to these

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probes. Our basic technique is to study the response of the bulk probe to a small change in boundary conditions. The formulation of the AdS-CFT correspondence in [2,3] then provides the expectation values of operators in the corresponding boundary states. This will also allow us to arrive at a quantized formulation of the bulk theory from the boundary perspective. In this section we will ignore some subtleties [9] which are unimportant for most of the considerations of this paper. A more careful treatment is provided in Appendix A.

We work with Poincaré coordinates for anti-de Sitter space. The metric of AdS<sub>d+1</sub> in these coordinates is

$$ds^2 = \frac{R^2}{z^2} (-dt^2 + d\vec{x}_{d-1}^2 + dz^2). \quad (1)$$

(We set  $R=1$  in this section.) Poincaré coordinates only cover a patch of the global spacetime, and  $z=0$  is the boundary of AdS spacetime while  $z=\infty$  is the horizon. (See [4] for more details and Penrose diagrams.) Euclidean AdS spacetime is obtained by taking  $t$  to  $it$ .

### A. Euclidean signature

The AdS-CFT correspondence is formulated in [2,3] as

$$Z(\phi_i) = e^{-S(\phi_i)} = \left\langle \exp \left( \int_B \phi_{0,i} \mathcal{O}^i \right) \right\rangle, \quad (2)$$

where  $S(\phi_i)$  is the effective action as a function of the bulk field  $\phi_i$ ,  $\phi_{0,i}$  is the boundary value of  $\phi_i$  (up to a scaling with the radial coordinate), and  $\mathcal{O}^i$  is the dual operator in CFT. The expectation value on the right is evaluated in the CFT vacuum. We can read Eq. (2) as saying that boundary conditions for the bulk theory are dual to sources in the boundary theory. In other words, field theory in Euclidean AdS spacetime, expanded around a background approaching  $\phi_{0,i}$  at the boundary, is described by a CFT deformed by the addition of a source. By functionally differentiating we find that

$$\frac{\delta}{\delta \phi_{0,i}(\mathbf{x})} [-S(\phi_i)] = \langle \mathcal{O}^i(\mathbf{x}) \rangle_{\phi_{0,i}}, \quad (3)$$

where the subscript  $\phi_{0,i}$  indicates that the expectation value on the right hand side is computed in the presence of the source term  $\int \phi_{0,i} \mathcal{O}^i$ . We will use this relation to learn about the expectation values of boundary operators in the presence of bulk probes.

*Massive scalar.* As an example, let us study a massive scalar field with a quadratic bulk action:

$$S(\phi) = \frac{1}{2} \int d^{d+1}x \sqrt{g} (|d\phi|^2 + m^2 \phi^2). \quad (4)$$

Consider a solution to the bulk equations of motion that approaches  $\phi_0(x)$  at the boundary (up to a scaling with the radial coordinate  $z$ ). The unique solution regular in the bulk

behaves as  $\phi(z, \mathbf{x}) = z^{2h_-} \phi_0(\mathbf{x})$  as  $z \rightarrow 0$  and can be written as

$$\phi(z, \mathbf{x}) = c \int d^d \mathbf{x}' \frac{z^{2h_+}}{(z^2 + |\mathbf{x} - \mathbf{x}'|^2)^{2h_+}} \phi_0(\mathbf{x}'), \quad (5)$$

where we have used the bulk-boundary propagator [3] and

$$h_{\pm} = \frac{d}{4} \pm \frac{\sqrt{d^2 + 4m^2}}{4} \equiv \frac{d}{4} \pm \frac{\nu}{2}. \quad (6)$$

The presence of this classical configuration corresponds to the addition of a source  $\int \phi_0 \mathcal{O}$  to the boundary theory.

We now apply Eq. (3) by computing the functional derivative on the left hand side. To do so we perturb around  $\phi(z, \mathbf{x})$  by a small fluctuation  $\delta\phi$  and evaluate the resulting change in the action. After integrating Eq. (4) by parts and using the equations of motion, the variation becomes a surface term at the boundary:

$$\delta S(\phi) = \int_B d\Sigma^\mu \partial_\mu \phi \delta\phi. \quad (7)$$

Evaluating this quantity near the boundary at  $z=0$  is delicate [9]. In this section we will follow [3] by considering contributions to the integrand of Eq. (5) from the region  $|\mathbf{x} - \mathbf{x}'| \neq 0$ . This procedure amounts to ignoring certain contact terms and normalization issues as we discuss in Appendix A and gives, as  $z \rightarrow 0$ ,

$$\frac{\partial \phi}{\partial z} = c(2h_+) z^{2h_+ - 1} \int d^d \mathbf{x}' \frac{\phi_0(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^{2(2h_+)}}. \quad (8)$$

The perturbation is written as  $\delta\phi = z^{2h_-} \delta\phi_0$ , and we use  $d\Sigma^0 = z^{1-d} d^d \mathbf{x}'$ . Then

$$\delta S(\phi) = c(2h_+) \int d^d \mathbf{x} d^d \mathbf{x}' \frac{\phi_0(\mathbf{x}') \delta\phi_0(\mathbf{x})}{|\mathbf{x} - \mathbf{x}'|^{2(2h_+)}}. \quad (9)$$

Using the relation (3) we derive

$$\langle \mathcal{O}(\mathbf{x}) \rangle_{\phi_0} = -c(2h_+) \int d^d \mathbf{x}' \frac{\phi_0(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^{2(2h_+)}}. \quad (10)$$

We have learned that in the presence of the source term  $\int \phi_0 \mathcal{O}$  the operator  $\mathcal{O}$  has acquired an expectation value given by the right hand side of Eq. (10). This matches what we expect from CFT by direct calculation:

$$\begin{aligned} \langle \mathcal{O}(\mathbf{x}) \rangle_{\phi_0} &= \left\langle \mathcal{O}(\mathbf{x}) \exp \left( \int \phi_0 \mathcal{O} \right) \right\rangle \\ &\approx \int d^d \mathbf{x}' \phi_0(\mathbf{x}') \langle \mathcal{O}(\mathbf{x}) \mathcal{O}(\mathbf{x}') \rangle \\ &\approx - \int d^d \mathbf{x}' \frac{\phi_0(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^{2(2h_+)}} \end{aligned} \quad (11)$$

where the form of the two-point correlator follows from scale invariance.

*Interpretation.* The AdS-CFT correspondence in Eq. (2) states that turning on a bulk mode which behaves as  $z^{2h-}\phi_0(\mathbf{x})$  near the boundary is dual to including a source term  $\int \phi_0 \mathcal{O}$  in the CFT. As discussed in [4], the growth of such modes near the boundary indicates that they are non-fluctuating classical backgrounds. In effect, the presence of the mode redefines the Hamiltonian of the theory, since fluctuations should take place on top of this background. This is mirrored in CFT by a modification of the action by the addition of a source term.

In the bulk, a mode with leading boundary behavior  $z^{2h-}\phi_0$  induces a *subleading* component behaving as  $z^{2h+}\tilde{\phi}$ . [This is seen by expanding Eq. (5) in powers of  $z$ .] The corresponding statement in CFT is that the addition of the source induces an expectation value for  $\mathcal{O}$ . In fact, our analysis showed that  $\langle \mathcal{O}(\mathbf{x}) \rangle_{\phi_0} \sim \tilde{\phi}(\mathbf{x})$ , so that operator expectation values and bulk field components behaving as  $z^{2h+}$  are precisely dual. This duality is the prevailing theme of the present work. A related connection between subleading terms in the AdS<sub>3</sub> metric and the stress tensor in the corresponding boundary theory has appeared recently in [10].

*Bulk sources.* In the above example, the solution  $\phi(z, \mathbf{x})$  was completely determined by the boundary value  $\phi_0$  and the requirement of regularity in the bulk. However, as we shall see in Sec. III, this uniqueness fails when we admit singular fields corresponding to sources in the bulk. Such bulk sources contribute subleading pieces to the fields at the boundary which modify Eq. (5) and contribute to operator expectation values. So once again we will find that subleading pieces of the bulk fields are dual to boundary expectation values. In this way, we will show in Sec. III that Eq. (3) implies that a bulk D-instanton in AdS<sub>5</sub> is dual to an instanton in boundary Yang-Mills theory.

### B. Lorentzian signature

The crucial new feature of the Lorentzian signature is that the bulk wave equation admits propagating, normalizable mode solutions. Such modes describe the physical, low-energy excitations of the spacetime; their explicit forms have been worked out in [11–13,4] and they behave as  $z^{2h+}$  near the boundary. These normalizable modes form the Hilbert space of the bulk theory. The possible boundary conditions for fields in AdS spacetime are encoded by the choice of non-normalizable mode solutions behaving as  $z^{2h-}$  near the boundary. As argued in [4], the normalizable and non-normalizable solutions are dual to states and sources respectively in the boundary conformal field theory. Here we make explicit the map between bulk and boundary states. So given a bulk field  $\phi_i$  approaching  $z^{2h-}\phi_{0,i}$  at the boundary, we write the Lorentzian bulk-boundary correspondence as

$$Z(\phi_i) = e^{iS(\phi_i)} = \left\langle s \left| \exp \left( i \int_B \phi_{0,i} \mathcal{O}^i \right) \right| s \right\rangle. \quad (12)$$

Here  $|s\rangle$  represents the CFT state that is dual to the bulk state. Operator expectation values in excited CFT states will

differ from their vacuum values; we write

$$\frac{\delta}{\delta \phi_{0,i}(\mathbf{x})} S(\phi_i) = \langle s | \mathcal{O}^i(\mathbf{x}) | s \rangle_{\phi_{0,i}}, \quad (13)$$

where the subscript indicates that the expectation is computed in the presence of a source term. We will also show that this can be turned into a statement relating quantized field operators in the bulk to operators in the boundary.

*Massive scalar.* As an example, we return again to the free massive scalar with action (4). Now a general nonsingular solution of the bulk equations approaching  $z^{2h-}\phi_0(\mathbf{x})$  at the boundary can be written as

$$\phi(z, \mathbf{x}) = \phi_n(z, \mathbf{x}) + c \int d^d \mathbf{x}' \frac{z^{2h+}}{(z^2 + |\mathbf{x} - \mathbf{x}'|^2)^{2h+}} \phi_0(\mathbf{x}'), \quad (14)$$

where  $\phi_n$  is a normalizable mode, and  $|\mathbf{x} - \mathbf{x}'|^2 = -(t - t')^2 + \sum_{i=1}^{d-1} (x_i - x'_i)^2$ . Here we have used a bulk-boundary propagator obtained by continuation from Euclidean space in [3]. There are ambiguities in this choice whose meaning is discussed in Sec. III C and Appendix A. We now repeat the procedure used in the Euclidean signature, dropping contact terms as before, and paying attention to the extra contribution from  $\phi_n$ . (There are subtleties in this procedure—see Appendix A.) Using  $\phi_n(z, \mathbf{x}) \rightarrow z^{2h+}\tilde{\phi}_n(\mathbf{x})$  as  $z \rightarrow 0$ , we find

$$\begin{aligned} \langle \tilde{\phi}_n | \mathcal{O}(\mathbf{x}) | \tilde{\phi}_n \rangle_{\phi_0} &= (2h_+) \tilde{\phi}_n(\mathbf{x}) \\ &+ c(2h_+) \int d^d \mathbf{x}' \frac{\phi_0(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^{2(2h_+)}} \end{aligned} \quad (15)$$

where we have indicated that CFT is in the excited state  $|\tilde{\phi}_n\rangle$ .<sup>1</sup> So  $\mathcal{O}$  gets an expectation value from two distinct contributions: from the excited state and from the source that has been turned on. Note that  $|\tilde{\phi}_n\rangle$  is a ‘‘coherent’’ state on the boundary in which operators have nonvanishing expectation values.

*Interpretation.* In Lorentzian AdS spacetime, normalizable and non-normalizable modes are dual to states and sources, respectively. As we have seen, operator expectation values are affected by both the state and the source. Nevertheless, as in the Euclidean case, the component of the total bulk field behaving as  $z^{2h-}$  defines the source while the component behaving as  $z^{2h+}$  gives rise to the boundary expectation value. We are free to set  $\phi_0$  to zero if we wish, so that the sources are turned off—then we are studying states of the original unmodified CFT.

*Bulk sources.* In the example above, we considered linearized wave equations for a massive scalar. It is possible to consider fully nonlinear solutions with possible bulk singularities due to sources. For instance, the field configuration arising from a D-brane in the AdS geometry would be of this

<sup>1</sup>The normalizations produced by this naive Lorentzian treatment are not correct—see Appendix A.

type. The treatment based on Eq. (13) is equally valid in this case—linearized fluctuations around the fully nonlinear solution lead by integration by parts to the same surface integrals. When  $\phi_0$  vanishes Eq. (15) will also continue to hold, but in general nonvanishing  $\phi_0$  will push the field configuration into the nonlinear regime and bulk interactions will become important. Equipped with Eq. (15), in the next sections we will determine the dual boundary description of various solitonic objects in the bulk.

### C. Operator formulation

We would also like a more microscopic mapping at the level of individual quantum states. This is obtained by regarding fields as quantized operators. In particular, we may write  $\phi_n$  in terms of a mode expansion,

$$\hat{\phi}_n = \sum_k [a_k \phi_{n,k} + a_k^\dagger \phi_{n,k}^*], \quad (16)$$

and similarly for the boundary operator  $\mathcal{O}$ ,

$$\hat{\mathcal{O}} = \sum_k [b_k \tilde{\phi}_{n,k} + b_k^\dagger \tilde{\phi}_{n,k}^*], \quad (17)$$

where, as before,  $\tilde{\phi}(\mathbf{x})$  is the boundary value of the component of the bulk mode  $\phi(z, \mathbf{x})$  scaling as  $z^{2h_+}$  as  $z \rightarrow 0$ . Note that the modes appearing in the expansion of  $\hat{\phi}_n$  satisfy an on-shell condition  $(\square - m^2)\phi_{n,k} = 0$ , whereas the modes  $\tilde{\phi}_{n,k}$  do not satisfy any wave equation on the boundary, but are instead a fully complete set of functions. Interpreting Eq. (15) as an operator statement we conclude that  $a_k = b_k$  and that  $a_k^\dagger |0\rangle = |k\rangle$  where  $|k\rangle$  is a ‘‘one particle’’ state created by a single application of  $b_k^\dagger$ . (It is intriguing that creation and annihilation operators of elementary bulk fields are related to composite operators on the boundary, leading us to identify  $\sum_k b_k^\dagger b_k$  as a particle number operator.) In other words, bulk states described by quanta occupying normalizable modes are dual to CFT states described by acting on the vacuum with modes of the appropriate boundary operator. This provides a direct correspondence between bulk and CFT states.<sup>2</sup>

*Choice of vacuum.* Quantum field theory in curved space can usually accommodate a variety of inequivalent vacua, corresponding to different definitions of positive frequency. In the present context, the choice of vacuum affects the formalism in two places. First, mode solutions in Eqs. (16), (17) have positive frequency with respect to a particular time coordinate, here taken to be Poincaré time. A different choice of time can lead to an inequivalent vacuum state, related to the original vacuum by a Bogoliubov transformation. Second, we have made a particular choice for the form of the bulk-boundary propagator in Eq. (14) which we obtained by continuation from Euclidean space. This is the appropriate propagator to use when perturbing around the Poincaré

vacuum. However, alternative vacua can be chosen by changing the mode expansions (16),(17) and modifying the bulk-boundary propagator. The latter modification will involve adding a normalizable mode to the original propagator, which leaves unchanged its  $z^{2h_-}$  dependence near  $z=0$ . These issues are discussed in greater detail in Appendix A.

### D. Radial isometry and boundary scale transformation

In subsequent sections we will use the formalism developed above to study the boundary representation of probes in the bulk. A recurring theme will be a duality between characteristic radial positions in the bulk and characteristic scales in the boundary. Let us review how this arises. The Poincaré metric (1) has a radial isometry:

$$(\vec{x}, t) = \mathbf{x} \rightarrow \lambda \mathbf{x}, \quad z \rightarrow \lambda z. \quad (18)$$

As we have discussed, boundary expectation values for a massive scalar are dual to the component of the bulk field scaling as  $z^{2h_+} \tilde{\phi}(\mathbf{x})$  near the boundary. So consider a one-parameter family of bulk solutions of the form

$$\phi^\lambda(z, \mathbf{x}) = \phi(\lambda z, \lambda \mathbf{x}). \quad (19)$$

According to Eq. (15), the expectation value of the boundary operator in the corresponding state behaves as

$$\langle \phi^\lambda | \mathcal{O}(\mathbf{x}) | \phi^\lambda \rangle = \lambda^{2h_+} \langle \mathcal{O}(\lambda \mathbf{x}) \rangle. \quad (20)$$

So the radial isometry generates scale transformations on the boundary. In the examples studied below we will explicitly see that the boundary configuration spreads out as the bulk probe falls towards the Poincaré horizon.

## III. INSTANTON PROBES

Our first example of a bulk probe is a D-instanton in  $\text{AdS}_5 \times S^5$ . The methods developed in the previous section will show that it is dual to a boundary Yang-Mills instanton. A particular consequence is a duality between the radial position of the bulk object and the scale size on the boundary. D-instanton solutions in AdS space have been discussed in [6–8] where the close similarity between the bulk dilaton profile and the boundary instanton was noted. Our main point in this section is that this fact follows from the general considerations of Sec. II. This *derives* the duality between the D-instanton and the Yang-Mills instanton from the fundamental formulation of the AdS-CFT correspondence in Eq. (2).

According to the methods of Sec. II, to determine the boundary expectation values corresponding to a D-instanton we require a form of the bulk solution near the boundary. We will use Poincaré coordinates for  $\text{AdS}_5$ , so that the metric is given by Eq. (1) with  $R^4 = 4\pi g_5 N \alpha'^2$ . D-instanton solutions have been presented in [6–8]; only the dilaton ( $\phi$ ) and the axion ( $\chi$ ) are turned on, while the  $\text{AdS}_5 \times S^5$  Einstein metric is unchanged. Dimensional reduction of these fields on  $S^5$  produces a Kaluza-Klein tower of modes on  $\text{AdS}_5$ . Here we are only interested in the behavior as  $z \rightarrow 0$  of the massless

<sup>2</sup>Related issues are discussed in [14].

five dimensional dilaton and axion that couple to  $\text{Tr}(F^2)$  and  $\text{Tr}(F\tilde{F})$  on the boundary.<sup>3</sup> This is given by<sup>4</sup>

$$e^\phi = g_s + c \frac{z^4 \tilde{z}^4}{[\tilde{z}^2 + |\vec{x} - \vec{x}_a|^2]^4} \dots, \quad (21)$$

$$\chi = \chi_\pm \pm (e^{-\phi} - 1/g_s). \quad (22)$$

Here  $\tilde{z}$  is the radial position of the D-instanton. The constant  $c$  in Eq. (21) can be determined by requiring that the D-instanton carry the correct axionic charge, that is,

$$\frac{1}{2\kappa_{10}^2} \int e^{2\phi} \partial_\mu \chi dS^\mu = 2\pi, \quad (23)$$

yielding

$$c = \frac{24\pi}{N^2}. \quad (24)$$

*Boundary expectations.* We can now use the methods of Sec. II to derive the expectation values of boundary operators. The five-dimensional dilaton action is

$$S_\phi = -\frac{1}{4\kappa_5^2} \int d^5x \sqrt{g} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + \dots, \quad (25)$$

where  $1/\kappa_5^2 = V_5 R^5 / \kappa_{10}^2$ ,  $\kappa_{10}$  is the ten-dimensional Newton constant, and  $V_5 = \pi^3$  is the volume of the unit five-sphere. We start with the dilaton background for the D-instanton (21) and add a small perturbation  $\delta\phi$ . The resulting change in the action is a boundary term

$$\delta S = -\frac{1}{2\kappa_5^2} \int d^4x \frac{R^3}{z^3} \delta\phi \partial_z \phi. \quad (26)$$

It follows that the functional derivative with respect to the boundary configuration  $\phi_0$  is  $(\delta S / \delta\phi_0) = -(1/2\kappa_5^2) R^3 \partial_z \phi$ . Using  $\phi$  in Eq. (21) and the relation  $R^4 = \kappa_{10} N / 2\pi^{5/2}$  gives

$$\frac{\delta S}{\delta\phi_0(\vec{x})} = -\frac{48}{4\pi g_s} \frac{\tilde{z}^4}{[\tilde{z}^2 + |\vec{x} - \vec{x}_a|^2]^4}. \quad (27)$$

<sup>3</sup>These couplings were first studied in [15].

<sup>4</sup>There is some disagreement between [6,7] and [8] concerning whether the the D-instanton should be localized on the  $S^5$ . The difference will lie in the excitation of the Kaluza-Klein harmonics that are massive fields on  $\text{AdS}_5$ . Including these excitations which fall off faster at the boundary will give expectation values to dual higher-dimension operators that were identified in [3]. The authors of [6–8] all agree on the asymptotic form of the massless  $\text{AdS}_5$  fields which is all that we require here.

We learn from [3,2] that the  $\text{AdS}_5$  massless dilaton couples to  $\text{Tr}(F^2)$  in boundary CFT. Choosing the normalization  $S_{YM} = (1/4g_{YM}^2) \int d^4x \text{Tr}(F^2) + \dots$  for the Yang-Mills action gives

$$\frac{\delta S_{YM}}{\delta\phi_0(\vec{x})} = -\frac{1}{4g_{YM}^2} \langle \text{Tr}[F^2(\vec{x})] \rangle. \quad (28)$$

Equating Eqs. (27) and (28) and using  $4\pi g_s = g_{YM}^2$ , we find

$$\frac{1}{4g_{YM}^2} \langle \text{Tr}F^2(\vec{x}) \rangle = \frac{48}{g_{YM}^2} \frac{\tilde{z}^4}{[\tilde{z}^2 + |\vec{x} - \vec{x}_a|^2]^4}, \quad (29)$$

which is exactly the classical Yang-Mills field strength in an instanton background. So, as advertised, a D-instanton in the bulk is precisely dual to a Yang-Mills instanton in the boundary theory.

The reader may be surprised that the expression for  $\langle \text{Tr}F^2 \rangle$  that we have computed agrees with *classical* instanton background although we are in a limit with large  $N$  and large  $g_{YM}^2 N$ . Many classical quantities receive large quantum corrections in such limits. However, the functional form of the instanton is essentially determined by conformal invariance, given the single length scale. The coefficient in front determines the topological charge and is protected by a Bogomol'nyi-Prasad-Sommerfield (BPS) bound. This seems to be the reason why the large- $N$  result implied by supergravity agrees with the weak-coupling classical Yang-Mills calculation.

This discussion of the dilaton can be extended to the axion which yields the expected  $\langle F\tilde{F} \rangle$  for an instanton. Since the  $\text{AdS}_5$  metric is unchanged, we learn that the expectation value of the stress tensor  $\langle T_{\mu\nu} \rangle$  vanishes in an instanton background. This is easily checked; the stress tensor is

$$T_{\mu\nu} = \frac{1}{4} g_{\mu\nu} \text{Tr}F^{\rho\sigma} F_{\rho\sigma} - \text{Tr}F_\mu^\rho F_{\rho\nu}. \quad (30)$$

The two terms cancel for the Yang-Mills instanton. Similarly, the Neveu-Schwarz–Neveu-Schwarz (NS-NS)  $B$  field in an  $S$  wave on  $S^5$  is known to be dual to a dimension-6 operator in Yang-Mills theory [16]. This operator was derived in [17,18]:

$$\mathcal{O}_{\mu\nu}^{(6)} \propto \text{tr} \left[ \frac{1}{2} F_{[\nu\alpha} F^{\alpha\beta} F_{\beta\mu]} + \frac{1}{8} F_{\alpha\beta} F^{\alpha\beta} F_{\mu\nu} \right], \quad (31)$$

where we have antisymmetrized the indices  $\mu, \nu$ . Again, it is simple to check that the first term vanishes upon antisymmetrization and the second term vanishes identically.<sup>5</sup> Finally,

<sup>5</sup>Note, however, that the symmetrized part of the first term does not vanish. In general we expect that there will be all sorts of combinations of  $F$  that will not vanish in this background. This is not a surprise—for example, in the presence of a classical bulk configuration, we expect that many operators corresponding to multiparticle bulk states will have expectation values.

the D-instanton action  $S_{D-inst} = 2\pi/g_s$  coincides with the YM instanton action  $S_{YM} = 8\pi^2/g_{YM}^2$  using  $4\pi g_s = g_{YM}^2$ .

*Scale-radius duality and bulk locality.* The above duality between the D-instanton and YM instanton provides the first example of a phenomenon we will call scale-radius duality. The characteristic radial position of the D-instanton is  $\tilde{z}$ . On the boundary  $\tilde{z}$  is the characteristic scale of the YM instanton. A D-instanton closer to the horizon at  $\tilde{z}=\infty$  is mapped into a fatter boundary object [5]. More specifically, the action of the isometries (18) translates the D-instanton both radially and parallel to the boundary. The corresponding conformal transformation of the boundary instanton rescales it and translates it at the same time.

This relation has interesting consequences for the emergence of local physics in the bulk when we apply our techniques to multi-instanton solutions. Consider two D-instantons at very different radial positions  $z$ , but at the same coordinate  $\vec{x}$  parallel to the boundary. These are dual to two coincident YM instantons with widely different scale sizes. Locality of the bulk objects is expected at large  $N$  when classical physics is valid. In this limit the collective coordinates of the boundary configuration should approximately decouple into two separate sets associated with instantons of two different scale sizes (i.e., the metric on the moduli space is block diagonal in this region). Such behavior typically occurs for instantons at large spatial distances. Here we learn that a large difference in scale size will also cause a separation of collective coordinates. Turning this around, the approximate noninteraction of collective coordinates of coincident instantons at widely different scales translates into approximate locality of the bulk physics.

We can also consider the interaction between instantons and anti-instantons. By evaluating the probe action of a D-instanton in the background of the anti-D-instanton, we find a bulk interaction of the form

$$\delta S \sim \frac{(z_1 z_2)^4}{(z_1 - z_2)^8}, \quad (32)$$

where the D-instantons are at coincident  $\vec{x}$  positions and  $z_{1,2}$  are their radial positions. In deriving Eq. (32) we have used the asymptotic form of the dilaton, which follows from the required falloff of the dilaton and SO(1,5) invariance (it is also consistent with [6–8]). From Eq. (32) it follows that in the conformal large- $N$  Yang-Mills theory we expect the interaction between coincident instantons and anti-instantons to fall off as the eighth power of difference in scale size. Such behavior is not evident in perturbative gauge theory, and presumably arises from the sum of planar diagrams in the large  $N$  limit.

#### IV. STRING PROBES

Fundamental strings and D-strings are particularly interesting probes of Lorentzian AdS<sub>5</sub>. Working in Poincaré coordinates (1), we will find the Yang-Mills description of string solitons stretched parallel to the AdS boundary and placed at fixed radial positions  $\tilde{z}$ . Once again, the character-

istic position in the bulk will be mapped to the characteristic scale on the boundary, and motion towards the Poincaré horizon appears as a fattening of the boundary flux tube. A string placed at a fixed radial position is not a solution to the equations of motion since it can reduce its potential energy by falling towards the horizon. Nevertheless, since the analysis of a static string is technically clearer, we imagine that it is stabilized by an external force. We will find that the corresponding boundary flux could reduce its energy by spreading and must be similarly stabilized.<sup>6</sup> This analysis is readily generalized to slowly moving strings.

*Strings as fluxes.* First, we establish that fundamental (F) and Dirichlet (D) strings in AdS<sub>5</sub> are described by electric and magnetic fluxes in the boundary gauge theory. This is easily shown by starting with the world volume action for D3-branes (with Higgs fields suppressed):

$$S_{D3} = -T_3 \text{Tr} \int d^4 \sigma \left\{ e^{-\phi} \sqrt{-\det(G_{mn} + 2\pi\alpha' F_{mn} + B_{mn})} - \frac{1}{4} \epsilon^{mnpq} C_{mn}^{(2)} F_{pq} \right\}. \quad (33)$$

(We have written the non-Abelian Born-Infeld action appropriate to commuting background fields [19].) The action includes terms of the form:

$$\int d^4 \sigma B^{mn} \text{Tr} F_{mn} \quad \text{and} \quad \int d^4 \sigma \epsilon^{mnpq} C_{mn}^{(2)} \text{Tr} F_{pq}, \quad (34)$$

where  $B_{mn}$  and  $C_{mn}^{(2)}$  are the NS-NS and Ramond-Ramond (RR) two-forms, respectively. An F-string extended in the  $x^i$  direction should couple to  $B_{ti}$ , and so is described by a nonvanishing value of  $E_i = \text{Tr} F_{ti}$ , in other words an electric flux.<sup>7</sup> Similarly, the D-string corresponds to a magnetic flux  $B_i = \epsilon_{ijk} \text{Tr} F_{jk}$ . This is consistent with S-duality, which interchanges F- and D-strings in the bulk, and electric and magnetic fields in Yang-Mills theory.

<sup>6</sup>Related discussions appear in [14].

<sup>7</sup>The source for the U(1) electric flux in the Yang-Mills theory is not associated with the dynamical part of  $B_{mn}$ , which couples to a dimension-6 operator in CFT [16,17], but rather with pure gauge degrees of freedom which contribute to surface integrals for conserved charges at infinity. It may seem surprising that Eq. (34) shows that bulk strings are related to the U(1) part of the U( $N$ ) boundary theory since this factor has been argued to decouple [20,3,21]. However, it is more accurate to consider the U(1) as ‘‘frozen’’ after inclusion of all external probes and vacuum expectation values (VEVs). The remaining SU( $N$ ) part of the theory is dynamical and the rest of this section studies the expectation of  $\text{Tr}(F^2)$  with a trace in SU( $N$ ). We are grateful to O. Aharony for correspondence regarding this issue.

*Bulk strings:* To describe bulk F- and D-strings we start with their world sheet actions, which include the terms

$$S_F = -\frac{1}{2\pi\alpha'} \int d^2\sigma \left\{ \sqrt{-\det g_{\mu\nu} \partial_\mu X^\mu \partial_\nu X^\nu} - \frac{1}{2} \epsilon^{mn} B_{\mu\nu} \partial_m X^\mu \partial_n X^\nu \right\}, \quad (35)$$

$$S_D = -\frac{1}{2\pi\alpha' g_s} \int d^2\sigma \left\{ e^{-\phi} \sqrt{-\det g_{\mu\nu} \partial_\mu X^\mu \partial_\nu X^\nu} - \frac{1}{2} \epsilon^{mn} C_{\mu\nu}^{(2)} \partial_m X^\mu \partial_n X^\nu \right\}. \quad (36)$$

We have included couplings due to the string frame metric  $g_{\mu\nu}$ , the dilaton  $\phi$ , and the two-forms  $B_{\mu\nu}$ ,  $C_{\mu\nu}^{(2)}$ . Now consider static strings extended in the  $x^1$  direction, parallel to the boundary. In a static gauge,

$$\begin{aligned} t(\sigma^m) &= \sigma^0, & z(\sigma^m) &= \tilde{z} = \text{const}, \\ x^1(\sigma^m) &= \sigma^1, & \vec{x}_\perp(\sigma^m) &= \vec{x}_{a\perp}. \end{aligned} \quad (37)$$

Here  $\vec{x}_{a\perp}$  are directions orthogonal to the string but parallel to the boundary. Evaluating  $S_F$ ,  $S_D$  in the AdS<sub>5</sub> background (1) gives the potential energy of the static strings (per unit coordinate length):

$$V_F = \frac{g_{YM}}{2\pi} N^{1/2} \frac{1}{\tilde{z}^2}, \quad V_D = \frac{2}{g_{YM}} N^{1/2} \frac{1}{\tilde{z}^2}. \quad (38)$$

Here we used the relations  $R^4 = 4\pi g_s N \alpha'^2$  and  $4\pi g_s = g_{YM}^2$  that are appropriate to AdS<sub>5</sub>. The F- and D-string potentials are related by S-duality:  $g_{YM}^2/4\pi \rightarrow (g_{YM}^2/4\pi)^{-1}$ .

*Boundary expectations.* A bulk string is dual to a boundary CFT state in which various operators have expectation values. For example, the actions (35),(36) will induce long-range fields for  $B_{tx^1}$ ,  $C_{tx^1}^{(2)}$  in the presence of F- and D-strings, respectively. The results of Sec. II and the couplings (34) then yield nonvanishing values for  $\langle \text{Tr} F_{tx^1} \rangle_F$  and  $\langle \text{Tr} F_{x^2 x^3} \rangle_D$ , corresponding to electric and magnetic fluxes. Rather than evaluating these explicitly, we focus on the expectation value for  $\text{Tr} F^2$ , which couples to the dilaton  $\phi$ . Note that this trace is in the  $SU(N)$  part of the gauge group. First we work out the long-range dilaton field produced by string sources via their linear coupling to the dilaton. Although it may appear from Eq. (35) that the F-string does not couple to the dilaton, this is simply because the actions are written in the string frame. Working instead in the Einstein frame, with  $g_{\mu\nu}^E = (g_s e^\phi)^{1/2} g_{\mu\nu}$ , we find the couplings

$$S_F = -\frac{g_{YM}}{4\pi} N^{1/2} \int d^2\sigma \frac{\phi}{\tilde{z}^2}, \quad S_D = \frac{1}{g_{YM}} N^{1/2} \int d^2\sigma \frac{\phi}{\tilde{z}^2}. \quad (39)$$

In order to obtain the long-range dilaton field we will need the asymptotic form of a Green's function, satisfying Dirichlet boundary conditions, for the equation

$$\frac{1}{2\kappa_5^2} \nabla^2 G_D(\vec{x}_\perp, z) = \frac{z^5}{R^5} \delta(z - \tilde{z}) \delta(\vec{x}_\perp - \vec{x}_{a\perp}), \quad (40)$$

where the right hand side contains a delta function transverse to the direction in which the string extends. This is readily solved to give

$$G_D(\vec{x}, z) = -\frac{4\pi}{N^2} \frac{z^4 \tilde{z}^4}{[\tilde{z}^2 + |\vec{x}_\perp - \vec{x}_{a\perp}|^2]^3}. \quad (41)$$

[This can also be obtained from our earlier results for the D-instanton by integrating Eq. (21) with respect to the positions  $t_a$  and  $x_a^1$  and multiplying by the appropriate normalization.] Now it follows from Eqs. (39) and (41) that

$$\phi_F = \frac{g_{YM}}{N^{3/2}} \frac{z^4 \tilde{z}^2}{[\tilde{z}^2 + |\vec{x}_\perp - \vec{x}_{a\perp}|^2]^3} \quad (42)$$

and

$$\phi_D = -\frac{4\pi}{g_{YM} N^{3/2}} \frac{z^4 \tilde{z}^2}{[\tilde{z}^2 + |\vec{x}_\perp - \vec{x}_{a\perp}|^2]^3}. \quad (43)$$

These asymptotic fields yield the expectation values

$$\langle \text{Tr} F^2 \rangle_F = \frac{2g_{YM}^3 N^{1/2}}{\pi^2} \frac{\tilde{z}^2}{[\tilde{z}^2 + |\vec{x}_\perp - \vec{x}_{a\perp}|^2]^3}, \quad (44)$$

$$\langle \text{Tr} F^2 \rangle_D = -\frac{8g_{YM} N^{1/2}}{\pi} \frac{\tilde{z}^2}{[\tilde{z}^2 + |\vec{x}_\perp - \vec{x}_{a\perp}|^2]^3}. \quad (45)$$

We see that the boundary configuration corresponding to a static string in the bulk is spread over a region with scale size  $\tilde{z}$ . This analysis can be generalized to a slowly moving string by using retarded Green's functions instead of Eq. (41);  $\tilde{z}$  is then replaced by its value at retarded time.

*Scale-radius duality and bulk locality.* We have learned that a bulk string placed at  $\tilde{z}$  is dual to a flux tube spread over a region with characteristic scale  $\tilde{z}$ —another example of scale-radius duality. In the bulk a string will fall towards the horizon (large  $\tilde{z}$ ) to minimize gravitational potential energy. Correspondingly, the gauge field strength in the boundary theory will spread out to minimize gradient energy, asymptotically going to zero. The AdS-CFT correspondence implies that the equation governing the spreading in the boundary is the geodesic equation for strings in the bulk. At present it is difficult to analyze this directly from the boundary perspective, but we gain some insights from [22],<sup>8</sup> where a system of  $p+2$  and  $p$  branes in flat space is studied. The authors found that a D-string can be included in a D3-brane  $SU(N)$  gauge theory as a  $Z_N$  flux after compactifying two directions transverse to the D-string. Their analysis showed

<sup>8</sup>We thank A. Sen for bringing this reference to our notice.

that the minimum energy configuration with fixed  $Z_N$  flux is pure gauge, with vanishing field strength. Taking a large compactification radius, this agrees with the picture in Eqs. (44),(45) where  $\langle \text{Tr}(F^2) \rangle$  vanishes as  $\tilde{z} \rightarrow \infty$ . For a classical, static field configuration which is purely electric or magnetic,  $(1/g_{YM}^2)\text{Tr}F^2$  is proportional to the energy density.<sup>9</sup> One expects that the total energy for a field configuration of fixed flux and size  $\tilde{z}$  goes like  $1/\tilde{z}^2$ , in agreement with the bulk potentials (38).

Two strings at large radial separations do not interact very much. This feature can be seen by examining the collective coordinates of the bulk solitons—each string has an approximately independent set. On the boundary the corresponding statement is that flux tubes of very different scale sizes have approximately independent collective fluctuations, even when they have coincident centers.<sup>10</sup> The interactions of bulk strings are also causal in the classical limit. For instance, a fluctuation on one string will only affect the other after a time lag. This translates into a typical interval required for the spread of boundary fluctuations from one scale to another. As in the case of D-instantons, the separation of boundary collective fluctuations and the time lag for interactions between scales are only expected to emerge in an approximate sense. At a more fundamental level, the exact dynamics dictated by the CFT description will imply deviations from bulk locality and causality.

## V. DILATON WAVE PACKET

Finally we study massless dilaton wave packets in the bulk of  $\text{AdS}_5$ . In previous sections we studied pointlike sources and found that the bulk position translated into a boundary scale. The situation is more complicated for wave packets because the bulk object already has a characteristic scale which will also get reflected on the boundary. The correct approach is to study a family of objects related in the bulk by the AdS isometry (18). For the D-instanton and string probes, this isometry simply translates the bulk objects. The component of the translation parallel to the boundary becomes a translation in CFT, while the radial translation becomes a *spatial* rescaling. We will see that the isometry (18) both translates and changes the size of dilaton wave packets. This is reflected in the boundary theory in the *spatiotemporal* width of  $\langle \text{Tr}(F^2) \rangle$ .

<sup>9</sup>Since we do not expect classical Yang-Mills theory to be accurate in this context, the following discussion is meant only to indicate the qualitative behavior of the field configuration.

<sup>10</sup>Of course, an object like a flux tube, not being a stable soliton, does not have collective coordinates in the usual sense. By ‘‘collective coordinates’’ we mean fluctuations of the boundary fields that preserve the overall scale and shape of the tube. For example, we can imagine endowing the tube with a ripple in its shape or a transverse velocity. Such motions would occur on time scales distinct from the rate of spreading of the tube.

*Bulk wavepackets.* A normalizable dilaton wave packet can be constructed by superimposing mode solutions [11–13,4]:

$$\phi_\lambda(z, \vec{x}, t) = \int d^3k d\rho C_\lambda(\vec{k}, \rho) (\rho z)^2 J_2(\rho z) e^{i(\vec{k} \cdot \vec{x} - \omega t)}. \quad (46)$$

Here  $J_2$  is a Bessel function and  $\omega^2 - \rho^2 - \vec{k}^2 = 0$ . The profile  $C_\lambda$  is a Gaussian centered at  $(\lambda \vec{k}_0, \lambda \rho_0)$  with a width  $\sigma \lambda^2$ :

$$C_\lambda(\vec{k}, \rho) = \frac{1}{\lambda^4} a \exp \left[ -\frac{(\vec{k} - \lambda \vec{k}_0)^2 + (\rho - \lambda \rho_0)^2}{2\lambda^2 \sigma} - i \frac{\rho \alpha}{\lambda} \right]. \quad (47)$$

With this definition, wave packets with different values of  $\lambda$  are related by the radial isometry (18):

$$\phi_\lambda(z, \vec{x}, t) = \phi_{(\lambda=1)}(\lambda z, \lambda \vec{x}, \lambda t). \quad (48)$$

So we expect to see a manifestation of the scale-radius duality in the dual boundary dynamics.

For sufficiently early or late times<sup>11</sup> we can use the stationary phase approximation and the asymptotic form of the Bessel function for large arguments to study Eq. (46). We find, self-consistently, that at large  $|t|$ , the packet  $\phi_\lambda$  is centered at

$$\vec{x} = \frac{\vec{k}_0}{\omega_0} t, \quad z = \frac{\rho_0}{\omega_0} |t| + \frac{\alpha}{\lambda}. \quad (49)$$

As  $\lambda$  increases, the center of the wave packet moves radially away from the horizon. At large times the detailed form of the wave packet is still somewhat complicated but the key features can be understood by setting  $\alpha = 0$ . This gives

$$\begin{aligned} \phi_\lambda(z, x_i, t) &= a \sqrt{\sigma} (\rho_0 z)^{3/2} \left( \frac{\omega_0}{t} \right)^{3/2} \\ &\times \exp \left( -\frac{\sigma \lambda^2}{2} (\vec{x} \cdot \hat{\zeta}_0 - t)^2 \right) \\ &\times \exp \left( -\frac{1}{2} \frac{x_T^2 \omega_0^2}{t^2 \sigma} \right) \exp \left( \frac{i}{2} \frac{x_T^2 \lambda \omega_0}{t} \right) \\ &\times \exp[i(\lambda \vec{\zeta}_0 \cdot \vec{x} - \lambda \omega_0 t)]. \end{aligned} \quad (50)$$

Here  $\vec{x}$  is the four-vector  $(z, x_i)$ ,  $\vec{\zeta}_0$  is the four-vector,  $(\rho_0, \vec{k}_0)$ , and  $x_T$  stands for the spatial distance in four dimensions transverse to the wave packet’s momentum along  $\vec{\zeta}_0$ . So as  $t \rightarrow -\infty$  the wave packet is a shock wave that emerges from the horizon travelling along  $\hat{\zeta}$ . Initially its energy is large and it travels like a massless particle along the light cone. But with time the pull of gravity gets stronger and begins to reflect the wave packet back. As this happens the shock wave contracts into a localized lump in the direction perpendicular to its motion. At this stage Eq. (50) is no longer valid. Eventually the state turns around completely, gathering itself into a shock wave again, this time hurtling

<sup>11</sup>We need  $|\lambda t| \gg \omega_0 / \rho_0^2$  and  $|\lambda t| \gg \omega_0 / \sigma$ .



towards the horizon in the far future and spreading out transverse to its direction of motion. This spreading transverse to the direction of motion will be familiar to reader as the standard behavior of relativistic wave packets in flat space.

In Eq. (50) we have set  $\alpha=0$ ; reinstating it does not change the qualitative features of the wave packet. In particular the widths  $\sigma\lambda^2$  and  $\omega_0^2/t^2\sigma$ , which govern the spreading parallel and perpendicular to the direction of motion stay the same. Thus, shock waves closer to the horizon [smaller  $\lambda$  from Eq. (49)] are also more spread out along their direction of motion.

*Boundary description.* Using the results of Sec. II we relate the asymptotic behavior of the dilaton wave packet to the expectation value of  $\langle \text{Tr}F^2 \rangle$ :

$$F^2(\vec{x}) \equiv \frac{1}{4g_{YM}^2} \langle \lambda | \text{Tr}[F^2(\vec{x})] | \lambda \rangle$$

$$= \int d^3k d\rho 4C_\lambda(\vec{k}, \rho) \rho^4 e^{i(\vec{k} \cdot \vec{x} - \omega t)}. \quad (51)$$

Here the states  $|\lambda\rangle$  are related to each other by a scale transformation dual to the isometry (18). Two limiting cases are instructive:  $\rho_0 \gg |\vec{k}_0|$  and  $\rho_0 \ll |\vec{k}_0|$ .

When  $\rho_0 \gg |\vec{k}_0|$  the wave packet starts at early times with most of its momentum in the radial direction. Then Eq. (51) gives

$$|\lambda t| \gg \frac{\omega_0}{\sigma}: \quad F^2(\vec{x}) \propto \left(\frac{\omega_0}{t}\right)^{3/2} \exp\left[-\frac{\sigma\lambda^2}{2}\left(t + \frac{\alpha}{\lambda}\right)^2\right] \exp\left(-\frac{1}{2}\frac{x^2\omega_0^2}{t^2\sigma}\right) \exp\left(\frac{i}{2}\frac{x^2\omega_0\lambda}{t}\right) \exp[-i\omega_0(\lambda t + \alpha)], \quad (52)$$

$$|\lambda t| \ll \frac{\omega_0}{\sigma}: \quad F^2(\vec{x}) \propto \exp\left[-\frac{\sigma\lambda^2}{2}\left(t + \frac{\alpha}{\lambda}\right)^2\right] \exp\left(-\frac{1}{2}(x^2\sigma\lambda^2)\right) \exp[-i\omega_0(\lambda t + \alpha)]. \quad (53)$$

At early and late times ( $|\lambda t| \gg \omega_0/\sigma$ ) the bulk state looks like a shock wave moving in the radial direction and at small intermediate times ( $|\lambda t| \ll \omega_0/\sigma$ ) the bulk state is being reflected by the AdS geometry and turned around.

When  $\rho_0 \ll |\vec{k}_0|$  the wave packet starts at early times with most of its momentum parallel to the boundary. Then we find

$$|\lambda t| \gg \frac{\omega_0}{\sigma}: \quad F^2(\vec{x}) \propto \left(\frac{\omega_0}{t}\right)^{3/2} \exp\left(-\frac{\sigma\lambda^2}{2}(\vec{x} \cdot \hat{k}_0 - t)^2\right) \exp\left(-\frac{1}{2}\frac{x_T^2\omega_0^2}{t^2\sigma}\right) \exp\left(\frac{i}{2}\frac{x_T^2\omega_0\lambda}{t}\right) \exp[i\lambda(\vec{k}_0 \cdot \vec{x} - \omega_0 t)], \quad (54)$$

$$|\lambda t| \ll \frac{\omega_0}{\sigma}: \quad F^2(\vec{x}) \propto \exp\left(-\frac{\sigma\lambda^2}{2}(\vec{x} \cdot \hat{k}_0 - t)^2\right) \exp\left(-\frac{1}{2}(x_T^2\lambda^2\sigma)\right) \exp[i\lambda(\vec{k}_0 \cdot \vec{x} - \omega_0 t)]. \quad (55)$$

Here  $x_T^2 = x^2 - (\vec{x} \cdot \hat{k}_0)^2 + (\alpha/\lambda)^2$ . At early and late times ( $|\lambda t| \gg \omega_0/\sigma$ ) the bulk state looks like a shock wave moving parallel to the boundary.

*Scale-radius duality.* We have just derived the boundary description of a class of wave packets  $\phi_\lambda$  that are related by the radial isometry (18). We learned from Eq. (49) that the characteristic spatial center of the packet depends linearly on  $1/\lambda$ . Packets that are closer to the horizon also have bigger bulk widths. The basic lesson we learn from Eqs. (52)–(55) is that these characteristic bulk features map on the boundary to a characteristic *spatiotemporal* scale. Packets which are characteristically closer to the horizon (and spatially wider) map to boundary fields with a greater spread in space and time.

To see this, first consider the case  $\rho_0 \gg |\vec{k}_0|$ . The bulk packet starts as a radial shock wave coming from the past horizon, reflects at intermediate times, and returns as a shock wave to the future horizon. From Eq. (52) the boundary  $F^2$  starts with a very small amplitude and a huge spread in the spatial directions. At intermediate times Eq. (53) tells us that

$F^2$  is a Gaussian in space and at late times the amplitude decreases again while  $F^2$  spreads in space. The temporal profile, like the intermediate-time spatial profile, is Gaussian with a scale set by  $\lambda$ . Since  $\lambda$  also indexes the radial isometries relating bulk packets, we are once again seeing a scale-radius duality.

As another example, consider the case  $\rho_0 \ll |\vec{k}_0|$ . Now the bulk packet starts as a shock wave parallel to the boundary, is reflected at intermediate times, and returns as a shock wave to the future horizon. From Eqs. (54) and (55) we learn that the boundary  $F^2$  is a shock wave spreading out spatially in the directions perpendicular to the motion at early and late times. Other than this, the behavior is exactly parallel to the case  $\rho_0 \gg |\vec{k}_0|$ . The profile in time, like the intermediate time spatial profile, is a Gaussian with a scale set by  $\lambda$ .

The analysis of this section is only a first step in a more complete study. For example, it would be interesting to understand how the bulk scattering of two shock waves in mirrored in the boundary theory. Understanding this would help uncover how bulk locality emerges from the boundary description.

## VI. DISCUSSION AND CONCLUSIONS

### A. Bulk versus boundary dynamics

*The classical limit.* Much of this article has addressed the CFT description of classical bulk probes. These states are ‘‘classical’’ because they contain a very large number of particles, and propagate in backgrounds with small curvature. There is a subtlety in defining such a limit in view of the ‘‘stringy exclusion principle’’ advocated in [24] for AdS<sub>3</sub>. Given the AdS-CFT correspondence, this ‘‘exclusion principle’’ imposes a bound on the occupancy of certain bulk states. Nevertheless, as  $N$  increases, the maximum occupancy increases also. So in the large- $N$  limit our considerations are valid.

*Hairy holography.* We have provided an explicit prescription for relating states in the bulk and boundary theories. Roughly speaking, the prescription works because normalizable modes in the bulk extend to the boundary and serve as a kind of ‘‘hair,’’ determining the boundary state. Each mode falls off exponentially (in physical distance), but the volume of the boundary grows exponentially as well, allowing for a significant effect. We have found that the asymptotic value of the ‘‘hair’’ appears directly in the expectation values of operators in the boundary state, giving a precise realization of the holographic proposal of ’t Hooft and Susskind [23]. Our analysis has been mainly at a linearized level and the back reaction on the metric was neglected. We hope to extend our work in the future to situations like the formation of a black hole where the nonlinearities of gravity are more important.

*S-matrix, bulk commutators, and local physics.* Using the operator relation between bulk and boundary fields, we can reiterate some points made in [5,4]. It is apparent that transition amplitudes between physical states in the bulk can be computed from the boundary theory. Specifically, prepare a bulk state  $|\Psi\rangle = a_{k_1}^\dagger \cdots a_{k_n}^\dagger |0\rangle$  and evolve it forward in time as  $|\Psi\rangle \rightarrow e^{-iHt} |\Psi\rangle$ . Precisely the same operation can be performed in the boundary theory: the expansion (17) and the equivalence  $a_k = b_k$  allow one to prepare the initial state, and time evolution with respect to the CFT Hamiltonian is identified with time evolution in the bulk. Thus transition amplitudes in the boundary theory can be reinterpreted as amplitudes in the bulk.

An unusual feature of this map is that the radial coordinate in the bulk does not appear in the boundary mode expansion (17). This makes it difficult to check aspects of bulk locality such as the commutation of operators at spacelike separation. However, as we have explicitly shown, the boundary theory has access to data on the radial position of localized bulk probes in the characteristic scale of their boundary images. As we have discussed, classical locality of the bulk physics is mapped, at large  $N$ , into the approximate independence of collective fluctuations of objects at different boundary scales. So the breakdown of locality in quantum gravity should be understood in terms of incomplete decoupling of scales in the boundary theory at finite  $N$ . We are investigating this issue and hope to report on it elsewhere.

### B. Holographic description of horizons

We have accumulated enough tools to suggest how space-time causal structure will be reflected in the boundary theory. Here we present a qualitative discussion—further details will appear elsewhere.

*Black holes and thermal states.* AdS-Schwarzschild black holes [25] and the Banados-Teitelboim-Zanelli (BTZ) black hole [26] (see [27] for a review) both have maximally extended solutions with two asymptotic regions, each with a timelike boundary at spatial infinity.<sup>12</sup> In such spacetimes, the bulk Hilbert space is a product of two identical copies, each accessed by a single asymptotic region [29]. For example, thermal states such as the Hartle-Hawking vacuum are written as correlated tensor products of states:

$$|\text{HH}\rangle = \sum_n e^{-\beta\omega_n} |n, \omega_n\rangle_I \otimes |n, \omega_n\rangle_{II}, \quad (56)$$

where Hilbert spaces  $I$  and  $II$  are formally identical. Tracing over one copy in the product produces the thermal ensemble accessible to the other asymptotic observer. In fact, this construction is a standard method of describing real-time, finite-temperature field theories; one studies operators that access one Hilbert space that is correlated appropriately with another. The auxiliary Hilbert space then functions as an external bath which thermalizes the system. The situation is entirely parallel from the boundary perspective. The boundary of spacetime has two disconnected components and so the CFT Hilbert space factorizes into a product of two identical pieces. The choice of bulk vacuum is reflect in the CFT vacuum as discussed in Sec. II C and Appendix A. Tracing over one boundary component leaves a thermal state accessible to one asymptotic region.

*Scale-radius duality.* We would like to use the scale-radius duality discussed in this paper to study motions towards the horizon from the boundary perspective. For pure AdS spacetime, scale-radius duality originates in the dual action of the bulk radial isometry and the boundary scale transformations. In fact, these are not symmetries of the black hole. The curvature of Schwarzschild and the discrete identifications of BTZ break the isometry group, and the corresponding thermal boundary state breaks conformal invariance. Nevertheless, there are arguments that the bulk-boundary duality continues to hold since the spacetime is *asymptotically* AdS [1,3,30,31]. What is more, BTZ black holes *locally* enjoy the same symmetries as AdS spacetime and so motions of local probes continue to map to motions in boundary scale. (These points can be made quantitatively using the methods of this paper, as we hope to discuss elsewhere.)

*Horizons from the boundary perspective.* Armed with the scale-radius duality, we introduce a bulk probe that starts near the boundary and falls towards the horizon. From the boundary perspective, the position of the horizon is repre-

<sup>12</sup>The disconnected topology of the boundary in the BTZ case can be seen directly from its orbifold construction [28].

sented by the thermal scale. The boundary probe starts life at a very high scale much above the thermal bath. This separation of scales allows it to spread unimpeded as though the bath was absent. As the bulk object falls through the horizon, its boundary dual reaches the thermal scale. Falling through the horizon in the bulk is reflected on the boundary as thermalization due to interaction with the thermal bath. Note that this does not mean that the probe state has ‘‘mixed’’ with the density matrix describing the black hole. Rather, interactions with the thermal bath make the probe state look like a typical state in the ensemble. As the bulk object penetrates to the singularity increasing the black hole mass, thermalization of its boundary dual raises the boundary temperature. The horizon as a causal construct preventing extraction of information is only ‘‘real’’ to the degree that thermalization obscures the history of a state.

*Black holes from collapse.* We can also consider black holes formed from collapsing shells of matter. Again, from the boundary perspective, the state will spread out until it reaches the scale characteristic of the temperature of the black hole it has created. The degree to which the resulting horizon is sharp will be the degree to which the final state is difficult to measure due to the complicated way that the information about the configuration is spread out over modes at low spatiotemporal scales. The causal structure of the black hole appears as a statistical phenomenon and is precise in the thermodynamic limit.

### C. Conclusions

In summary, we have developed techniques for describing bulk probes from the boundary perspective. Using our methods, properties of solutions to the bulk equations of motion can be translated into properties of states and expectation values in the boundary theory. We have argued that there is a map between the quantized mode expansions of bulk fields and boundary operators and used our approach to demonstrate a scale-radius duality for several probes. Finally, we outlined the application of our methods to the study of black hole causal structure. This work constitutes a preliminary attempt to address the emergence and eventual breakdown of local spacetime from the gauge theory perspective towards quantum gravity.

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## APPENDIX A: CORRELATORS AND PROPAGATOR AMBIGUITIES

In this appendix we will give a more careful discussion of the methods illustrated in Sec. II. In particular we will discuss the meaning of the ambiguities in defining the bulk-boundary propagator for Lorentzian spacetimes. We will continue to work in Poincaré coordinates (1).

### 1. Euclidean signature

As in Sec. II we study the bulk-boundary correspondence (2) for a classical massive scalar with action (4) that couples to an operator with dimension  $2h_+$  Eq. (6). In Sec. II we worked in position space and followed the procedure of [3] relating the boundary contribution to the bulk action to CFT correlators. In fact, as discussed in [9] this procedure violates Ward identities and must be modified. The correct algorithm is to evaluate a suitably normalized bulk action at  $z = \epsilon$  prior to taking  $\epsilon \rightarrow 0$ .

*Improved procedure in momentum space.* The improved procedure of [9] is easiest to implement in momentum space. The unique solution to the wave equation  $(\square - m^2)\phi = 0$  with momentum  $\vec{k}$  parallel to the boundary is the Bessel function [2,9,4]

$$\phi \propto e^{i\vec{k}\cdot\vec{x}} z^{d/2} K_\nu(|k|z) \phi_0(\vec{k}). \quad (\text{A1})$$

At the boundary  $z \rightarrow 0$ ,  $\phi \rightarrow A(z) + B(z)$  where  $A = z^{2h_-}(1 + \dots)$  and  $B = z^{2h_+}(1 + \dots)$  and the ellipses indicate series in even powers of  $z$ .<sup>13</sup> For  $m^2 > 0$  this is divergent as  $z \rightarrow 0$  and requires regulation. According to [9] we work at  $z = \epsilon$  with the normalization  $\phi = C(\epsilon) \phi_0(\vec{k}) e^{i\vec{k}\cdot\vec{x}}$ . The authors of [9] set  $C(\epsilon) = 1$ , but since the scaling of  $\phi$  as it approaches the boundary is important, we choose  $C = \epsilon^{2h_-}$ :

$$\phi(z, \vec{k}) = \epsilon^{2h_-} \frac{z^{d/2} K_\nu(|k|z)}{\epsilon^{d/2} K_\nu(|k|\epsilon)} \phi_0(\vec{k}) e^{i\vec{k}\cdot\vec{x}}. \quad (\text{A2})$$

The bulk action (4) then reduces to a boundary term [2,9]:

$$S = \lim_{z \rightarrow \epsilon} z^{1-d} \delta(\vec{k} + \vec{k}') \phi_0(\vec{k}') \epsilon^{2h_-} \frac{(|k'|z)^{d/2} K_\nu(|k'|z)}{(|k'|\epsilon)^{d/2} K_\nu(|k'|\epsilon)} \\ \times \partial_z \epsilon^{2h_-} \frac{(|k|z)^{d/2} K_\nu(|k|z)}{(|k|\epsilon)^{d/2} K_\nu(|k|\epsilon)}. \quad (\text{A3})$$

As we discussed,  $z^{d/2} K_\nu(z) \rightarrow A(z) + B(z) = z^{2h_-}(1 + \dots) + z^{2h_+}(1 + \dots)$ . Putting this in Eq. (A3) as  $\epsilon \rightarrow 0$  yields some singular contact terms that we drop and a finite term arising from the mixture of the  $A$  and  $B$  terms in the numerator and denominator. These give the expected behavior of the two-point function:

<sup>13</sup>For integral  $\nu$  there are some logarithm terms also, but they do not change the basic argument.

$$\langle \mathcal{O}(\vec{k}') \mathcal{O}(\vec{k}) \rangle \propto \delta(\vec{k}' + \vec{k}) |k|^{2\nu}. \quad (\text{A4})$$

The position space procedure used in Sec. II gives different normalizations because the bulk-boundary propagator used there approaches a delta function at the  $z=0$  boundary rather than at  $z=\epsilon$ .

*Operator expectation values* From Sec. II we know that turning on source in the boundary theory will lead to non-trivial operator expectation values. These are given by the first variation of  $S(\phi)$ :

$$\delta S(\phi) = \int_{z \rightarrow 0} d\Sigma^\mu \partial_\mu \phi \delta \phi, \quad (\text{A5})$$

where, as in Sec. II, we have added a small perturbation of the form  $\delta\phi = z^{2h} \delta\phi_0$ . In momentum space, with the cutoff procedure prescribed above,  $\delta\phi(z=\epsilon, \vec{k}) = \epsilon^{2h} \delta\phi_0(\vec{k})$ . Combining this with  $\partial_z \phi$  at  $z=\epsilon$ , using Eq. (A5), and dropping contact terms as before, we get a finite one-point function

$$\langle \mathcal{O}(\vec{k}) \rangle_{\phi_0(\vec{k})} = \langle \mathcal{O}(\vec{k}) \mathcal{O}(-\vec{k}') \rangle_{\phi_0(\vec{k}')}, \quad (\text{A6})$$

where the two-point function was given in Eq. (A4) and is evaluated here in the absence of a source. So the interesting part of the one-point function—the part which does not come from coincidence of the source and the operator insertion—arises from the subleading part of the source term which scales as  $z^{2h+}$  at the boundary. The bulk of this paper rests on the independent specification of this subleading part near  $z=0$  via the addition of bulk probes and the consequent modifications of operator expectations on the boundary.

## 2. Lorentzian signature

We again begin by discussing free, massive scalar fields. In Lorentzian signature, normalizable solutions to the wave equation exist; so specifying the fields at  $z=0$  does not uniquely specify the field configuration in the bulk. As discussed in [4] the normalizable solutions form the bulk Hilbert space which is dual to the space of boundary states. There is also a spectrum of non-normalizable modes that implement boundary conditions and are dual to boundary sources.

In the supergravity effective action, the normalizable solutions can appear in two places. First, classical field theory in the bulk involves expanding the bulk action around classical, normalizable backgrounds. We will see that this corresponds to turning on expectation values for CFT operators. Second, the bulk-boundary propagator is not uniquely specified by the asymptotic behavior  $z^{2h} \delta^d(\vec{x} - \vec{x}')$  as  $z \rightarrow 0$ , since a normalizable solution vanishing at the boundary can always be added to it. This ambiguity is related to the choice of vacuum for the theory and will affect the correlation functions.

Working in momentum space and in Poincaré coordinates (1), we write  $\vec{k} = (\omega, \vec{q})$  for the momentum parallel to boundary ( $\vec{y} = (t, \vec{x})$ ). For  $k^2 > 0$  (spacelike momenta) the solutions

are identical to the Euclidean case and there is no normalizable solutions. For  $k^2 < 0$ , there are two solutions which are smooth in the interior [4]. One solution is

$$\phi^-(\vec{k}, z) \propto z^{d/2} J_{-\nu}(|k|z) e^{i\vec{k} \cdot \vec{y}} \quad (\text{A7})$$

when  $\nu$  is nonintegral.<sup>14</sup> This solution is not normalizable (for  $\nu > 1$ —see [4] for a discussion of the case  $\nu < 1$ ); it behaves as  $z^{2h-}$  when  $z \rightarrow 0$ . So the mode is nonfluctuating and is dual to a source term at the boundary. An independent solution is

$$\phi^+(\vec{k}, z) = z^{d/2} J_\nu(|k|z) e^{i\vec{k} \cdot \vec{y}}, \quad (\text{A8})$$

which is normalizable and behaves as  $z^{2h+}$  when  $z \rightarrow 0$ .

*Operator expectation values.* The most general solution  $\phi(\vec{k}, z)$  which is asymptotic to  $\epsilon^{2h-} \phi_0(\vec{k}) e^{i\vec{k} \cdot \vec{y}}$  when  $z = \epsilon$  is

$$\begin{aligned} \phi(\vec{k}, z) = & \Phi(\vec{k}) e^{i\vec{k} \cdot \vec{y}} \phi^+(\vec{k}, z) \\ & + \epsilon^{2h-} \frac{\phi^-(\vec{k}, z) + A(\vec{k}) \phi^+(\vec{k}, z)}{\phi^-(\vec{k}, \epsilon) + A(\vec{k}) \phi^+(\vec{k}, \epsilon)} e^{i\vec{k} \cdot \vec{y}} \phi_0(\vec{k}). \end{aligned} \quad (\text{A9})$$

Here  $\Phi(\vec{k})$  is the Fourier component of the classical field configuration we wish to study and

$$K_A(\vec{k}, \epsilon, z) = \epsilon^{2h-} \frac{\phi^-(\vec{k}, z) + A(\vec{k}) \phi^+(\vec{k}, z)}{\phi^-(\vec{k}, \epsilon) + A(\vec{k}) \phi^+(\vec{k}, \epsilon)} \quad (\text{A10})$$

is the bulk-boundary Green's function. As we can see,  $K_A$  has an ambiguity which is encoded by  $A(\vec{k})$ . Once again, we insert this into the (quadratic) action and integrate by parts.<sup>15</sup> The result is

$$\begin{aligned} S_A(\Phi, \phi_0) = & \frac{1}{2} \int_{z=\epsilon} d^d k d^d k' \delta^d(\vec{k} + \vec{k}') z^{1-d} \partial_z \\ & \times [\Phi(\vec{k}) \phi^+(\vec{k}, z) K_A(\vec{k}', \epsilon, z) \phi_0(\vec{k}') \\ & + \phi_0(\vec{k}) K_A(\vec{k}', \epsilon, z) \phi_0(\vec{k}')]. \end{aligned} \quad (\text{A11})$$

From this we can read off the one-point function

$$\begin{aligned} \langle \mathcal{O}(\vec{k}) \rangle_{\phi_0} = & \lim_{z \rightarrow \epsilon} \partial_z \{ [\Phi(-\vec{k}) \phi^+(-\vec{k}, z) \\ & + \phi_0(-\vec{k})] K(\vec{k}, \epsilon, z) \}. \end{aligned} \quad (\text{A12})$$

When  $\phi_0=0$ , it is easy to see that

$$\langle \mathcal{O}(\vec{k}) \rangle = \frac{d}{2} \Phi(-\vec{k}). \quad (\text{A13})$$

The two-point function is

<sup>14</sup>When  $\nu$  is integral,  $\phi^-(\vec{k}, z) \propto z^{d/2} Y_\nu(|k|z) e^{i\vec{k} \cdot \vec{y}}$ .

<sup>15</sup>In fact the integration by parts gives an oscillating surface term at  $z \rightarrow \infty$ . This should be understood in terms of an infrared cutoff in the gauge theory.

$$\begin{aligned} \langle \mathcal{O}(\vec{k}) \mathcal{O}(\vec{k}') \rangle_{\phi_0} &= \langle \mathcal{O}(\vec{k}) \rangle_{\phi_0} \langle \mathcal{O}(\vec{k}') \rangle_{\phi_0} \\ &+ \lim_{z \rightarrow \epsilon} \delta^d(\vec{k} + \vec{k}') \partial_z K(\vec{k}, \epsilon, z). \end{aligned} \quad (\text{A14})$$

In this case, the parts of  $K$  which behave as  $z^{2h_+}$  and  $z^{2h_-}$  are explicitly labeled by  $\phi^+$  and  $\phi^-$ . As in the Euclidean case, after dropping contact terms the finite piece as  $\epsilon \rightarrow 0$  comes combinations of  $\phi^-$  terms in the numerator and  $\phi^+$  terms in the denominator (and vice versa) of Eq. (A10). Thus the connected part of the two-point function will depend quite strongly on  $A(\vec{k})$ .

*Interpretation of propagator ambiguities.* The dependence of these correlators on  $\Phi$  is easy to understand. Turning on such a classical background implies that the bulk theory is in a ‘‘coherent’’ state in which many modes have an expectation value. The map between bulk and boundary states discussed in Sec. II implies that the CFT should also be in a ‘‘coherent’’ state built from the modes of the dual boundary operator. Not surprisingly, the CFT operator has an expectation value in this state. Roughly speaking, the expectation value is dual the part of  $\phi$  that behaves as  $z^{2h_+}$  at the boundary.

In addition to the freedom to specify a classical background-coherent state via  $\Phi$ , there is an ambiguity arising from the freedom to specify  $A(\vec{k})$ .<sup>16</sup> The origin of the ambiguity is the freedom to specify a vacuum state. In standard field theory, Wick rotation from a Euclidean signature specifies a certain propagator. But if we were simply to ask that the two-point function satisfy the appropriate inhomogeneous differential equation, we would have a much wider range of choices; we could pick some combination of the advanced, retarded, and Feynman propagators, and we could add propagators that solved the associated homogeneous differential equation. Our choice would depend on boundary conditions and the choice of vacuum. Here we choose  $A$  via analytic continuation from the Euclidean case; the combination of  $J_\nu(|\vec{k}|z)$  and  $J_{-\nu}(|\vec{k}|z)$  will be that equal to  $K_\nu(i|\vec{k}|z)$ :

$$K_\nu(i|\vec{k}|z) = \frac{\pi}{2} \frac{e^{i\pi\nu/2} J_{-\nu}(-|\vec{k}|z) - e^{-i\pi\nu/2} J_\nu(-|\vec{k}|z)}{\sin(\nu\pi)}. \quad (\text{A15})$$

In some spacetimes (e.g., those containing black holes) there are inequivalent vacua defined with respect to different times. Presumably this would be reflected in the the bulk-boundary propagator.

## APPENDIX B: THE BULK RESPONSE: ANOTHER METHOD

The relation between the dilaton field and the vev of  $\text{Tr}F^2$  was obtained in Sec. III by considering the response of the bulk and boundary theories to small perturbations. An alternate method for computing the bulk response in the presence of branes is to do things in the opposite order. Instead of

starting with the bulk soliton and evaluating the effect of a perturbation, we could consider a perturbation of AdS and ask about the change in the action on including the soliton. Here we study the D-instanton using the latter technique.

Consider a perturbation of the dilaton in AdS<sub>5</sub> which reduces to a delta function at  $\vec{x} = \vec{x}_a$  on the boundary. The bulk value of this perturbation is

$$\delta\phi = \frac{6}{\pi^2} \frac{z^4}{[z^2 + |\vec{x} - \vec{x}_a|^2]^4}. \quad (\text{B1})$$

The D-instanton is a source for the dilaton and axion and so the linearized effects of a D-instanton at  $z = \tilde{z}$  and  $\vec{x} = \vec{x}_1$  can be incorporated by adding a term in the action of the form

$$S_{D\text{-inst}} = 2\pi \int J(e^{-\phi} + i\chi), \quad (\text{B2})$$

where  $J = \delta(x_0 - \tilde{x}_0) \delta(\vec{x} - \vec{x}_a)$  stands for the D instanton source. The coefficient in front of the integral in Eq. (B2) is obtained by requiring that the D-instanton action be given by  $S = 2\pi/g_s$ . Writing  $e^{-\phi} = (1/g_s)(1 - \delta\phi)$  and substituting for  $\delta\phi$  from Eq. (B1) gives

$$S_{D\text{-inst}} = -\frac{48}{4\pi g_s} \frac{x_0^4}{[x_0^2 + |\vec{x} - \vec{x}_a|^2]^4}. \quad (\text{B3})$$

This is exactly equal to the bulk response in Eq. (27).

This method gives the same bulk response as in Sec. III because we are working in the linearized limit. An analogy with electrodynamics is useful. Introduce a small perturbation of the electrostatic potential surrounding a system of charges. The energy of the resulting system can be computed from the action for the electromagnetic field and the source coupling to the field. Then the bulk contribution vanishes by the equations of motion and we are left with a surface integral like Eq. (26). This method is followed in most of this paper. Alternatively, the energy is  $E = \sum_i q_i V_i$ , where  $q_i$  and  $V_i$  are the charge and the potential at the position of each charge, respectively. So the change in energy is determined by the perturbation of the potential at the location of each charge. This is exactly analogous to Eq. (B2).

The same reasoning works for a general linear system. The supergravity theory under consideration here is certainly not linear, but the D-branes act as small sources. Since they carry a charge of order  $\sim 1/g_s$ , the corresponding changes in the supergravity fields are of order  $\mathcal{O}(g_s^2 \times 1/g_s)$ . In the large- $N$  limit that we are working in, these changes are of order  $\mathcal{O}(1/N)$  and are therefore small.<sup>17</sup> Thus to leading order in  $1/N$  (and for purposes of calculating the one-point functions) one can work with a supergravity action expanded to quadratic order about the AdS background. The resulting system is in effect linear and the two methods for evaluating the bulk response must then agree.

<sup>16</sup>We are grateful to Tom Banks and Emil Martinec for a discussion of this issue.

<sup>17</sup>For example, from Eq. (21), Eq. (23), we see that the dilaton is changed from its asymptotic value by an amount  $\delta\sigma = (1/N) \times (24\pi/g_s^2 N) x_0^4 x_0^{-4} / [x_0^{-2} + |\vec{x} - \vec{x}_a|^2]^4$ . In the large- $N$  limit where  $g_s N$  is kept fixed this is of  $\mathcal{O}(1/N)$ .

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