Vacuum energy, variational methods, and the Casimir energy

Remo Garattini*
Mécanique et Gravitation, Université de Mons-Hainaut, Faculté des Sciences, 15 Avenue Maistriau, B-7000 Mons, Belgium and Facoltà di Ingegneria, Università degli Studi di Bergamo, Viale Marconi, 5, 24044 Dalmine (Bergamo) Italy
(Received 4 September 1998; published 26 April 1999)

Following the subtraction procedure for manifolds with boundaries, we calculate by variational methods the Schwarzschild and flat space energy difference. The one-loop approximation for TT tensors is considered here. An analogy between the computed energy difference in momentum space and the Casimir effect is illustrated. We find a singular behavior in the UV limit, due to the presence of the horizon when $r=2m$. When $r>2m$ this singular behavior disappears, which is in agreement with various other models previously presented.
[S0556-2821(99)02910-0]

PACS number(s): 04.62.+v, 04.20.Fy

I. INTRODUCTION

An interesting problem appearing in Einstein gravity is the computation of quantum corrections to a classical energy. A possible approach is the analysis of the thermodynamical quantities that characterize the system under consideration. This analysis can be carried out by computing the system free energy at a given volume and temperature by means of a partition function and the Euclidean action. Following the background method, we fix a metric and look at quantum fluctuations with respect to such a background with the appropriate boundary conditions; then we functionally integrate such metric fluctuations which are strictly periodic in Euclidean time t. In particular, the only feasible way to treat functional integration is by saddle-point methods. This is adequate for the treatment of the small perturbation concerning Minkowski space and for a semiclassical analysis of vacuum stability. However, a different point of view based on the Hamiltonian approach could be considered. In this framework, quantum corrections to classical energy can be computed by means of expectation values of the total Hamiltonian with respect to some states. It is clear that the problem is too large to be completely solved. To this end we might take into consideration the simplest nontrivial saddle point we can extract from vacuum Einstein equations, the Schwarzschild solution

$$ds^2=-\left(1-\frac{2MG}{r}\right)dt^2+\left(1-\frac{2MG}{r}\right)^{-1}dr^2+r^2d\Omega^2, \tag{1}$$

where $d\Omega^2=d\theta^2+\sin^2\theta d\phi^2$ is the line element of the unit sphere, G is Newton's constant, and M is a parameter representing the mass of the wormhole. This metric is asymptotically flat. The apparent singularity located at $r=2MG$ can be removed by a suitable definition of the coordinates, e.g., the Kruskal-Szekeres coordinates, which is written as

$$\left(\frac{r}{2MG}-1\right)\exp\left(\frac{r}{2MG}\right)=xy,$$

$$\exp\left(\frac{t}{2MG}\right)=\frac{x}{y}. \tag{2}$$

In terms of these coordinates we have

$$ds^2=\frac{32(MG)^3}{r}\exp\left(-\frac{r}{2MG}\right)dxdy+r^2d\Omega^2. \tag{3}$$

The only true singularities are at curves $xy=-1$, where $r=0$. The region $\{x>0,y>0\}$ is the "*outside region*," the only region from which distant observers can obtain any information. The line $y=0$, where $r=2MG$, is the "*future horizon*"; the line $x=0$ where also $r=2MG$, is the "*past horizon*." We will consider a slice Σ of the Schwarzschild manifold representing a constant time section of $\mathcal{M}$. This surface Σ is an Einstein-Rosen bridge with wormhole topology $S^2\times R^1$ which defines a bifurcation surface, dividing Σ into two parts denoted by Σ_+ and Σ_-. Our purpose is to consider perturbations at Σ with t constant, which naturally define quantum fluctuations of the Einstein-Rosen bridge. In particular we will focus our attention on the Σ_+ sector of the manifold, corresponding to the "*outside region*" of the Kruskal manifold. The explicit expression of the Hamiltonian can be calculated by considering the line element

$$ds^2=-N^2(dx^0)^2+g_{ij}(N^idx^0+dx^i)(N^jdx^0+dx^j), \tag{4}$$

where N is called the *lapse* function and N_i is the *shift* function. When $N=\sqrt{1-2MG/r}$, $N_i=0$, and $g_{ij}dx^idx^j=(1-2MG/r)^{-1}dr^2+r^2d\Omega^2$, we recover the Schwarzschild solution. On the slice Σ, deviations from the Schwarzschild metric spatial section will be considered:

$$g_{ij}=\bar{g}_{ij}+h_{ij}, \tag{5}$$

with $N_i=0$ and $N\equiv N(r)$. Then the line element (4) becomes

$$ds^2=-N^2(r)(dx^0)^2+g_{ij}dx^idx^j \tag{6}$$

and the total Hamiltonian is

$$H_T=H_\Sigma+H_{\partial\Sigma}=\int_\Sigma d^3x(N\mathcal{H}+N_i\mathcal{H}^i)+H_{\partial\Sigma}, \tag{7}$$

*Email address: Garattini@mi.infn.it

here

$$\mathcal{H}=G_{ijkl}\pi^{ij}\pi^{kl}\left(\frac{l_p^2}{\sqrt{g}}\right)-\left(\frac{\sqrt{g}}{l_p^2}\right)R^{(3)} \quad \text{(super Hamiltonian)}, \tag{8}$$

$$\mathcal{H}^i=-2\pi^{ij}_{|j} \quad \text{(super momentum)}, \tag{9}$$

while $H_{\partial\Sigma}$ represents the energy stored in the boundaries. In this respect, we will follow the Arnowitt-Deser-Misner (ADM) approach [1], even though the quasilocal energy context gives a more general treatment with the possibility of looking at the gravitational thermodynamics [2,3]. Moreover, since the space under investigation is asymptotically flat in spacelike directions, the quasilocal energy agrees with the results of the ADM approach in the limit that the boundary tends to spatial infinity. In any case, to correctly compute $H_{\partial\Sigma}=H_{\mathrm{ADM}}$ we have to fix a reference frame to normalize the energy value on the boundary to zero. This brings about the problem of the subtraction procedure investigated in Refs. [3,4]. In this paper we would like to apply such a procedure extended to the volume term, at least at one loop. Since the reference space for the Schwarzschild metric is flat space, the contribution to the energy term is

$$H_{\mathrm{ADM}}=\lim_{r\to\infty}\int_{\partial\Sigma}\sqrt{\hat{g}}\hat{g}^{ij}[\hat{g}_{ik,j}-\hat{g}_{ij,k}]dS^k=M, \tag{10}$$

where $\hat{g}_{ij}$ is the metric induced on a spacelike hypersurface $\partial\Sigma$ which has a boundary at infinity like S^2. Following the result of Ref. [4], we see that H_{ADM} is completely equivalent to

$$-\frac{1}{8\pi G}\int_{\partial\Sigma}[{}^2K-{}^2K_0], \tag{11}$$

where the subtraction structure is evident. The one-loop contribution to the zero-point energy for gravitons embedded in flat space is

$$2\times\frac{1}{2}\int\frac{d^3k}{(2\pi)^3}\sqrt{k^2}. \tag{12}$$

It is clear that this term is UV divergent. We will show that the same kind of divergence is present when the curved background is considered. In the spirit of the subtraction procedure we will compute the difference between zero-point energies. Their difference at one loop represents a Casimir-like computation. The paper is structured as follows: in Sec. II we define the Gaussian wave functional for gravity and we analyze the orthogonal decomposition of the metric deformations, in Sec. III we give some of the basic rules to perform the functional integration and we define the Hamiltonian approximated up to second order, and in Sec. IV we analyze the spin-2 operator acting on transverse traceless tensors, only for positive values of E^2. We summarize and conclude in Sec. V.

II. ENERGY DENSITY CALCULATION IN SCHRÖDINGER REPRESENTATION

As already mentioned, we would like to discuss the possibility of generalizing the boundary subtraction procedure. To this end, by looking at the Hamiltonian structure, we see that there are two classical constraints

$$\mathcal{H}=0,$$

$$\mathcal{H}^i=0, \tag{13}$$

which are satisfied both by the Schwarzschild and flat metrics and two *quantum* constraints

$$\mathcal{H}\tilde{\Psi}=0,$$

$$\mathcal{H}^i\tilde{\Psi}=0. \tag{14}$$

$\mathcal{H}\tilde{\Psi}=0$ is known as the *Wheeler-DeWitt* (WDW) equation. Nevertheless, we are interested in assigning a meaning to

$$\frac{\langle\Psi|H_\Sigma^{\mathrm{Schw}}-H_\Sigma^{\mathrm{flat}}|\Psi\rangle}{\langle\Psi|\Psi\rangle}+\frac{\langle\Psi|H_{\mathrm{ADM}}|\Psi\rangle}{\langle\Psi|\Psi\rangle}, \tag{15}$$

where Ψ is a wave functional whose structure will be determined later and $H_\Sigma^{\mathrm{Schw}}(H_\Sigma^{\mathrm{flat}})$ is the total Hamiltonian referred to the different spacetimes for the volume term. This has to be meant in this way: it is true that the WDW equation refers to the space of metrics, but the space of metrics possesses different sectors [5] and we are considering the sector of asymptotically flat metrics, in which the zero-point energy is defined with respect to Minkowski space. For the de Sitter sector, we have to substract the energy of de Sitter background and so on. Note that if the expectation value is calculated on the wave functional solution of the WDW equation, we obtain only the boundary contribution. However, in this context boundaries are at infinity in spacelike directions; that is, it is equivalent to considering the unphysical situation of computing energy excitation in the asymptotic region. Then to give meaning to Eq. (15), we adopt the semiclassical strategy of the WKB expansion. By observing that the kinetic part of the super Hamiltonian is quadratic in the momenta, we expand the three-scalar curvature $\int d^3x\sqrt{g}R^{(3)}$ up to $o(h^3)$ and we get

$$\int d^3x\left[-\frac{1}{4}h\Delta h+\frac{1}{4}h^{li}\Delta h_{li}-\frac{1}{2}h^{ij}\nabla_l\nabla_i h^l_j+\frac{1}{2}h\nabla_l\nabla_i h^{li}-\frac{1}{2}h^{ij}R_{ia}h^a_j+\frac{1}{2}hR_{ij}h^{ij}\right], \tag{16}$$

where h is the trace of h_{ij}. On the other hand, following the usual WKB expansion, we will consider $\tilde{\Psi} \simeq C \exp(iS)$. In this context, the approximated wave functional will be substituted by a *trial wave functional* according to the variational approach we would like to implement as regards this problem.

III. GAUSSIAN WAVE FUNCTIONAL FOR TRANSVERSE-TRACELESS TENSORS

To actually make such calculations, we need an orthogonal decomposition for both π_{ij} and h_{ij} to disentangle gauge modes from physical deformations. We define the inner product

$$\langle h,k\rangle := \int_{\mathcal{M}} \sqrt{g} G^{ijkl} h_{ij}(x) k_{kl}(x) d^3x, \tag{17}$$

by means of the inverse WDW metric G_{ijkl}, to have a metric on the space of deformations, i.e., a quadratic form on the tangent space at h, with

$$G^{ijkl} = (g^{ik}g^{jl} + g^{il}g^{jk} - 2g^{ij}g^{kl}). \tag{18}$$

The inverse metric is defined on co-tangent space and it assumes the form

$$\langle p,q\rangle := \int_{\mathcal{M}} \sqrt{g} G_{ijkl} p^{ij}(x) q^{kl}(x) d^3x, \tag{19}$$

so that

$$G^{ijnm} G_{nmkl} = \frac{1}{2}(\delta^i_k \delta^j_l + \delta^i_l \delta^j_k). \tag{20}$$

Note that in this scheme the ''inverse metric'' is actually the WDW metric defined on phase space. Now, we have the desired decomposition on the tangent space of three-metric deformations [6,7]:

$$h_{ij} = \frac{1}{3} h g_{ij} + (L\xi)_{ij} + h^{\perp}_{ij}, \tag{21}$$

where the operator L maps ξ_i into symmetric trace-free tensors,

$$(L\xi)_{ij} = \nabla_i \xi_j + \nabla_j \xi_i - \frac{2}{3} g_{ij} (\nabla \cdot \xi). \tag{22}$$

Then the inner product between three-geometries becomes

$$\langle h,h\rangle := \int_{\mathcal{M}} \sqrt{g} G^{ijkl} h_{ij}(x) h_{kl}(x) d^3x = \int_{\mathcal{M}} \sqrt{g} \left[-\frac{2}{3} h^2 + (L\xi)^{ij} (L\xi)_{ij} + h^{ij\perp} h^{\perp}_{ij} \right]. \tag{23}$$

With the orthogonal decomposition in hand we can define a ''*vacuum trial state*''

$$\Psi[h_{ij}(\vec{x})] = \mathcal{N} \exp\left\{ -\frac{1}{4l_p^2} [\langle hK^{-1}h\rangle^{\perp}_{x,y} + \langle (L\xi)K^{-1}(L\xi)\rangle^{\parallel}_{x,y} + \langle hK^{-1}h\rangle^{Trace}_{x,y}] \right\}, \tag{24}$$

which will be used as a probe for the gravitational ground state. This particular expression is useful because the functional can be represented as a product of three functionals defined on the decomposed tensor field

$$\Psi[h_{ij}(\vec{x})] = \mathcal{N} \Psi[h^{\perp}_{ij}(\vec{x})] \Psi[(L\xi)_{ij}] \Psi\left[\frac{1}{3} g_{ij} h(\vec{x})\right]. \tag{25}$$

$h^{\perp}_{ij}$ is the trace-free transverse part of the 3D quantum field, $(L\xi)_{ij}$ is the longitudinal part, and finally h is the trace part of the same field. $\langle \cdot,\cdot \rangle_{x,y}$ denotes space integration and K^{-1} is the inverse propagator containing variational parameters. The main reason for a similar ''*ansatz*'' comes from the observation that the quadratic part in the momenta of the Hamiltonian decouples in the same way as Eq. (23). Note that the decomposition related to the momenta is independent of the choice of the functional. To calculate the energy density, we need to know the action of some basic operators on $\Psi[h_{ij}]$. The action of the operator h_{ij} on $|\Psi\rangle = \Psi[h_{ij}]$ is realized by

$$h_{ij}(x)|\Psi\rangle = h_{ij}(\vec{x}) \Psi[h_{ij}]. \tag{26}$$

The action of the operator π_{ij} on $|\Psi\rangle$, in general, is

$$\pi_{ij}(x)|\Psi\rangle = -i \frac{\delta}{\delta h_{ij}(\vec{x})} \Psi[h_{ij}]. \tag{27}$$

The inner product is defined by the functional integration

$$\langle \Psi_1 | \Psi_2 \rangle = \int [\mathcal{D}h_{ij}] \Psi_1^*\{h_{ij}\} \Psi_2\{h_{kl}\}, \tag{28}$$

and the energy eigenstates satisfy the stationary Schrödinger equation

$$\int d^3x \mathcal{H}\left\{ -i \frac{\delta}{\delta h_{ij}(\vec{x})}, h_{ij}(\vec{x}) \right\} \Psi\{h_{ij}\} = E\Psi\{h_{ij}\}, \tag{29}$$

where $\mathcal{H}\{-i[\delta/\delta h_{ij}(x)], h_{ij}(x)\}$ is the Hamiltonian density. Note that the previous equation in the general context of Einstein gravity is devoid of meaning, because of the constraints. However, in the semiclassical context, we can give a meaning to Eq. (29), where a *semiclassical time* is introduced in the same manner of Refs. [8,9]. There, a Schrödinger equation of the form

$$i \frac{\partial \Psi^{\perp}}{\partial t} = H_{|2} \Psi^{\perp} \tag{30}$$

is recovered by the WDW equation approximated to second order for a perturbed minisuperspace Friedmann model without boundary terms. When asymptotically flat boundary terms are present we have to take account of such contributions in the WKB expansion such as in Ref. [10]. However, in this paper only gravitational transverse-traceless (TT) modes are considered on the fixed curved background and $\Psi^{\perp}$ is substituted by a trial wave functional. To further proceed, instead of solving Eq. (29), which is of course impossible, we can formulate the same problem by means of a variational principle [14]. We demand that

$$\frac{\langle\Psi|H|\Psi\rangle}{\langle\Psi|\Psi\rangle}=\frac{\int[\mathcal{D}g_{ij}^{\perp}]\int d^3x\,\Psi_1^*\{g_{ij}^{\perp}\}\mathcal{H}\Psi\{g_{kl}^{\perp}\}}{\int[\mathcal{D}g_{ij}^{\perp}]|\Psi\{g_{ij}^{\perp}\}|^2} \tag{31}$$

be stationary against arbitrary variations of $\Psi\{h_{ij}\}$. The form of $\langle\Psi|H|\Psi\rangle$ can be computed as follows. We define normalized mean values

$$\overline{g_{ij}^{\perp}}(\vec{x})=\frac{\int[\mathcal{D}g_{ij}^{\perp}]\int d^3x\,g_{ij}^{\perp}(\vec{x})|\Psi\{g_{ij}^{\perp}\}|^2}{\int[\mathcal{D}g_{ij}^{\perp}]|\Psi\{g_{ij}^{\perp}\}|^2}, \tag{32}$$

$$\times\overline{g_{ij}^{\perp}}(\vec{x})\overline{g_{kl}^{\perp}}(\vec{y})+K_{ijkl}^{\perp}(\vec{x},\vec{y}) \tag{33}$$

$$=\frac{\int[\mathcal{D}g_{ij}^{\perp}]\int d^3x\,g_{ij}^{\perp}(\vec{x})g_{kl}^{\perp}(\vec{y})|\Psi\{g_{ij}^{\perp}\}|^2}{\int[\mathcal{D}g_{ij}^{\perp}]|\Psi\{g_{ij}^{\perp}\}|^2}. \tag{34}$$

It follows that, by defining $h_{ij}^{\perp}=g_{ij}-\overline{g}_{ij}$, we have

$$\int[\mathcal{D}h_{ij}^{\perp}]h_{ij}^{\perp}(\vec{x})|\Psi\{h_{ij}^{\perp}+\overline{g_{ij}^{\perp}}\}|^2=0 \tag{35}$$

and

$$\int[\mathcal{D}h_{ij}^{\perp}]\int d^3x\,h_{ij}^{\perp}(\vec{x})h_{kl}^{\perp}(\vec{y})|\Psi\{h_{ij}^{\perp}+\overline{g_{ij}^{\perp}}\}|^2$$
$$=K_{ijkl}^{\perp}(\vec{x},\vec{y})\int[\mathcal{D}h_{ij}^{\perp}]|\Psi\{h_{ij}^{\perp}+\overline{g_{ij}^{\perp}}\}|^2. \tag{36}$$

Nevertheless, the application of the variational principal on arbitrary wave functional does not improve the situation described by Eq. (29). To this purpose, we give to the trial wave functional the form

$$\Psi[h_{ij}^{\perp}]=\mathcal{N}\exp\left\{-\frac{1}{4l_p^2}\langle(g-\overline{g})K^{-1}(g-\overline{g})\rangle_{x,y}^{\perp}\right\}. \tag{37}$$

We immediately conclude that

$$\langle\Psi|\pi_{ij}^{\perp}(\vec{x})|\Psi\rangle=0, \tag{38}$$

where $\pi_{ij}^{\perp}$ is the TT momentum. In Appendix B, we will show that

$$\langle\Psi|\pi_{ij}^{\perp}(\vec{x})\pi_{kl}^{\perp}(\vec{y})|\Psi\rangle=\frac{1}{4}K_{ijkl}^{-1}(\vec{x},\vec{y}). \tag{39}$$

Choice (37) is related to the form of the Hamiltonian approximated to quadratic order in the metric deformations. Indeed, up to this order we have a harmonic oscillator whose ground state has a Gaussian form. By means of decomposition (21), we extract the TT sector contribution in the previous expression. Moreover, the functional representation (25) eliminates every interaction between gauge and the other terms. Then for the TT sector (spin 2), one gets

$$\int_{\Sigma}d^3x\sqrt{g}R^{(3)}\simeq\frac{1}{4l_p^2}\int_{\Sigma}d^3x\sqrt{g}[h^{\perp ij}(\Delta_2)_j^a h_{ia}^{\perp}-2hR_{ij}h^{\perp ij}], \tag{40}$$

where $(\Delta_2)_j^a:=-\Delta\delta_j^a+2R_j^a$. The latter term disappears because the Gaussian integration does not mix the components. Then by collecting together Eqs. (40) and (39), one obtains the one-loop-like Hamiltonian form for TT deformations:

$$H^{\perp}=\frac{1}{4l_p^2}\int_{\mathcal{M}}d^3x\sqrt{g}G^{ijkl}[K^{-1\perp}(x,x)_{ijkl}$$
$$+(\Delta_2)_j^aK^{\perp}(x,x)_{iakl}]. \tag{41}$$

The propagator $K^{\perp}(x,x)_{iakl}$ comes from a functional integration and it can be represented as

$$K^{\perp}(\vec{x},\vec{y})_{iakl}:=\sum_N\frac{h_{ia}^{\perp}(\vec{x})h_{kl}^{\perp}(\vec{y})}{2\lambda_N(p)}, \tag{42}$$

where $h_{ia}^{\perp}(\vec{x})$ are the eigenfunctions of Δ_{2j}^{a} and $\lambda_N(p)$ are infinite variational parameters.

IV. SPECTRUM OF THE SPIN-2 OPERATOR AND THE EVALUATION OF THE ENERGY DENSITY IN MOMENTUM SPACE

The spin-2 operator is defined by

$$(\Delta_2)_j^a:=-\Delta\delta_j^a+2R_j^a, \tag{43}$$

where Δ is the curved Laplacian (Laplace-Beltrami operator) on a Schwarzschild background and R_j^a is the mixed Ricci tensor whose components are

$$R_j^a=\text{diag}\left\{\frac{-2m}{r^3},\frac{m}{r^3},\frac{m}{r^3}\right\}, \tag{44}$$

where $2m=2MG$. This operator is similar to the Lichnerowicz operator provided that we substitute the Riemann tensor by the Ricci tensor. This is essentially due to the fact that the Riemann tensor in three dimensions is a linear combination of the Ricci tensor. In Eq. (45) the Ricci tensor acts as a potential on the space of TT tensors; for this reason we are led to study the following eigenvalue equation:

$$(-\Delta\delta_j^a+2R_j^a)h_a^i=E^2h_j^i, \tag{45}$$

where E^2 is the eigenvalue of the corresponding equation. In doing so, we follow Regge and Wheeler in analyzing the equation as modes of definite frequency, angular momentum, and parity. We can specialize to the case where the quantum number corresponding to the projection of the angular momentum on the z axis is zero, without altering the contribution to the total energy because of the spherical symmetry of the problem. In this case, Regge-Wheeler decomposition [11] shows that the even-parity three-dimensional perturbation is

$$\begin{aligned} h_{ij}^{even}&(r,\vartheta,\phi)\\ &=\mathrm{diag}\left[H(r)\left(1-\frac{2m}{r}\right)^{-1},r^2K(r),r^2\sin^2\vartheta K(r)\right]\\ &\quad\times Y_{l0}(\vartheta,\phi). \end{aligned} \tag{46}$$

Representation (46) shows a gravitational perturbation decoupling. For a generic value of the angular momentum l, one gets

$$\begin{aligned} &-\Delta_l H(r)-\frac{4m}{r^3}H(r)=E_l^2H(r),\\ &-\Delta_l K(r)+\frac{2m}{r^3}K(r)=E_l^2K(r),\\ &-\Delta_l K(r)+\frac{2m}{r^3}K(r)=E_l^2K(r). \end{aligned} \tag{47}$$

The Laplacian in this particular geometry can be written as

$$\Delta_l=\left(1-\frac{2m}{r}\right)\frac{d^2}{dr^2}+\left(\frac{2r-3m}{r^2}\right)\frac{d}{dr}-\frac{l(l+1)}{r^2}. \tag{48}$$

Defining reduced fields, such as

$$H(r)=\frac{h(r)}{r},\quad K(r)=\frac{k(r)}{r}, \tag{49}$$

and changing variables to

$$x=2m\left\{\sqrt{\frac{r}{2m}}\sqrt{\frac{r}{2m}-1}+\ln\left(\sqrt{\frac{r}{2m}}+\sqrt{\frac{r}{2m}-1}\right)\right\}, \tag{50}$$

the system (47) becomes

$$\begin{aligned} &-\frac{d^2}{dx^2}h(x)+V^-(x)h(x)=E_l^2h(x),\\ &-\frac{d^2}{dx^2}k(x)+V^+(x)k(x)=E_l^2k(x),\\ &-\frac{d^2}{dx^2}k(x)+V^+(x)k(x)=E_l^2k(x), \end{aligned} \tag{51}$$

where

$$V^{\mp}(x)=\frac{l(l+1)}{r^2(x)}\mp\frac{3m}{r(x)^3}. \tag{52}$$

This new variable represents the proper geodesic distance from the wormhole throat such that

$$\text{when}\quad r\to\infty,\quad x\simeq r\ \text{ and }\ V^{\mp}(x)\to 0$$

$$\text{when}\quad r\to r_0,\ x\simeq 0\quad\text{and}\ V^{\mp}(x)\to\frac{l(l+1)}{r_0^2}\mp\frac{3m}{r_0^3}=\text{const}, \tag{53}$$

where r_0 satisfies the condition $r_0>2m$. The solution of Eq. (51), in both cases (flat and curved one), is the spherical Bessel function of the first kind:

$$j_0(px)=\sqrt{\frac{2}{\pi}}\sin(px). \tag{54}$$

This choice is dictated by the requirement that

$$h(x),k(x)\to 0\quad\text{when}\ x\to 0\ \ (\text{alternatively}\ \ r\to 2m). \tag{55}$$

Then

$$K(x,y)=\frac{j_0(px)j_0(py)}{2\lambda}\frac{1}{4\pi}. \tag{56}$$

Substituting Eq. (56) into Eq. (41) one gets (after normalization in spin space and after a rescaling of the fields in such a way as to absorb l_p^2)

$$E(m,\lambda)=\frac{V}{2\pi^2}\sum_{l=0}^{\infty}\sum_{i=1}^{2}\int_0^\infty dp\,p^2\left[\lambda_i(p)+\frac{E_i^2(p,m,l)}{\lambda_i(p)}\right], \tag{57}$$

where

$$E_{1,2}^2(p,m,l)=p^2+\frac{l(l+1)}{r_0^2}\mp\frac{3m}{r_0^3}, \tag{58}$$

$\lambda_i(p)$ are variational parameters corresponding to the eigenvalues for a (graviton) spin-two particle in an external field, and V is the volume of the system.

By minimizing Eq. (57) with respect to $\lambda_i(p)$ one obtains $\overline{\lambda}_i(p)=[E_i^2(p,m,l)]^{1/2}$ and

$$E(m,\overline{\lambda})=\frac{V}{2\pi^2}\sum_{l=0}^{\infty}\sum_{i=1}^{2}\int_0^{\infty}dp\,2\sqrt{E_i^2(p,m,l)},\tag{59}$$

with

$$p^2+\frac{l(l+1)}{r_0^2}>\frac{3m}{r_0^3}.$$

Thus, in presence of the curved background, we get

$$E(m)=\frac{V}{2\pi^2}\frac{1}{2}\sum_{l=0}^{\infty}\int_0^{\infty}dp\,p^2\left(\sqrt{p^2+c_-^2}+\sqrt{p^2+c_+^2}\right),\tag{60}$$

where

$$c_{\mp}^2=\frac{l(l+1)}{r_0^2}\mp\frac{3m}{r_0^3},$$

while when we refer to the flat space, in the spirit of the subtraction procedure, we have $m=0$ and $c^2=l(l+1)/r_0^2$. Then

$$E(0)=\frac{V}{2\pi^2}\frac{1}{2}\sum_{l=0}^{\infty}\int_0^{\infty}dp\,p^2\left(2\sqrt{p^2+c^2}\right).\tag{61}$$

Now, we are in position to compute the difference between Eqs. (60) and (61). Since we are interested in the *UV* limit, we have

$$\begin{aligned}\Delta E(m)&=E(m)-E(0)\\&=\frac{V}{2\pi^2}\frac{1}{2}\sum_{l=0}^{\infty}\int_0^{\infty}dp\,p^2\left[\sqrt{p^2+c_-^2}+\sqrt{p^2+c_+^2}-2\sqrt{p^2+c^2}\right]\\&=\frac{V}{2\pi^2}\frac{1}{2}\sum_{l=0}^{\infty}\int_0^{\infty}dp\,p^3\left[\sqrt{1+\left(\frac{c_-}{p}\right)^2}+\sqrt{1+\left(\frac{c_+}{p}\right)^2}-2\sqrt{1+\left(\frac{c}{p}\right)^2}\right]\end{aligned}\tag{62}$$

and, for $p^2\gg c_{\mp}^2,c^2$, we obtain

$$\begin{aligned}&\frac{V}{2\pi^2}\frac{1}{2}\sum_{l=0}^{\infty}\int_0^{\infty}dp\,p^3\left[1+\frac{1}{2}\left(\frac{c_-}{p}\right)^2-\frac{1}{8}\left(\frac{c_-}{p}\right)^4+1+\frac{1}{2}\left(\frac{c_+}{p}\right)^2-\frac{1}{8}\left(\frac{c_+}{p}\right)^4-2-\left(\frac{c}{p}\right)^2-\frac{1}{4}\left(\frac{c}{p}\right)^4\right]\\&=-\frac{V}{2\pi^2}\frac{c^4}{8}\int_0^{\infty}\frac{dp}{p}.\end{aligned}\tag{63}$$

We will use a cutoff Λ to keep under control the UV divergence:[1]

$$\int_0^{\infty}\frac{dp}{p}\sim\int_0^{\Lambda/c}\frac{dx}{x}\sim\ln\left(\frac{\Lambda}{c}\right).\tag{64}$$

Thus $\Delta E(m)$ for high momenta becomes

$$\Delta E(m)\sim-\frac{V}{2\pi^2}\frac{c^4}{16}\ln\left(\frac{\Lambda^2}{c^2}\right)=-\frac{V}{2\pi^2}\left(\frac{3m}{r_0^3}\right)^2\frac{1}{16}\ln\left(\frac{r_0^3\Lambda^2}{3m}\right).\tag{65}$$

At this point we can compute the total energy, namely, the classical contribution plus the quantum correction up to second order. Recalling the definition of asymptotic energy for an asymptotically flat background, such as the Schwarzschild, one gets

$$\begin{aligned}M-\frac{V}{2\pi^2}\left(\frac{3m}{r_0^3}\right)^2\frac{1}{16}\ln\left(\frac{r_0^3\Lambda^2}{3m}\right)\\=M-\frac{V}{2\pi^2}\left(\frac{3MG}{r_0^3}\right)^2\frac{1}{16}\ln\left(\frac{r_0^3\Lambda^2}{3MG}\right).\end{aligned}\tag{66}$$

One can observe that

$$\Delta E(m)\to\infty\quad\text{when}\quad m\to 0,\ \text{for}\ r_0=2m=2GM\tag{67}$$

and

$$\Delta E(m)\to 0\quad\text{when}\quad m\to 0,\ \text{for}\ r_0\neq 2m=2GM.\tag{68}$$

We would like to explain the reasons that support the results of formula (65). In that formula we introduced a particular value of the radius, which behaves as a regulator with respect to the horizon approach of the potential. The meaning of this particular value is related to the necessity of explaining the dynamical origin of black hole entropy by the entanglement entropy mechanism and by the so-called ''*brick wall model*'' [12,13]. Indeed, the same mechanism is present when one has to regularize entropy by imposing a kind of cutoff, which in coordinate space means $r_0>2m$. In fact, r_0 can be seen as $2m+h$, where h assumes the same meaning of Ref. [12]. However, to explicitly relate this quantity we have to compare the Bekenstein-Hawking entropy with the result deriving from the evaluation of the free energy for gravitons, in this case. The only difference from the original calculation is the spin contribution not present for scalar fields.

---

[1]It is known that at one-loop level gravity is renormalizable only in flat space. In a dimensional regularization scheme its contribution to the action is, on shell, proportional to the Euler character of the manifold that is nonzero for the Schwarzschild instanton. Although in our approach we are working with sections of the original manifold to deal with these divergences, one must introduce a regulator that indeed appears in the contribution of energy density.

V. SUMMARY AND CONCLUSIONS

We started from the problem of defining quantum corrections (semiclassical) to a gravitational energy. By means of a variational approach with Gaussian wave functionals an attempt to calculate such a correction was made. Despite the constraint equations, this calculation is based on an extension of the subtraction procedure involving volume terms in the semiclassical regime. Excitations coming from boundary terms have been neglected to avoid the unphysical situation of having contributions deriving from infinity. In this context the extended subtraction procedure corresponds to the difference between zero-point energies calculated in an asymptotically flat curved background referring to a flat background. This procedure eliminates the UV divergence of the free gravitons, leaving the contribution of the curved background related to an *imposed by hand* UV cutoff. A strong analogy with the Casimir vacuum energy calculation is revealed, opening the possibility of understanding several configurations and their relationship with the vacuum stability. Indeed, this apparatus can be applied also to the Schwarzschild–de Sitter background which asymptotically approaches the de Sitter space and so on. Although this evaluation has to be completed with a careful study of the spin-2 operator, we can conclude that the variational approach for the computation of quantum corrections (semiclassical) to a classical energy can be thought of as a tool for testing zero-point energy (Casimir-like energy) in a complicated theory such as Einstein gravity.

ACKNOWLEDGMENTS

I wish to thank G. Esposito, V. Frolov, and D.L. Rapoport for helpful discussions. I also thank S. Liberati and B. Jensen for their suggestions on how to justify the horizon approach.

APPENDIX A: CONVENTIONS

Riemann tensor, Ricci tensor, and the scalar curvature in 3D:

$$R^l_{ijm}=\Gamma^l_{mi,j}-\Gamma^l_{ji,m}+\Gamma^l_{ja}\Gamma^a_{mi}-\Gamma^l_{ma}\Gamma^a_{ji} \quad \text{Riemann tensor.}$$

Because of the vanishing of the Weyl tensor in 3D, that is, $C^l_{ijm}=0$, the Riemann tensor is completely determined by the Ricci tensor:

$$R_{lijm}=g_{lj}R_{im}-g_{lm}R_{ij}-g_{ij}R_{lm}+g_{im}R_{lj}\,,$$

$$R_{im}=R^l_{ilm} \quad \text{Ricci tensor,}$$

$$R=g^{lj}R_{lj} \quad \text{scalar curvature.}$$

APPENDIX B: THE KINETIC TERM

The Schrödinger picture representation of the kinetic term is

$$G_{ijkl}\pi^{ij}\pi^{kl}=G_{ijkl}\left(-\frac{\delta^2}{\delta h_{ij}(x)\,\delta h_{kl}(x)}\right). \tag{B1}$$

We have to apply this quantity to the Gaussian wave functional $|\Psi\rangle$. This means that

$$\begin{aligned}\pi^{ij}(x)\pi^{kl}(x)|\Psi\rangle=&-\frac{\delta^2\Psi[h]}{\delta h_{ij}(x)\,\delta h_{kl}(x)}\\ =&\frac{1}{2}K^{-1(kl)(ij)}(x,x)[\sqrt{g(x)}]^2\Psi[h]\\ &-\frac{1}{4}\int d^3y'd^3y''[\sqrt{g(x)}]^2\sqrt{g(y')}\\ &\times\sqrt{g(y'')}K^{-1(kl)(k'l')}(x,y')h_{k'l'}(y')\\ &\times K^{-1(ij)(k''l'')}(x,y'')h_{k''l''}(y'')\Psi[h].\end{aligned} \tag{B2}$$

By functional integrating,

$$\langle\Psi|h_{k'l'}(y')h_{k''l''}(y'')|\Psi\rangle=K_{(k'l')(k''l'')}(y',y'')\langle\Psi|\Psi\rangle. \tag{B3}$$

Thus

$$\langle\Psi|\pi^{ij}(x)\pi^{kl}(x)|\Psi\rangle$$

becomes

$$\begin{aligned}&\frac{1}{2}K^{-1(kl)(ij)}(x,x)[\sqrt{g(x)}]^2\\ &\quad-\frac{1}{4}\int d^3y'd^3y''[\sqrt{g(x)}]^2\sqrt{g(y')}\sqrt{g(y'')}\\ &\quad\times K^{-1(kl)(k'l')}(x,y')K^{-1(ij)(k''l'')}(x,y'')\\ &\quad\times K_{(k'l')(k''l'')}(y',y'')\langle\Psi|\Psi\rangle\\ &=\frac{1}{4}K^{-1(kl)(ij)}(x,x)[\sqrt{g(x)}]^2\langle\Psi|\Psi\rangle.\end{aligned} \tag{B4}$$

Then the expectation value of the kinetic term, with the Planck length reinserted, is

$$\langle T\rangle=\frac{1}{4l_p^2}\int d^3x\sqrt{g}[G_{ijkl}K^{-1(kl)(ij)}(x,x)]. \tag{B5}$$

[1] R. Arnowitt, S. Deser, and C.W. Misner, in *Gravitation: An Introduction to Current Research,* edited by L. Witten (Wiley, New York, 1962); B.S. DeWitt, Phys. Rev. **160**, 1113 (1967); C.W. Misner, K.S. Thorne, and J.A. Wheeler, *Gravitation* (Freeman, San Francisco, 1973), p. 842; M.S. Morris and K.S. Thorne, Am. J. Phys. **56**, 395 (1988).

[2] J.D. Brown and J.W. York, Phys. Rev. D **47**, 1407 (1993).
[3] V.P. Frolov and E.A. Martinez, Class. Quantum Grav. **13**, 481 (1996).
[4] S.W. Hawking and G.T. Horowitz, Class. Quantum Grav. **13**, 1487 (1996).
[5] T. Regge and C. Teitelboim, Ann. Phys. (N.Y.) **88**, 286 (1974).
[6] M. Berger and D. Ebin, J. Diff. Geom. **3**, 379 (1969).
[7] J.W. York, Jr., J. Math. Phys. **14**, 4 (1973); Ann. Inst. Henri Poincaré, Sect. A **21**, 319 (1974).
[8] J.J. Halliwell and S.W. Hawking, Phys. Rev. D **31**, 1777 (1985).
[9] J.J. Halliwell, in *Jerusalem Winter School for Theoretical Physics: Quantum Cosmology and Baby Universes*, edited by S. Coleman, J.B. Hartle, T. Piran, and S. Weinberg (World Scientific, Singapore, 1991), Vol. 7, pp. 159–243.
[10] T. Brotz, Phys. Rev. D **57**, 2349 (1998).
[11] T. Regge and J.A. Wheeler, Phys. Rev. **108**, 1063 (1957).
[12] G. 't Hooft, Nucl. Phys. **B256**, 727 (1985).
[13] V.P. Frolov and I. Novikov, Phys. Rev. D **48**, 4545 (1993).
[14] A.K. Kerman and D. Vautherin, Ann. Phys. (N.Y.) **192**, 408 (1989); J. M. Cornwall, R. Jackiw, and E. Tomboulis, Phys. Rev. D **10**, 2428 (1974); R. Jackiw, in ''Séminaire de Mathéma-tiques Supérieures,'' Montréal, Québec, Canada, 1988, notes by P. de Sousa Gerbert; M. Consoli and G. Preparata, Phys. Lett. **154B**, 411 (1985).