Matrix D-brane dynamics, logarithmic operators, and quantization of noncommutative spacetime

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We describe the structure of the moduli space of σ-model couplings for the worldsheet description of a system of N D-particles, in the case that the couplings are represented by a pair of logarithmic recoil operators. We derive expressions for the canonical momenta conjugate to the D-particle couplings and the Zamolodchikov metric to the first few orders in σ-model perturbation theory. We show, using only very general properties of the operator product expansion in logarithmic conformal field theories, that the canonical dynamics on moduli space agree with the predictions of the non-Abelian generalization of the Born-Infeld effective action for D-particles with a symmetrized trace structure. We demonstrate that the Zamolodchikov metric naturally encodes the short-distance structure of spacetime, and from this we derive uncertainty relations for the D-particle coordinates directly from the quantum string theory. We show that the moduli space geometry naturally leads to new forms of spacetime indeterminacies involving only spatial coordinates of target space and illustrate the manner in which the open string interactions between D-particles lead to a spacetime quantization. We also derive appropriate non-Abelian generalizations of the string-modified Heisenberg uncertainty relations and the space-time uncertainty principle. The non-Abelian uncertainties exhibit decoherence effects suggesting the interplay of quantum gravity in multiple D-particle dynamics. [S0556-2821(99)01510-6]

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I. INTRODUCTION

Dirichlet-branes are solitonic backgrounds of superstring theory whose discovery [1] has drastically changed the understanding of the non-perturbative and target space structures of string theory. Their dynamics can be simply described by open strings whose worldsheets are discs with Dirichlet boundary conditions for the collective coordinates of the soliton [2], and they are related to ordinary closed string backgrounds by duality transformations [1]. In this paper we shall study the dynamics of a many-body system of D-particles.

The effective field theory for a system of N parallel D-branes, with infinitesimal separation between them, is a good probe of the short-distance structure of the spacetime implied by string theory [3]. The main characteristic behind this property of D-brane dynamics is the observation [4] that the low energy effective field theory for a system of N D-branes is ten-dimensional maximally supersymmetric $U(N)$ Yang-Mills theory dimensionally reduced to the world-volume of the D-branes. For the case of D-particles the Yang-Mills potential is

$$V_{D0}[Y] = \frac{\mathcal{T}^2}{4g_s} \sum_{i,j=1}^{9} \mathrm{tr}[Y^i, Y^j]^2 \qquad (1.1)$$

where $\mathcal{T} = 1/2\pi\alpha'$ is the elementary string tension, with α' the string Regge slope whose square root is the intrinsic string length l_s, and g_s is the (dimensionless) string coupling constant. The fields $Y^i(t)$ are $N \times N$ Hermitian matrices in the adjoint representation and the trace is taken in the fundamental representation of the gauge group $U(N)$. In the free string limit $g_s \to 0$, the field theory involving the potential (1.1) localizes onto those matrix configurations satisfying

$$[Y^i, Y^j] = 0, \quad i,j = 1, \ldots, 9 \qquad (1.2)$$

and so the D-brane coordinate fields can be simultaneously diagonalized by a gauge transformation. The corresponding eigenvalues y_a^i, $a = 1, \ldots, N$, of Y^i then represent the collective transverse coordinates of the N D-branes. In this limit the parallel D-branes are very far apart from each other and massless vector states emerge only when fundamental strings start and end on the same D-particle (Fig. 1). The gauge group is then $U(1)^N$. Since the energy of a string which stretches between two D-branes is

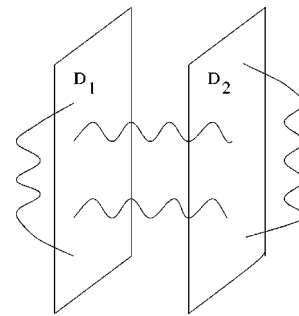$$M \propto \mathcal{T}|y_a - y_b| \qquad (1.3)$$

FIG. 1. Emergence of the enhanced $U(N)$ gauge symmetry for bound states of $N = 2$ parallel D-branes (planes). An oriented fundamental string (wavy lines) can start and end either at the same or different D-brane, giving four massless vector states in the limit of coinciding branes. These states form a representation of $U(2)$.

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more massless vector states emerge when the branes are practically on top of each other. The collection of all massless states corresponding to an elementary string starting and ending on either the same or different D-brane forms a $U(N)$ multiplet (Fig. 1). The off-diagonal components of the Y^i and the remnant gauge fields describe the dynamics of the short open strings interacting with the branes through the Dirichlet condition.

Thus when the D-branes are very far apart the classical vacuum solution of the field theory has unbroken supersymmetry (or zero energy) and the spacetime coordinates are represented by a set of *commutative* Y^i. When the branes are very close to each other, the full quantum $U(N)$ gauge theory must be taken into account, whose spectrum consists of D-brane bound states with broken supersymmetry ($[Y^i, Y^j] \neq 0$ for $i \neq j$) and at very short distances the spacetime is described by *noncommutative* structures. The gauge symmetry is interpreted as a symmetry generalizing the statistics symmetry for identical particles in quantum mechanics and the D-brane coordinates are viewed as adjoint Higgs fields in this description. D-brane field theory therefore explicitly realizes the old ideas of string theory that at very short distance scales (smaller than the target space Planck length or the finite size of the string) the classical concepts of spacetime geometry break down. The noncommutative structure of the spacetime is controlled by the strength of the string interactions among the constituent D-branes. This is precisely the structure inherent in the noncommutative geometry formalisms of stringy spacetimes [5], in which the target space geometry is represented by the algebra of observables (such as a vertex operator algebra) corresponding to the interacting states of the theory.

The dimensionally reduced Yang-Mills theory is the relevant field theory for the description of matrix theory [6], which hypothesizes that the D-particles of type-IIA superstring theory are the partons and the supersymmetric gauge theory the exact quantum field theory in the infinite momentum frame of 11-dimensional spacetime. However, this is not the case in other regimes, for instance in the weak-coupling limit where the relevant effective action is the disc generating functional. In this paper we shall be interested in the description of N D-particle dynamics from an elementary point of view of the bosonic part of a worldsheet σ-model for the string interactions. In this formalism, the D-brane coordinate fields appear as coupling constants associated with boundary deformation vertex operators on the worldsheet of a free σ-model. Already at the tree level in the string coupling g_s (the disc diagram) and in flat target space, the effective action for N D-branes is a highly non-local object that is not known in closed form. This complexity is due to the fact that, even at the tree level, correlation functions on the disc receive contributions from the massive string states which yield a non-local functional of the massless modes.

The low-energy effective field theory for the σ-model couplings $y^i(t)$ in the case of a single D-particle is well-known to be described by the Born-Infeld action [7]

$$\Gamma_{\mathrm{BI}}[y] = \frac{1}{g_s l_s} \int dt \sqrt{1 - (\dot{y}^i)^2} \qquad (1.4)$$

which is just the relativistic free particle action for the D0-brane. The appropriate generalization of Eq. (1.4) to the case of non-Abelian (Chan-Paton) σ-model couplings appropriate to the description of multi-D-brane dynamics has been a point of ambiguity in recent literature. Although it is established that the appropriate global gauge invariant structure in the action is a trace in the fundamental representation of $U(N)$, the ambiguity arises in choosing a particular matrix ordering prescription for the action. The original proposal in [8], which employs a symmetrized matrix ordering, has been argued to hold *only* when one incorporates worldsheet supersymmetry [9], or alternatively when one imposes certain energy-minimizing Bogomol'nyi-Prasad-Sommerfield-(BPS)-type conditions on the form of the action [10]. The $U(N)$ Yang-Mills theory should appear as a ''non-relativistic'' approximation to the non-Abelian Born-Infeld action. An interesting closed-form expression for the symmetrized action in the case $N = 2$ has been obtained recently in [11].

In the following we shall show how an appropriate worldsheet formalism yields the symmetrized form of the effective bosonic action functional for multi-D-brane dynamics, without the need of resorting to supersymmetry arguments. A crucial feature of the D-brane couplings we shall use is that, not only do they define a (non-marginal) perturbation about a truly marginal deformation, but the deformed worldsheet field theory has logarithmic scaling violations, coming from logarithmic divergences on the worldsheet, and defines not a conventional two-dimensional conformal field theory, but rather a *logarithmic* conformal field theory [12]. Logarithmic conformal field theories lie on the border between conformal field theories and generic two-dimensional renormalizable field theories, and they correspond to the emergence of hidden continuous symmetries [13]. It has been suggested [14] that the appropriate worldsheet description of the collective coordinates (zero modes) of a soliton in string theory is given by logarithmic operators. The normalizable target space zero modes for D-branes arise from translations and rotations (in both spacetime and isospin space) of the background, and there is a family of backgrounds connected by the symmetries which act on the moduli space of σ-model couplings characterizing the background. These modes are an important ingredient for the proper incorporation of recoil effects during the scattering of closed string states off the D-brane background when the soliton state changes during the process of scattering [15–19]. These effects are important aspects of the quantization of the collective coordinates of D-branes.

Logarithmic operators have conformal dimensions which are degenerate with those of the usual primary fields, and as a result of this degeneracy one can no longer completely diagonalize the usual Virasoro operator L_0. Together with the standard operators they form the basis of a Jordan cell for L_0. For a logarithmic pair (C, D) of conformal dimension Δ, the operator product expansion of the worldsheet stress-energy tensor T with these fields is non-trivial and involves a mixing [12]

$$T(z)C(w) \sim \frac{\Delta}{(z-w)^2} C(w) + \frac{1}{z-w} \partial C(w) + \cdots$$

$$T(z)D(w) \sim \frac{\Delta}{(z-w)^2} D(w) + \frac{1}{(z-w)^2} C(w)$$
$$+ \frac{1}{z-w} \partial D(w) + \cdots \qquad (1.5)$$

where an appropriate normalization of the D operator has been chosen. Defining the Virasoro operator L_0 through the Laurent series expansion $T(z) = \Sigma_{n \in \mathbb{Z}} L_n z^{-n-2}$, it follows that the corresponding states $|C\rangle = C(0)|0\rangle$ and $|D\rangle = D(0)|0\rangle$ generate a 2×2 Jordan block for L_0,

$$L_0|C\rangle = \Delta|C\rangle, \quad L_0|D\rangle = \Delta|D\rangle + |C\rangle. \qquad (1.6)$$

This mixing is a consequence of the behavior of the conformal blocks of the underlying worldsheet theory which exhibit logarithmic scaling violations. It is the characteristic non-trivial property of theories involving logarithmic operators.

In this paper we shall study the disc amplitude in a worldsheet boundary auxilliary field formalism [20–23] which can be thought of as an ''Abelianization'' of the $U(N)$ theory. In this framework, before the auxilliary fields are integrated out, the only difference from the Abelian situation is an extra explicit dependence on the variables parametrizing the boundary of the string worldsheet. This representation of the Wilson loop operator enables one to carry out σ-model perturbation theory in much the same way as in the Abelian (single D-brane) case. Within this formalism, we will construct the moduli space of the σ-model couplings which represents the effective spacetime of the D-particles and whose geometry is determined by the Zamolodchikov metric [24]. The dynamics on moduli space is determined by the Zamolodchikov C-theorem and a set of conditions which ensure the possibility of canonical quantization [25]. The crucial observation is that, because of the logarithmic nature of the D-brane couplings, the worldsheet deformations become slightly relevant, which in the recoil problem is precisely the property that leads to a change of state of the 0-brane background. To restore marginality, we dress the worldsheet theory with two-dimensional quantum gravity, i.e. Liouville theory [26]. We demonstrate explicitly that the canonical form of the moduli space dynamics coincides with that of the symmetrized non-Abelian Born-Infeld theory. Physically, the dynamical theory describes the non-relativistic motion of open strings in the background of a ''fat brane,'' as described in [27]. Although in this framework the explicit form of the D-brane couplings is associated with those relevant to the recoil problem, we shall see that the derivation of our results are based only on very general properties of the operator product expansion in generic logarithmic conformal field theories. The derivation of the appropriate non-Abelian Born-Infeld dynamics in the kinematical region of interest thereby represents a highly non-trivial application of the theory of logarithmic operators.

Quantization of the moduli space is then achieved by summing over worldsheet topologies, in the pinched approximation which gives the dominant terms [14,15,18,25]. In the case of a single D-particle, it was shown in [28] that, to leading order in the σ-model coupling constant expansion, one recovers the usual canonical quantum phase space with position and momentum having a constant commutator and ''Planck constant'' given in terms of the string coupling g_s. Incorporating stringy effects reproduces the generalized string uncertainty principle [29,30]

$$\Delta y^i \Delta p_j \geq \frac{\hbar}{2} \delta_j^i (1 + \mathcal{O}(\alpha'_s)(\Delta p_i)^2 + \cdots) \qquad (1.7)$$

which corresponds to adding corrections to the Heisenberg commutation relations of the form

$$[[\hat{y}^i, \hat{p}_j]] = i\hbar \delta_i^j (1 + \mathcal{O}(\alpha'_s)\hat{p}_i^2 + \cdots) \qquad (1.8)$$

where $\alpha'_s = g_s^2 \alpha'$ is the 0-brane scale. The result (1.7) can also be derived from a Heisenberg microscope approach to the uncertainty principle for D-branes [17]. Minimizing the modified uncertainty relation (1.7) leads to a minimal measurable length $\Delta y^i \geq \mathcal{O}(\sqrt{\alpha'_s})$. Note that this length scale vanishes in the weak-coupling regime $g_s \to 0$, in which case there is no lower bound on the measurability of distances in the spacetime and free D-particles can probe distances smaller than the string length.

In the multi-D-particle case we shall find that the fluctuating worldsheet topologies yield the appropriate non-Abelian generalization of the result (1.7), and in addition lead to a proper quantization of the noncommutative spacetime implied by the D-brane field theory. As we will see, this leads to new forms of uncertainty relations involving only the coordinates of spacetime, in the spirit of [31,32], which are superior to the phase space uncertainty relation (1.7). The simplest such relation has the form [33]

$$\Delta y^i \Delta t \geq l_s^2 = \alpha'. \qquad (1.9)$$

The space-time uncertainty principle (1.9) follows from the energy-time uncertainty relation of quantum mechanics applied to strings, and it can be derived from very basic worldsheet conformal symmetry arguments. The same relation can be derived within the framework of the effective field theory for D-instantons [34] and it is also naturally encoded in the effective supersymmetric Yang-Mills theory for D-particles [35]. It can be shown [35] that, for the nonrelativistic scattering of two D-particles of BPS mass $1/\sqrt{\alpha'_s}$ with impact parameter of order Δy^i, the space-time uncertainty relation (1.9) gives the minimal spatial and temporal lengths

$$\Delta y^i \geq g_s^{1/3} l_s \equiv l_{\mathrm{P}}^{(11)}, \quad \Delta t \geq g_s^{-1/3} l_s \qquad (1.10)$$

where $l_{\mathrm{P}}^{(11)}$ is the 11-dimensional Planck length which is the characteristic distance scale of M theory [6]. The space-time uncertainty principle therefore implies that, for each state of a D-particle, no information can be stored below the Planck distance in the transverse space.

The following results represent the first examples of such relations within the framework of a flat space worldsheet D-brane field theory. In this σ-model formalism we shall find the appearence of quantum smearing of multi-D-particle coordinates arising from the string interactions between constituent branes. The appearance of minimal measurable spacetime lengths in this way is reminiscent of the lower bounds which arise from the existence of internal (ultraviolet regularization) symmetries of the target space [36]. The internal symmetry group is the enhanced $U(N)$ gauge symmetry which comes from the string interactions. For each constituent D-particle we shall obtain phase space and spacetime uncertainty relations of the form of Eqs. (1.7) and (1.9) when string interactions are turned on. There is no noncommutativity between different directions on a given brane and one obtains the standard stringy smearings of the coordinates. However, among the matrix off-diagonal components, representing the fundamental string degrees of freedom, there are uncertainties between different directions of the fundamental string, in addition to the usual smearing, which leads to a proper quantum noncommutativity among the D-brane fields. The open string interactions are in this way responsible for non-trivial quantum mechanical correlations between different spatial coordinate directions of the D-particles. As discussed in [37], these noncommutative uncertainty relations are determined entirely by the geometry of moduli space. The Zamolodchikov metric on this space involves the various non-trivial kinematical quantities characterizing the multi-D-brane dynamics, and it naturally encodes the small-scale structure of spacetime. The noncommutative structures of spacetime are determined by the transformations which diagonalize the Zamolodchikov metric. These noncommutative smearings arise from an expansion of the moduli space around the background of a (Lie algebraic) commutative spacetime determined as in Eq. (1.2) which has the effect of encoding the noncommutative string interactions into a gauge transformation. The gauge field interactions are then ultimately responsible for the occurence of the quantum noncommutativity. This is reminiscent of the matrix string framework for nonperturbative string theory [38,39], which encodes the geometry of the genus expansion through singular gauge transformations of commutative spacetime coordinates and naturally yields the characteristic spatial scale in Eq. (1.10) [39]. The following results therefore yield a geometric picture of the string interactions among D-branes and hence of the short-distance noncommutativity of spacetime.

The present worldsheet framework thus gives an explicit realization of the spacetime noncommutativity described in [32], where, based on very general requirements arising from the Heisenberg uncertainty principle and classical general relativity, uncertainty relations among different coordinate directions are postulated in the form

$$\sum_{i<j} \Delta y^i \Delta y^j \geqslant l_P^2. \qquad (1.11)$$

However, there are several crucial differences in the present approach. The first one is that all of our uncertainties are *derived* from statistical distribution functions that are induced from the worldsheet genus expansion, without the need of postulating auxilliary relations. In particular, we shall find uncertainties of the sort (1.11) as implied by a stronger smearing of the coordinates involving a statistical connected correlation function of the matrix fields. The present approach therefore distinguishes the quantum non-commutativity of spacetime from the algebraic one, in contrast to the approaches of [31,32,34] where the two structures are identified. Secondly, the noncommutative smearings that we find depend on the energy content of the system and suggest the emergence of quantum decoherence in multi-D-brane dynamics. In particular, we shall derive a triple spacetime uncertainty relation which implies that the scattering of D-particles at high energies can probe very small distances through their open string interactions. The emergence of decoherence effects is characteristic of certain approaches to spacetime quantum gravity, so that the present formulation of matrix D-brane dynamics seems to naturally encode the effects of quantum gravity.

The structure of the remainder of this paper is as follows. In Sec. II we briefly describe the formalism of coupling constant quantization in Liouville string theory. In Sec. III we describe the relevant brane configurations that we shall study, introducing their low-energy effective field theory (the non-Abelian Born-Infeld action) and the associated logarithmic recoil operators. In Sec. IV we carry out a detailed perturbative calculation, up to third order in the σ-model coupling constants, of the canonical momentum of the multi-D-brane system and show that the result coincides with the predictions of the symmetrized form of the non-Abelian Born-Infeld action. In Sec. V we show that the resulting moduli space dynamics takes the canonical form of that in Liouville string theory. With this correspondence established, in Sec. VI we carry out the sum over worldsheet topologies in the pinched approximation which leads to a quantization of the D-particle couplings. Then we derive the spacetime uncertainty relations and discuss their physical significances. Section VII contains some concluding remarks and possible physical tests of the noncommutativity of spacetime. At the end of the paper there are four appendices containing some of the more technical calculations. In Appendix A we describe the structure of generic correlation functions of the logarithmic operators, and in Appendix B we describe the technical details of the computation of the canonical momentum of Sec. IV, including a description of a particular renormalization scheme that must be used for the auxilliary field representation of the Wilson loop operator. Appendix C summarizes the complicated boundary integrations used in the paper, and finally in Appendix D we show how to cancel the leading modular divergences in the genus expansion of Sec. VI by imposing momentum conservation in the scattering of string states off the multiple D-brane background.

II. HELMHOLTZ CONDITIONS AND COUPLING CONSTANT QUANTIZATION FOR TWO-DIMENSIONAL σ-MODELS

In this section we will briefly review the formalism of coupling constant quantization for two-dimensional

σ-models. Consider a worldsheet σ-model that is given by a deformed conformal field theory on a compact Riemann surface Σ with metric $\gamma_{\alpha\beta}$. The deformation is described by a set of coupling constants g^i associated with vertex operators $V_i(z,\bar{z})$ that have conformal dimensions $(\Delta_i,\bar{\Delta}_i)$ and operator product expansion coefficients c^i_{jk}. The action is of the form

$$S_\sigma[x;g] = S_0[x] + \int_\Sigma d^2z\sqrt{\gamma}\, g^i V_i \tag{2.1}$$

where $S_0[x]$ is the action of the unperturbed conformal field theory and an implicit sum over repeated indices is always understood. The vertex operators V_i are constructed from the fields of $S_0[x]$. As we will discuss, because of special properties of the Zamolodchikov renormalization group flow [24], the summation over worldsheet genera leads to a canonical quantization of the system of moduli space variables $\{g^i\}$ in a non-trivial way [18,25]. In this picture the ultraviolet worldsheet renormalization group scale $\log\Lambda$ plays the role of time for the quantum mechanical system of variables $\{g^i\}$.[1]

When the vertex operators V_i describe a relevant deformation (i.e. $\Delta_i+\bar{\Delta}_i<2$), the running coupling constants $g^i(\Lambda)$ acquire non-trivial flow under the renormalization group which is described by the flat worldsheet β-function

$$\beta^i[g] \equiv \frac{dg^i}{d\log\Lambda} = (\Delta_i+\bar{\Delta}_i-2)g^i - \pi c^i_{jk}g^j g^k. \tag{2.2}$$

The flows in the space of running coupling constants interpolate between various two-dimensional renormalizable quantum field theories. Conformally invariant theories are infrared or ultraviolet fixed points of these flows. Studying the global aspects of this moduli space leads to a geometrical understanding of certain equivalences between various conformal field theories and their associated target spaces.

One can restore conformal invariance at the quantum level by including worldsheet gravitational effects and dressing the action (2.1) by Liouville theory. This amounts to dressing the vertex operators in Eq. (2.1) as $V_i \rightarrow [V_i]_\varphi$, where φ is the Liouville field which scales the worldsheet metric as

$$\gamma_{\alpha\beta} = e^{(2/\sqrt{\alpha'}\,Q)\varphi}\hat{\gamma}_{\alpha\beta} \tag{2.3}$$

with $\hat{\gamma}_{\alpha\beta}$ a fixed fiducial metric on Σ and Q is a constant related to the central charge c of the two-dimensional quantum gravity. In the Liouville framework, $\log\Lambda$ is therefore identified with the worldsheet zero mode of the Liouville field [40]. This dressing can be viewed as a renormalization of the corresponding coupling constants in Eq. (2.1) as

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$$g^i(\varphi) = g^i\, e^{\alpha_i\varphi/\sqrt{\alpha'}} + \frac{\pi/\sqrt{\alpha'}}{Q+2\alpha_i}c^i_{jk}g^j g^k \varphi e^{\alpha_i\varphi/\sqrt{\alpha'}} + \cdots. \tag{2.4}$$

The dressed deformation $[V_i]_\varphi$ is then truly marginal provided that

$$\frac{1}{2}\alpha_i(\alpha_i+Q) = \Delta_i+\bar{\Delta}_i-2. \tag{2.5}$$

The gravitationally dressed version of Eq. (2.1) is $S_0[x] + S_L[x;\varphi]$, where

$$S_L[x;\varphi] = \frac{1}{4\pi\alpha'}\int_\Sigma d^2z\sqrt{\hat{\gamma}}[\,\hat{\gamma}^{\alpha\beta}\partial_\alpha\varphi\partial_\beta\varphi - Q\varphi R^{(2)}(\hat{\gamma})]$$
$$- \frac{Q}{4\pi\alpha'}\oint_{\partial\Sigma}d\hat{s}\,\varphi K(\hat{\gamma}) + \int_\Sigma d^2z\sqrt{\hat{\gamma}}g^i(\varphi)V_i \tag{2.6}$$

is the Liouville action coupled to the "matter" part of Eq. (2.1) [26]. Here $R^{(2)}$ is the scalar curvature of the worldsheet Σ and K is the extrinsic curvature at the worldsheet boundary $\partial\Sigma$.

The most general renormalization group flow for a σ-model coupling g^i, corresponding to a vertex operator V_i, in Liouville string theory has the form of a friction equation of motion [25,41,42]

$$\alpha'\,\ddot{g}^i(\phi) + \sqrt{\alpha'}\,Q\,\dot{g}^i(\phi) = -\beta^i[g] = G^{ij}\frac{\partial}{\partial g^j}C[g;\phi] \tag{2.7}$$

where the dots denote differentiation with respect to the Liouville zero mode

$$\phi = -\sqrt{\alpha'}\,Q\log\Lambda \tag{2.8}$$

and

$$Q = (\tfrac{1}{3}|c_* - C[g;\phi]| + \tfrac{1}{4}\beta^i G_{ij}\beta^j)^{1/2} \tag{2.9}$$

is the central charge deficit with c_* the central charge at the critical dimension. The quantity $C[g;\phi]$ is the Zamolodchikov C-function [24]. It interpolates in moduli space among two-dimensional field theories on Σ according to the C-theorem, which for flat worldsheets reads

$$\frac{\partial C}{\partial\log\Lambda} = -\beta^i G_{ij}\beta^j \tag{2.10}$$

where

$$G_{ij} = \Lambda^4\langle V_i(z,\bar{z})V_j(z,\bar{z})\rangle_L \tag{2.11}$$

is the Zamolodchikov metric on moduli space. Here $\langle\,\cdot\,\rangle_L$ denotes the average in the non-critical σ-model (2.1) dressed with the Liouville action (2.6), and G^{ij} denotes the matrix inverse of Eq. (2.11).

In Eq. (2.7) we took into account the gradient flow property of the β-functions

$$\frac{\partial}{\partial g^i} C = G_{ij} \beta^j \qquad (2.12)$$

which is an off-shell corollary of the flat worldsheet C-theorem [24]. When the C-function is regarded as an effective action in moduli space, the corresponding classical equations of motion therefore coincide with the renormalization group equations $\beta^i[g]=0$. The Zamolodchikov metric (2.11) can also be used to determine the short distance behavior of 3-point correlation functions of the σ-model. For a scale-invariant field theory, the short-distance operator product expansion is

$$V_i(z_1,\bar{z}_1)V_j(z_2,\bar{z}_2) \sim c_{ij}^k z_{12}^{\Delta_i+\Delta_j-\Delta_k} \bar{z}_{12}^{\bar{\Delta}_i+\bar{\Delta}_j-\bar{\Delta}_k}$$
$$\times V_k(\tfrac{1}{2}(z_1+z_2),\tfrac{1}{2}(\bar{z}_1+\bar{z}_2)) \qquad (2.13)$$

for $|z_1| \sim |z_2|$, where

$$z_{ij}=z_i-z_j. \qquad (2.14)$$

Then the three-point function of the deformation operators

$$\langle V_i(z_1,\bar{z}_1)V_j(z_2,\bar{z}_2)V_k(z_3,\bar{z}_3)\rangle_{\mathrm{L}}$$
$$=C_{ijk}z_{12}^{\Delta_i+\Delta_j-\Delta_k}\bar{z}_{12}^{\bar{\Delta}_i+\bar{\Delta}_j-\bar{\Delta}_k}z_{23}^{\Delta_j+\Delta_k-\Delta_i}$$
$$\times \bar{z}_{23}^{\bar{\Delta}_j+\bar{\Delta}_k-\bar{\Delta}_i}z_{13}^{\Delta_i+\Delta_k-\Delta_j}\bar{z}_{13}^{\bar{\Delta}_i+\bar{\Delta}_k-\bar{\Delta}_j} \qquad (2.15)$$

can be determined as

$$C_{ijk}=c_{ij}^l G_{lk} \qquad (2.16)$$

in the limit $|z_{23}| \sim |z_{12}| \gg |z_{13}|$. The coefficients C_{ijk} are completely symmetric in their indices. From Eq. (2.16) it follows that the asymptotic behaviors of the first three sets of correlation functions of the vertex operators can be related as

$$\langle V_iV_j\rangle_{\mathrm{L}} \sim c_{ij}^k \langle V_k\rangle_{\mathrm{L}} \sim G^{kl}\langle V_iV_jV_k\rangle_{\mathrm{L}}\langle V_l\rangle_{\mathrm{L}}. \qquad (2.17)$$

It is well-known that higher-genus effects will quantize the effective coupling constants $g^i(\phi)$ [18,25]. For a full quantum description, we must ensure that Eq. (2.7), which is characteristic of frictional motion in a potential $C[g;\phi]$, is consistent with the canonical quantization conditions. We therefore need an action formalism for the renormalization group flow. In general such equations of motion cannot be cast in a Lagrangian form, but in the case of non-critical strings this is possible due to the non-trivial metric G_{ij}. In this framework, the Liouville zero mode (2.8) is identified as the physical time coordinate [25,43], observed in standard units.

The conditions for the existence of an underlying Lagrangian L whose equations of motion are equivalent (but not necessarily identical) to Eq. (2.7) are determined by the existence of a non-singular matrix ω_{ij} with

$$\omega_{ij}(\alpha'\ddot{g}^j+\sqrt{\alpha'}Q\dot{g}^j+\beta^j)=\frac{d}{d\phi}\left(\frac{\partial L}{\partial \dot{g}^i}\right)-\frac{\partial L}{\partial g^i} \qquad (2.18)$$

which obeys the Helmholtz conditions [44]

$$\omega_{ij}=\omega_{ji} \qquad (2.19)$$

$$\frac{\partial \omega_{ij}}{\partial \dot{g}^k}=\frac{\partial \omega_{ik}}{\partial \dot{g}^j} \qquad (2.20)$$

$$\frac{1}{2}\frac{D}{D\phi}\left(\omega_{ik}\frac{\partial f^k}{\partial \dot{g}^j}-\omega_{jk}\frac{\partial f^k}{\partial \dot{g}^i}\right)=\omega_{ik}\frac{\partial f^k}{\partial g^j}-\omega_{jk}\frac{\partial f^k}{\partial g^i} \qquad (2.21)$$

$$\frac{D}{D\phi}\omega_{ij}=-\frac{1}{2\alpha'}\left(\omega_{ik}\frac{\partial f^k}{\partial \dot{g}^j}+\omega_{jk}\frac{\partial f^k}{\partial \dot{g}^i}\right) \qquad (2.22)$$

where

$$f^i \equiv -\sqrt{\alpha'}Q\dot{g}^i-\beta^i[g], \qquad \frac{D}{D\phi}\equiv\frac{\partial}{\partial \phi}+\dot{g}^i\frac{\partial}{\partial g^i}+\frac{f^i}{\alpha'}\frac{\partial}{\partial \dot{g}^i}. \qquad (2.23)$$

If the conditions (2.19)–(2.22) are met, then

$$\alpha'\omega_{ij}=\frac{\partial^2 L}{\partial \dot{g}^i \partial \dot{g}^j} \qquad (2.24)$$

and the Lagrangian in Eq. (2.24) can be determined up to total derivatives according to [44]

$$\mathcal{S}\equiv\int d\phi L=-\int d\phi \int_0^1 d\kappa g^i E_i(\phi,\kappa g,\kappa\dot{g},\kappa\ddot{g})$$

$$E_i(\phi,g,\dot{g},\ddot{g})\equiv\omega_{ij}(\alpha'\ddot{g}^j+\sqrt{\alpha'}Q\dot{g}^j+\beta^j). \qquad (2.25)$$

In the case of non-critical strings one can identify [25]

$$\omega_{ij}=-\frac{1}{\sqrt{\alpha'}}G_{ij}. \qquad (2.26)$$

Near a fixed point in moduli space, where the variation of Q is small, the action (2.25) then becomes [18,25]

$$\mathcal{S}=\int d\phi\left(-\frac{\sqrt{\alpha'}}{2}\dot{g}^i G_{ij}[g;\phi]\dot{g}^j-\frac{1}{\sqrt{\alpha'}}C[g;\phi]+\cdots\right) \qquad (2.27)$$

where the dots denote terms that can be removed by a change of renormalization scheme. Within a critical string (on-shell) approach, the action (2.25), (2.27) can be considered as an effective action generating the string scattering amplitudes. Here it should be considered as a target space "off-shell"

action for non-critical strings [25]. From Eq. (2.27) it follows that the canonical momenta p_i conjugate to the couplings g^i are given by

$$p_i = \sqrt{\alpha'} \, G_{ij} \, \dot{g}^j. \qquad (2.28)$$

Let us briefly sketch the validity of the conditions (2.19)–(2.22) for the choice (2.26). Since G_{ij} is symmetric, the first Helmholtz condition (2.19) is satisfied. The conditions (2.20) and (2.21) hold automatically because of the gradient flow property (2.12) of the β-function, and the fact that G_{ij} and $C[g;\phi]$ are functions of the coordinates g^i and not of the conjugate momenta. Finally, the fourth Helmholtz condition (2.22) yields the equation

$$\frac{D}{D\phi} G_{ij} = \frac{Q}{\sqrt{\alpha'}} G_{ij} \qquad (2.29)$$

which implies an expanding scale factor for the metric in moduli space

$$G_{ij}[\phi;g(\phi)] = e^{Q\phi/\sqrt{\alpha'}} \hat{G}_{ij}[\phi;g(\phi)] \qquad (2.30)$$

where $\hat{G}_{ij}$ is a Liouville renormalization group invariant function, i.e. a fixed fiducial metric on moduli space. This is exactly the form of the Zamolodchikov metric for Liouville strings [18,40]. Thus there is an underlying Lagrangian dynamics in the non-critical string problem.

The action (2.27) allows canonical quantization, which as we have mentioned is induced by including higher genus effects in the string theory [18,25]. In the canonical quantization scheme the couplings g^i and their canonical momenta (2.28) are replaced by quantum mechanical operators (in target space) $\hat{g}^i$ and $\hat{p}_i$ obeying

$$[[\hat{g}^i, \hat{p}_j]] = i\hbar_{\mathcal{M}} \delta^i_j \qquad (2.31)$$

where the quantum commutator $[[\cdot, \cdot]]$ is defined on the moduli space $\mathcal{M}$ of deformed conformal field theories of the form (2.1), and $\hbar_{\mathcal{M}}$ is an appropriate "Planck constant." We can use the Schrödinger representation in which the canonical momentum operators obey [25]

$$\langle \hat{p}_i \rangle_{\mathrm{L}} = \left\langle -i \frac{\delta}{\delta g^i} \right\rangle_{\mathrm{L}} = \langle V_i \rangle_{\mathrm{L}}. \qquad (2.32)$$

Thus the canonical commutation relation (2.31) in general yields, on account of Eq. (2.32), a non-trivial commutator between the couplings g^i and the associated vertex operators of the (genera resummed) σ-models.

III. MATRIX σ-MODELS AND FAT BRANE DYNAMICS

To describe the moduli space dynamics of a multi-D-brane system, we shall use the construction described in [27] which for the present purposes lends the best physical interpretation. In this picture, the assembly of D-branes, including all elementary string interactions, is regarded as a composite "fat brane" which couples to a single fundamental string

with a matrix-valued coupling. In a T-dual (Neumann boundary conditions) framework,[2] the resulting effective theory is described by a σ-model on an "effective" topology of a disc, propagating in the background of a non-Abelian $U(N)$ Chan-Paton gauge field.

Consider the $U(N)$-invariant matrix σ-model action

$$S_N[X;A] = \frac{1}{4\pi\alpha'} \int_{\Sigma\{z_{ab}\}} d^2z \, \mathrm{tr} \, \eta_{\mu\nu} \partial X^\mu \bar{\partial} X^\nu$$

$$- \frac{1}{2\pi\alpha'} \oint_{\partial\Sigma\{z_{ab}\}} \mathrm{tr} \, Y_i(x^0(s)) dX^i(s)$$

$$+ \oint_{\partial\Sigma\{z_{ab}\}} \mathrm{tr} \, A^0(x^0(s)) dX^0(s) \qquad (3.1)$$

where $\eta_{\mu\nu}$ is a (critical) flat 9+1-dimensional spacetime metric. The worldsheet fields X, Y and A are $N \times N$ Hermitian matrices which transform in the adjoint representation of $U(N)$.[3] The traces in Eq. (3.1) are taken in the fundamental representation.[4] The surface $\Sigma\{z_{ab}\}$ is a sphere with a set of marked points z_{ab}, $1 \leq a,b \leq N$, on it. For each $a=b$ it has the topology of a disc Σ, while for each pair $a \neq b$ it has the topology of an annulus. The variable $s \in [0,1]$ parametrizes the circle $\partial\Sigma$. In [27] it was shown that the action (3.1) describes an assembly of N parallel D-particles with fundamental oriented open strings stretching between each pair of them. The diagonal component Y_{aa} of the matrix field Y parametrizes the Dirichlet boundary condition on D-particle a, while the off-diagonal component $Y_{ab} = Y^*_{ba}$ represents the Dirichlet boundary condition for the fundamental oriented open string whose endpoints attach to D-particles a and b. The matrix field A^0 parametrizes the usual Neumann boundary conditions in the temporal direction of the target space. The action (3.1) is written in terms of Neumann boundary conditions on the configuration fields, which is the correct description of the dynamics of the D-branes in this way, but it is straightforward to apply a functional T-duality transformation on the fields of Eq. (3.1) to express it in the usual, equivalent Dirichlet parametrization [27]. The configuration

$$A^\mu = \left(A^0, -\frac{1}{2\pi\alpha'} Y^i \right) \qquad (3.2)$$

---

[2] For subtleties in applying the T-dual picture see [9,23]. In this paper, as in [27], we assume that the Neumann picture is the fundamental picture to describe the propagation of strings in fat brane backgrounds. The Dirichlet picture is then *derived* by applying T-duality as a canonical functional integral transformation.

[3] In this paper we shall consider only the case of oriented open strings. For unoriented open strings, the global symmetry group $U(N)$ is replaced with $O(N)$ everywhere.

[4] Repeated upper and lower spacetime indices, which are raised and lowered with the flat metric $\eta_{\mu\nu}$, are always assumed to be summed over. We also normalize the generators T^a of $U(N)$ as $\mathrm{tr} \, T^a T^b = \delta^{ab}$ and hence use the flat metric δ^{ab} to raise and lower color indices.

can be interpreted as a ten-dimensional $U(N)$ isospin gauge field dimensionally reduced to the worldline of the D-particle [1,45].

However, the action (3.1) on its own does not properly take into account the interactions between the D-particles and the fundamental strings. To do so we must transform it in two ways [27]. First, we must include the sum over all worldsheet topologies, incorporating the Liouville dressing discussed in the previous section. Due to the induced quantum fluctuations of the couplings Y_i^{ab}, this provides an infinitesimal separation between the N constituent D-particles proportional to the string coupling g_s,[5] and also allows the endpoints of the fundamental strings to fluctuate in space-time. We must then integrate out all the fluctuations among the fat brane constituents, i.e. over all of the marked points of $\Sigma\{z_{ab}\}$. This necessarily makes the action non-local. By $U(N)$-invariance, the resulting σ-model partition function then becomes the expectation value, in a free (scalar) σ-model, of the path-ordered $U(N)$ Wilson loop operator $W[\partial\Sigma;A]$ along the boundary of the worldsheet disc Σ,

$$Z_N[A] \equiv \sum_{\text{genera}} \int [dX] \int_\Sigma \prod_{a,b=1}^N d^2 z_{ab} e^{-S_N[X;A]}$$

$$\simeq \langle W[\partial\Sigma;A]\rangle_0 \equiv \int Dx\, e^{-N^2 S_0[x]} \text{tr}\, P$$

$$\times \exp\left(ig_s \oint_{\partial\Sigma} A_\mu(x^0(s)) dx^\mu(s) \right) \quad (3.3)$$

where dX is the normalized invariant Haar measure for integration on the Lie algebra of $N\times N$ Hermitian matrices and

$$S_0[x] = \frac{1}{4\pi\alpha'} \int_\Sigma d^2 z\, \eta_{\mu\nu} \partial x^\mu \bar\partial x^\nu \quad (3.4)$$

is the free σ-model action for the fundamental string. The path integral measure Dx is normalized so that $\langle 1\rangle_0 = 1$. The partition function (3.3) describes the dynamics of a fat brane, which is depicted in Fig. 2.

The low-energy effective action for the D-brane configurations is now obtained by integrating out the fundamental string configurations x in Eq. (3.3). To lowest order in the gauge-covariant derivative expansion, the result is $Z_N[A] \simeq e^{-N^2\Gamma_{\text{NBI}}[A]}$, where

$$\Gamma_{\text{NBI}}[A] = \frac{c_0}{\sqrt{2\pi\alpha'}\, g_s} \int dt\, \text{tr}(\text{Sym}+i\zeta\,\text{Asym})(\text{det}_{\mu,\nu}[\,\eta_{\mu\nu}I_N$$

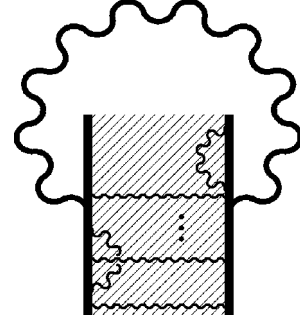$$+ 2\pi\alpha' g_s^2 F_{\mu\nu}])^{1/2} \quad (3.5)$$

FIG. 2. Schematic representation of a fat brane. The bold strips denote the assembly of N parallel D-branes and the thin wavy lines represent the fundamental strings which start and end on them. The shading represents the integration over all of these string interactions, as well as the sum over worldsheet genera. The matrix σ-model describes the interaction of the fat brane with a single fundamental string, represented by the thick wavy line, which starts and ends on the fat brane with a matrix-valued coupling constant Y.

is the non-Abelian Born-Infeld action for the dimensionally-reduced gauge field A_μ. Here c_0 is a numerical constant and $t=x^0(s=0)$ is the worldsheet zero-mode of the temporal embedding field. I_N is the $N\times N$ identity matrix, Sym denotes the symmetrized matrix product

$$\text{Sym}(M_1,\ldots,M_n) = \frac{1}{n!} \sum_{\pi\in S_n} M_{\pi_1}\cdots M_{\pi_n} \quad (3.6)$$

and Asym is the antisymmetrized matrix product

$$\text{Asym}(M_1,\ldots,M_n) = \frac{1}{n!} \sum_{\pi\in S_n} (\text{sgn}\,\pi) M_{\pi_1}\cdots M_{\pi_n}. \quad (3.7)$$

The symmetric product (and similarly for the Asym operation) on functions $f(M_1,\ldots,M_n)$ of n matrices M_k is defined by first formally expanding f as a Taylor series and then applying the Sym operation to each monomial,

$$\text{Sym}\, f(M_1,\ldots,M_n)$$

$$= \sum_{k_1,\ldots,k_n \geq 0} \frac{f^{(k_1,\ldots,k_n)}(0,\ldots,0)}{k_1!\cdots k_n!} \text{Sym}(M_1^{k_1},\ldots,M_n^{k_n}). \quad (3.8)$$

The symmetrization and antisymmetrization operations have the effect of removing the ambiguity in the definition of the spacetime determinant in Eq. (3.5) for matrices with noncommuting entries.

The components of the field strength tensor in Eq. (3.5) are given by

$$2\pi\alpha' F_{0i} = \frac{d}{dt} Y_i - ig_s[A_0, Y_i], \quad (2\pi\alpha')^2 F_{ij} = g_s[Y_i, Y_j]$$

(3.9)

and the constant $\zeta \in \mathbb{R}$ is left arbitrary so that it interpolates among the proposals for the true trace structure inherent in the non-Abelian generalization of the Born-Infeld action. The case $\zeta = 0$ corresponds to the original proposal in [8] while the trace structure with $\zeta = 1$ was suggested (in a different context) in [46]. In [9] the two-loop worldsheet β-function for the model (3.3) was calculated to be

$$\beta_i^{ab} \equiv \frac{\partial Y_i^{ab}}{\partial \log \Lambda} = -(2\pi\alpha' g_s)^2 (D^\mu F_{\mu i})^{ab}$$
$$+ 2(2\pi\alpha' g_s)^3 (D^\mu [F_{\mu\nu}, F_i^\nu])^{ab} + \mathcal{O}((\alpha' g_s)^4)$$

(3.10)

where

$$D_0 = \frac{d}{dt} - ig_s[A_0, \cdot], \quad D_i = \frac{ig_s}{2\pi\alpha'}[Y_i, \cdot] \quad (3.11)$$

are the components of the dimensionally reduced gauge-covariant derivative. It is readily seen that Eq. (3.10) coincides with the variation of the action (3.5) with $\zeta = 1$ up to the order indicated in Eq. (3.10), so that the worldsheet renormalization group equations $\beta_i^{ab} = 0$ coincide with the equations of motion of the D-branes. The first term in Eq. (3.10) yields the (reduced) Yang-Mills equations of motion, while the second term represents the first order stringy correction to the Yang-Mills dynamics. We shall return to this issue in the next section.

In this paper we will study the target space quantum dynamics from the worldsheet σ-model point of view, which will provide dynamical worldsheet origins for the noncommutativity of spacetime and matrix D-brane dynamics in general. We shall study the simplest background of a Galilean-boosted D-brane,

$$Y_i(x^0)^{ab} = Y_i^{ab} + U_i^{ab} x^0 \qquad (3.12)$$

corresponding to the case of non-relativistic heavy D-particles. The velocity matrix U_i describes the velocities of the constituent D-branes in the fat brane. Alternatively, the choice of couplings (3.12) can be thought of as parametrizing the action of the spacetime Euclidean group on the fat brane. However, this background is trivial from the point of view of the dynamics of the D-branes. In the Neumann picture the D-brane configurations are essentially gauge fields, so the only part of Eq. (3.12) which contributes to the action (3.1) is the velocity operator. But we can also Galilean transform to the rest frame where $U_i = 0$. We shall see explicitly in the next section that the quantum dynamics determined by the configuration (3.12) are trivial.

The problem is resolved by considering again the genus expansion of the matrix σ-model (3.1). An analysis of the

annulus amplitude reveals that there are logarithmic divergences arising from modular parameter integrations of the form $\int dq/q$ [14–16]. These divergences can be removed by replacing the velocity operator in Eq. (3.12) by $\lim_{\epsilon \to 0^+} U_i^{ab} D(x^0; \epsilon)$, where

$$D(x^0; \epsilon) = x^0 \Theta(x^0; \epsilon) \qquad (3.13)$$

and

$$\Theta(s; \epsilon) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{dq}{q - i\epsilon} e^{iqs} \qquad (3.14)$$

is the regulated step function, with $\Theta(s) \equiv \lim_{\epsilon \to 0^+} \Theta(s; \epsilon) = 0$ for $s < 0$ and $\Theta(s) = 1$ for $s > 0$. The infinitesimal parameter ϵ regulates the ambiguous value of $\Theta(s)$ at $s = 0$, and the integral representation (3.14) is used since x^0 will eventually be a quantum operator. When this velocity term is inserted into the boundary integral of the σ-model action, the $\epsilon \to 0^+$ divergences arising from the regulated step function can be used to cancel the logarithmic divergences of the annulus amplitudes [14–16]. This relates the target space regularization parameter ϵ to the worldsheet ultraviolet scale Λ by [17]

$$\epsilon^{-2} = -2\alpha' \log \Lambda. \qquad (3.15)$$

We shall describe these cancellations explicitly in Sec. VI.

This new velocity operator is called the impulse operator [16] and it has non-zero matrix elements between different states of the fat brane. It describes recoil effects from the emission or scattering of closed string states off the fat brane, and in an impulse approximation, it ensures that (classically) the fat brane starts moving only at time $x^0 = 0$. But this is not all that is required. The operator (3.13) on its own does not lead to a closed conformal algebra. Computing its operator product expansion with the stress-energy tensor shows [17] that it is only the *pair* of operators $D(x^0; \epsilon), C(x^0; \epsilon)$, where

$$C(x^0; \epsilon) = \epsilon \Theta(x^0; \epsilon) \qquad (3.16)$$

that define a closed algebra under the action of the worldsheet stress-energy tensor. They form a pair of logarithmic operators of the conformal field theory [12]. Thus, in order to maintain conformal invariance of the worldsheet theory, one cannot just work with the operator (3.13), because Eq. (3.16) will be induced by conformal transformations. If we rescale the worldsheet cutoff

$$\Lambda \to \Lambda' = \Lambda e^{-t/\sqrt{\alpha'}} \qquad (3.17)$$

by a linear renormalization group scale t, then Eq. (3.15) induces a transformation

$$\epsilon^2 \to \epsilon'^2 = \frac{\epsilon^2}{1 - 4\sqrt{\alpha'}\epsilon^2 t} \qquad (3.18)$$

and we find

$$D(x^0;\epsilon')=D(x^0;\epsilon)+t\sqrt{\alpha'}\,C(x^0;\epsilon),$$

$$C(x^0;\epsilon')=C(x^0;\epsilon). \qquad (3.19)$$

If we now modify the initial position of the fat brane to $\lim_{\epsilon\to 0^+}\sqrt{\alpha'}\,Y_i^{ab}C(x^0;\epsilon)$, then this scale transformation will induce, by conformal invariance, a transformation of the velocities and positions as

$$U_i\to U_i, \quad Y_i\to Y_i+U_i t \qquad (3.20)$$

i.e. a Galilean evolution of the fat brane in target space.

To properly incorporate non-trivial dynamics of the fat brane, one must therefore consider instead of Eq. (3.12) the recoil operator

$$Y_i(x^0)^{ab}=\lim_{\epsilon\to 0^+}(\sqrt{\alpha'}\,Y_i^{ab}C(x^0;\epsilon)+U_i^{ab}D(x^0;\epsilon)).$$
$$(3.21)$$

The conformal algebra reveals that the operators (3.13) and (3.16) have the same conformal dimension [17]

$$\Delta_\epsilon=-\alpha'|\epsilon|^2/2 \qquad (3.22)$$

which vanishes as $\epsilon\to 0^+$. For finite ϵ the operator (3.21), when inserted into the action (3.1), yields a deformation operator of conformal dimension $1-\alpha'|\epsilon|^2/2$ which therefore describes a relevant deformation of the σ-model and the resulting string theory is non-critical. From Eq. (3.20) it follows that the corresponding matrix-valued β-functions are

$$\beta_{Y_i}=\Delta_\epsilon Y_i+\sqrt{\alpha'}\,U_i, \quad \beta_{U_i}=\Delta_\epsilon U_i. \qquad (3.23)$$

As the ''dressing'' by the operators C and D is determined entirely by the temporal coordinate x^0, we identify this field as the Liouville field φ. Marginality of the deformation is then restored by taking the limit $\epsilon\to 0^+$. In this sense, the gravitational dressing is provided by the temporal embedding fields of the string, giving a natural interpretation to the Liouville zero mode as the time coordinate $t=x^0(s=0)$ that appears in Eq. (3.5). The relation with the worldsheet renormalization scale is then set by Eq. (3.15). Thus, if we consider the initial velocity matrix U_i of the fat brane as an unrenormalized coupling, then Eq. (3.21) is interpreted as the Liouville-dressed renormalized coupling constants (2.4) of the matrix σ-model. We shall make this correspondence somewhat more precise in Sec. VI. Some properties of the correlators of the logarithmic pair C,D, which will be required in the following, are described in Appendix A.

IV. CANONICAL MOMENTUM OF COLLECTIVE D-BRANE CONFIGURATIONS

In this section we shall compute, as prescribed in Sec. II, the canonical momenta conjugate to the matrix-valued cou-

plings Y_i in the σ-model of the previous section. The necessity to carry out this complicated calculation is many-fold. For instance, we shall see that the perturbative theory requires a renormalization of the D-brane couplings, which is unambiguously fixed by the momentum. This will be important in the following sections where we shall map the fat brane problem onto the Liouville string problem of Sec. II. Furthermore, this quantity enables the most direct comparison with the non-Abelian Born-Infeld theory and illustrates the usage of the generic features of logarithmic conformal field theory in the calculation of matrix D-brane dynamical quantities.

A. Perturbation expansion

We shall need a proper path integral representation of the $U(N)$ Wilson loop operator, representing the pertinent vertex operator for the description of a system of N D-branes in the σ-model framework. We introduce one-dimensional complex auxilliary fields $\bar{\xi}_a(s),\xi_a(s)$ on the boundary $\partial\Sigma$ of the worldsheet. They transform in the fundamental representation of the $U(N)$ gauge group, and their propagator is

$$\langle\!\langle\bar{\xi}_a(s_1)\xi_b(s_2)\rangle\!\rangle\equiv\lim_{\epsilon\to 0^+}\int D\bar{\xi}D\xi\,\bar{\xi}_a(s_1)\xi_b(s_2)$$

$$\times\exp\left(-\sum_{c=1}^N\int_0^1 ds\,\bar{\xi}_c(s-\epsilon)\frac{d}{ds}\xi_c(s)\right)$$

$$=\delta_{ab}\Theta(s_2-s_1) \qquad (4.1)$$

where again ϵ regulates the ambiguous value of $\Theta(s)$ at $s=0$.

Using the propagator (4.1) and Wick's theorem we can undo the path ordering in the Wilson loop operator in Eq. (3.3) by writing it as [20–23]

$$W[\partial\Sigma;A]=\frac{1}{N\,\mathcal{W}_{U(1)}[\partial\Sigma;A]}\lim_{\epsilon\to 0^+}$$

$$\times\left\langle\!\left\langle\sum_{c=1}^N\bar{\xi}_c(0)\exp\left(ig_s\sum_{a,b=1}^N\int_0^1 ds\,\bar{\xi}_a\right.\right.\right.$$

$$\left.\left.\left.\times(s-\epsilon)A_\mu^{ab}(x^0(s))\xi_b(s)\frac{d}{ds}x^\mu(s)\right)\xi_c(1)\right\rangle\!\right\rangle.$$
$$(4.2)$$

This representation of the Wilson loop operator also requires a renormalization scheme for the auxilliary quantum field theory which we describe in Appendix B. It puts the partition function (3.3) into the form of a functional integral over a local action. Note that it corresponds to the partition function for the boundary fields $\bar{\xi},\xi$ minimally coupled to the gauge field A_μ. The additional factor

$$\mathcal{W}_{U(1)}[\partial\Sigma;A] = \lim_{\epsilon\to 0^+}\left\langle\!\!\left\langle\exp\!\left(ig_s\sum_{a,b=1}^{N}\int_0^1 ds\,\bar{\xi}_a\right.\right.\right.$$

$$\left.\left.\left.\times(s-\epsilon)A_\mu^{ab}(x^0(s))\xi_b(s)\frac{d}{ds}x^\mu(s)\right)\right\rangle\!\!\right\rangle \tag{4.3}$$

is induced by the vacuum graphs of the auxilliary quantum field theory. With a periodic definition of the step function $\Theta(s)$ on the circle $\partial\Sigma$ [for instance with a discretized version of Eq. (3.14)], the auxilliary fields induce loop contractions of the color indices of the gauge field A_μ leading to the $U(1)$ subgroup projection (4.3) of the Wilson loop operator.

This formalism gives a one-parameter family of Dirichlet boundary conditions for the fundamental string fields, labelled by $s\in[0,1]$, in the corresponding T-dual formalism [21,22], i.e. the dual configuration fields are

$$\tilde{Y}_i(x^0;s) = \lim_{\epsilon\to 0^+}\sum_{a,b=1}^{N}\bar{\xi}_a(s-\epsilon)Y_i^{ab}(x^0(s))\xi_b(s). \tag{4.4}$$

Now, instead of being forced to sit on a unique hypersurface as in the Abelian D-brane case, there is an infinite set of hypersurfaces on which the string endpoints are situated. Alternatively, we obtain a one-parameter family of bare matrix-valued vertex operators

$$V_{ab}^i(x;s) = \frac{ig_s}{2\pi\alpha'}\frac{d}{ds}x^i(s)\lim_{\epsilon\to 0^+}\bar{\xi}_a(s-\epsilon)\xi_b(s) \tag{4.5}$$

and renormalized matrix couplings (3.21). Thus the trade-off for removing the non-locality of the effective theory (3.3) is the extra explicit boundary dependence of operators involved.

We will use the representation (4.2) to compute the classical canonical momentum $\Pi_{ab}^j(s)$ in the moduli space of the collective D-brane configurations $Y_j^{ab}(s)$. According to Eq. (2.32), the momentum can be computed as the one-point function of the deformation vertex operators (4.5) in the statistical ensemble (3.3),

$$\Pi_{ab}^j(s) \equiv N\mathcal{W}_{U(1)}[\partial\Sigma;A]\left(-\frac{\delta}{\delta Y_j^{ab}(x^0(s))}Z_N[A]\right)$$

$$= \tilde{\Pi}_{ab}^j(s) - N\left\langle W[\partial\Sigma;A]\right.$$

$$\left.\times\left(-\frac{\delta}{\delta Y_j^{ab}(x^0(s))}\mathcal{W}_{U(1)}[\partial\Sigma;A]\right)\right\rangle_0 \tag{4.6}$$

where

$$\tilde{\Pi}_{ab}^j(s) = \lim_{\epsilon\to 0^+}\sum_{c=1}^{N}\left\langle\!\!\left\langle\!\!\left\langle\bar{\xi}_c(0)V_{ab}^j(x;s)\right.\right.\right.$$

$$\times\exp\!\left(ig_s\sum_{d,e=1}^{N}\int_0^1 ds'\,\bar{\xi}_d(s'-\epsilon)\right.$$

$$\left.\times A_\mu^{de}(x^0(s'))\xi_e(s')\frac{d}{ds'}x^\mu(s')\right)\xi_c(1)\left.\right\rangle\!\!\right\rangle\!\!\right\rangle_0 \tag{4.7}$$

is the contribution from the $SU(N)$ part of the gauge group. The second term in Eq. (4.6) involves traces of the gauge field A_μ which we identify as the center of mass coordinates of the fat brane, i.e. $Y_j^{cm}\equiv(1/N)\mathrm{tr}\,Y_j$. The expression (4.6) thus shows that the momenta of the collective center of mass motion of the fat brane and of the constituent D-branes comprising the fat brane completely decouple. In this paper we shall be interested in only the former contribution, since the latter one essentially represents the dynamics of a single D-brane [i.e. gauge group $U(1)$] and here we are interested in the non-Abelian modification determined by the constituent D-particles. In effect we restrict attention to unimodular Wilson loops [i.e. gauge group $SU(N)$]. For these terms the statistics of the auxilliary boundary fields $\bar{\xi},\xi$ are irrelevant.

From now on we shall work in the static gauge $A_0=0$ for the dimensionally reduced gauge field. Then the canonical momentum (4.7) can be expanded as the power series

$$\tilde{\Pi}_{ab}^j(s) = -\sum_{n=1}^{\infty}\frac{(-i)^{n+1}}{n!}\left(\frac{g_s}{2\pi\alpha'}\right)^{n+1}\mathcal{P}_{ab}^{(n)j}[Y;s] \tag{4.8}$$

where the $\mathcal{O}(Y(x^0)^n)$ contribution is

$$\mathcal{P}_{ab}^{(n)j}[Y;s] = \lim_{\epsilon\to 0^+}\sum_{c=1}^{N}\sum_{\substack{a_1,\ldots,a_n\\b_1,\ldots,b_n}}\int_0^1\prod_{k=1}^{n}ds_k$$

$$\times\left\langle\!\!\left\langle\bar{\xi}_c(0)\bar{\xi}_a(s-\epsilon)\xi_b(s)\left(\prod_{k=1}^{n}\bar{\xi}_{a_k}(s_k-\epsilon)\right.\right.\right.$$

$$\left.\left.\left.\times\xi_{b_k}(s_k)\right)\xi_c(1)\right\rangle\!\!\right\rangle$$

$$\times\left\langle\frac{d}{ds}x^j(s)\prod_{k=1}^{n}Y_{i_k}^{a_kb_k}(x^0(s_k))\frac{d}{ds_k}x^{i_k}(s_k)\right\rangle_0. \tag{4.9}$$

The correlation functions appearing in Eq. (4.9) can be evaluated using Wick's theorem and the propagator (4.1) to write the auxilliary field averages as a sum over permutations,

$$\left\langle\left\langle \prod_{k=1}^{m} \bar{\xi}_{a_k}(s_k)\xi_{b_k}(s'_k) \right\rangle\right\rangle = \sum_{P\in S_m} \prod_{k=1}^{m} \delta_{a_k,b_{P(k)}}\Theta(s'_{P(k)}-s_k).$$

(4.10)

The evaluation of the momentum contribution (4.9) is rather technically involved and is presented in Appendix B. It is also shown there that one must further specify a renormalization of the auxilliary quantum field theory in order to remove step function ambiguities which come from the correlation functions (4.10). The resulting renormalized expression is finite at $s=0$. This point defines the (renormalized) target space coordinates $\bar{\Pi}_{ab}^j\equiv\bar{\Pi}_{ab}^j(s=0)^{\text{ren}}$ as the zero modes of the worldsheet fields. As shown in Appendix B, the order n contribution is given by

$$\mathcal{P}_{ab}^{(n)j}[Y;0]^{\text{ren}} = \int_0^1 \prod_{k=1}^n ds_k \left\langle \frac{dx^j(s)}{ds}\bigg|_{s=0} \right.$$
$$\left. \times \text{Sym}\left[\prod_{k=1}^n Y_{i_k}(x^0(s_k))\frac{d}{ds_k}x^{i_k}(s_k)\right]_{ba} \right\rangle_0.$$

(4.11)

This is to be compared with the corresponding expression in the Abelian case (corresponding to a single D-particle, $N=1$) for which there is no matrix-ordering problem and the expansion of the Abelian Wilson loop operator proceeds directly without the need of an auxilliary field representation. We see that the properly renormalized momentum (4.11) is a natural non-Abelian generalization of the corresponding single D-particle quantity, to which it reduces in the limit $N=1$. Physically, the symmetrization of the amplitude occurs because the correlators involve bosonic fields.

In the following we will, for simplicity, normalize $\eta^{00}=1$ and assume that the target space temporal and spatial embedding fields are uncorrelated, i.e. $\eta^{0i}=\eta^{i0}=0$. The resulting time-space factorization of correlators implies that Eq. (4.11) is non-vanishing only when n is odd. Note that for the configurations (3.12) we have $\tilde{\Pi}^i\equiv 0$, since all the periodic boundary integrations in Eq. (4.11) then vanish. When the fat brane configuration is given by the non-trivial recoil operator (3.21), the correlation functions of the logarithmic operators C and D can be evaluated using the results of Appendix A. In particular, the correlation functions involving only $C(x^0;\epsilon)$ operators vanish as $\epsilon\to 0^+$. This means that the canonical momentum vanishes at zero velocities, as expected from physical considerations. It is a nontrivial function which mixes the velocities and positions of the fat brane. This explicit vanishing of the correlators of the C operator is required to consistently yield the correct Abelian limit in which the momentum depends only on the velocity.

B. Velocity renormalization

In this subsection we will consider the lowest non-trivial contribution (4.11), which using Eqs. (A17) and (A4) can be written as

$$\mathcal{P}_{ab}^{(1)j}(Y,U;0)^{\text{ren}} = \lim_{\epsilon\to 0^+} [U_i]_{ba}\int_0^1 ds' \langle D(s';\epsilon)\rangle_0$$
$$\times\left\langle \frac{d}{ds}x^j(s)\bigg|_{s=0} \frac{d}{ds'}x^i(s') \right\rangle_0$$
$$= \lim_{\epsilon\to 0^+} \frac{4\pi^2 a\alpha'}{\epsilon}U_{ba}^j\frac{1}{\pi\tan\pi s_\Lambda}$$

(4.12)

where a is an arbitrary constant and s_Λ is the short-distance cutoff (A16) on $\partial\Sigma$. The divergent term as $\Lambda\to 0$ is the boundary version of the bulk logarithmic divergence $\log\Lambda$. Naively, the boundary integral in Eq. (4.12) vanishes since its integrand is a total derivative. However, the one-point function of the logarithmic D operator is divergent as $\epsilon\to 0^+$ and one must carefully regularize the boundary integration. The boundary regulator s_Λ is also correlated with the target space regularization parameter ϵ as in the bulk equation (3.15). Although s_Λ is given explicitly by Eq. (A16), we shall assume that the bulk and boundary cutoffs are independent and take $\tan\pi s_\Lambda\sim\epsilon^2$. This usage of the logarithmic correlation functions will be the key feature in the determination of the matrix D-brane dynamics.

The resulting expression (4.12) diverges as $\epsilon\to 0^+$. Part of this divergence can be removed by renormalizing the velocity matrix of the D-branes as

$$U_i = \sqrt{\alpha'}\,\epsilon\bar{U}_i.$$

(4.13)

From Eqs. (3.15) and (3.23) we see that this renormalized coupling constant is truly marginal,

$$\frac{d\bar{U}_i}{dt} = 0$$

(4.14)

where $t=-\sqrt{\alpha'}\log\Lambda$, and it therefore plays the role of a uniform velocity for the fat brane dynamics. From Eq. (4.8) we see that the remaining ϵ^{-2} divergence can be absorbed into a renormalization of the string coupling constant as

$$g_s = \sqrt{\alpha'}\,\epsilon\,\bar{g}_s.$$

(4.15)

As we will see in Sec. VI, $\bar{g}_s$ is also a truly marginal coupling. Thus we see that, after a suitable renormalization of the logarithmic deformation, the leading order contribution to the canonical momentum (4.8) is just the constant velocity of the Galilean boosted fat brane, which coincides with the corresponding result for a single non-relativistic heavy D-particle [28].

C. Logarithmic algebra

We now examine the leading order corrections to the velocity of the fat brane, which are given by

$$\mathcal{P}_{ab}^{(3)j}(Y,U;0)^{\mathrm{ren}} = \lim_{\epsilon\to 0^+} \int_0^1 ds_1\, ds_2\, ds_3 \left\langle \frac{d}{ds}x^j(s)\bigg|_{s=0} \frac{d}{ds_1}x^{i_1}(s_1)\frac{d}{ds_2}x^{i_2}(s_2)\frac{d}{ds_3}x^{i_3}(s_3)\right\rangle_0$$

$$\times \mathrm{Sym}\big[\,\alpha'\langle C(s_1;\epsilon)C(s_2;\epsilon)D(s_3;\epsilon)\rangle_0 (Y_{i_1}Y_{i_2}U_{i_3}+Y_{i_1}U_{i_3}Y_{i_2}+U_{i_3}Y_{i_2}Y_{i_1})$$

$$+\sqrt{\alpha'}\langle C(s_1;\epsilon)D(s_2;\epsilon)D(s_3;\epsilon)\rangle_0 (Y_{i_1}U_{i_2}U_{i_3}+U_{i_2}Y_{i_1}U_{i_3}+U_{i_3}U_{i_2}Y_{i_1})$$

$$+\langle D(s_1;\epsilon)D(s_2;\epsilon)D(s_3;\epsilon)\rangle_0 U_{i_1}U_{i_2}U_{i_3}\big]_{ba}. \tag{4.16}$$

Using Wick's theorem, the propagator (A17) and the three-point functions (A12)–(A14) of the logarithmic operators, after some lengthy tedious algebra we can write Eq. (4.16) as

$$\mathcal{P}_{ab}^{(3)j}(Y,U;0)^{\mathrm{ren}} = \lim_{\epsilon\to 0^+} (4\pi^2\alpha')^2\Big[\,\alpha'(d\epsilon+c\epsilon^3\alpha'\log\Lambda)I_0(6\,Y^jY_iU^i+3\,Y_iY^iU^j+3Y^j[U_i,Y^i]+Y_i[U^j,Y^i]+[Y_iU^i,Y^j]$$

$$+[U^j,Y_iY^i]+[U_i,Y^j]Y^i+[U_iY^i,Y^j]+[Y_i,Y^j]U^i)+\frac{c}{2}\epsilon^3\alpha'^2 I_u^{(1)}(3\,Y_iY^iU^j-6\,Y^jY_iU^i-[Y_i,Y^j]U^i$$

$$+Y_i[U^j,Y^i]-[Y_iU^i,Y^j]+[U^j,Y_iY^i]-[U_i,Y^j]Y^i-[U_iY^i,Y^j]-3Y^j[U_i,Y^i])-c\epsilon^3\alpha'^2 I_c^{(1)}(3\,Y_iY^iU^j$$

$$+Y_i[U^j,Y^i]+[U^j,Y_iY^i])+\sqrt{\alpha'}\Big(\frac{e}{\epsilon}+2d\epsilon\alpha'\log\Lambda+c\epsilon^3\alpha'^2(\log\Lambda)^2\Big)I_0(6\,Y_iU^iU^j+3\,Y^jU_iU^i+Y_i[U^j,U^i]$$

$$+[U^j,Y_iU^i]+U_i[U^j,Y^i]+3[U_i,Y^i]U^j+[U^j,U_iY^i]+[U_iU^i,Y^j]+[U_i,Y^j]U^i)+\sqrt{\alpha'}\Big(c\epsilon^3\alpha'^2 I_q^{(1)}-(d\epsilon\alpha'$$

$$+c\epsilon^3\alpha'^2\log\Lambda)I_u^{(1)}+\frac{c}{4}\,\epsilon^3\alpha'^2 I_u^{(2)}\Big)(3\,Y^jU_iU^i+[U_i,Y^j]U^i+[U_iU^i,Y^j])+\sqrt{\alpha'}\Big(\frac{c}{2}\epsilon^3\alpha'^2(I_q^{(2)}+I_q^{(3)})$$

$$-(d\epsilon\alpha'+c\epsilon^3\alpha'^2\log\Lambda)I_c^{(1)}+\frac{c}{4}\epsilon^3\alpha'^2 I_c^{(2)}\Big)(6Y_iU^iU^j+Y_i[U^j,U^i]+[U^j,Y_iU^i]+U_i[U^j,Y^i]+3[U_i,Y^i]U^j$$

$$+[U^j,U_iY^i])-(U_iU^iU^j+U_iU^jU^i+U^jU_iU^i)\Big\{I_0\Big(\frac{f}{\epsilon^3}+\frac{3e}{\epsilon}\alpha'\log\Lambda+3d\epsilon\alpha'^2(\log\Lambda)^2+c\epsilon^3\alpha'^3(\log\Lambda)^3\Big)$$

$$-(I_u^{(1)}+2I_c^{(1)})\Big(\frac{e}{2\epsilon}+d\epsilon\alpha'^2\log\Lambda+\frac{c}{2}\epsilon^3\alpha'^3(\log\Lambda)^3\Big)+\frac{3}{4}(2I_u^{(2)}+I_c^{(2)})(d\epsilon\alpha'^2-c\epsilon^3\alpha'^3\log\Lambda)$$

$$+c\epsilon^3\alpha'^3\Big(\Big(I_m^{(1)}+\frac{1}{2}I_m^{(2)}\Big)\log\Lambda-\frac{1}{2}I_t^{(1)}-\frac{3}{4}(I_t^{(2)}+I_t^{(3)}+I_t^{(4)})+\frac{1}{8}(2I_t^{(5)}+I_t^{(6)})\Big)\Big\}\Big]_{ba}. \tag{4.17}$$

The quantities denoted by I in Eq. (4.17) are the various boundary integrals that arise and are summarized in Appendix C. The constants $c,d,\ldots$ come from the correlation functions of the logarithmic operators. These constants are for the most part arbitrary integration constants, the remaining ones being fixed by the leading logarithmic terms in the conformal blocks. We shall eliminate the arbitrary ones by demanding that, in the limit $N=1$, Eq. (4.17) reproduce the appropriate result anticipated from Abelian Born-Infeld theory, i.e. that only the U^jU^2 term in Eq. (4.17) survives in the Abelian reduction. In doing so, we assume a more general logarithmic deformation structure than that given by the recoil operators of the previous section, but the qualitative (and most quantitative) features remain the same.

Let us start with the first set of Y^2U type terms. From the discussion of the previous subsection and Eq. (3.15) it follows that the bulk and boundary ultraviolet cutoff scales are related as

$$4\mu\log\Lambda = \frac{1}{\tan\pi s_\Lambda} \tag{4.18}$$

where μ is a real-valued constant to be determined. Using Eqs. (C4)–(C6), it then follows that the Y^2U part of Eq. (4.17) reduces to

$$\lim_{\epsilon \to 0^+} \frac{4\alpha'}{\pi^3} (4\pi^2\alpha')^2 \frac{\log\Lambda}{\tan\pi s_\Lambda} [(d\epsilon + c\,\epsilon^3\alpha'\log\Lambda)(6\,Y^jY_iU^i + 3\,Y_iY^iU^j + 3Y^j[U_i,Y^i] + Y_i[U^j,Y^i] + [Y_iU^i,Y^j] + [U^j,Y_iY^i]$$

$$+ [U_i,Y^j]Y^i + [U_iY^i,Y^j] + [Y_i,Y^j]U^i) - c\,\epsilon^3\alpha'\log\Lambda((3-6\mu)Y_iY^iU^j - 6\,Y^jY_iU^i - [Y_i,Y^j]U^i + (1-2\mu)Y_i[U^j,Y^i]$$

$$- [Y_iU^i,Y^j] + (1-2\mu)[U^j,Y_iY^i] - [U_i,Y^j]Y^i - [U_iY^i,Y^j] - 3\,Y^j[U_i,Y^i])]_{ba}. \tag{4.19}$$

In the Abelian limit $N=1$, all commutators in Eq. (4.19) vanish. Requiring that the coefficients of the $Y^jY_iU^i$ and $Y_iY^iU^j$ terms vanish leads, respectively, to the equations

$$6d\epsilon + 12c\,\epsilon^3\alpha'\log\Lambda = 0, \quad 3d\epsilon + 6\mu c\,\epsilon^3\alpha'\log\Lambda = 0 \tag{4.20}$$

which for finite ϵ have unique solution

$$\mu = 1, \quad d = -2c\,\epsilon^2\alpha'\log\Lambda. \tag{4.21}$$

Here we have used the fact that the constant c is determined by the leading logarithmic terms in the conformal blocks of the logarithmic conformal algebra generated by the C and D operators, and hence that $c \neq 0$. We see that the arbitrariness of certain integration constants which appear from the logarithmic conformal algebra can be fixed by the appropriate Abelian reduction requirement. Substituting Eq. (4.21) into Eq. (4.19) we see that the set of Y^2U type terms in fact vanishes identically.

Next we examine the second set of YU^2 type terms in Eq. (4.17). Using Eqs. (4.18), (4.21), the integrals (C1)–(C11) and dropping those terms which vanish as $\epsilon \to 0^+$ relative to the rest, we arrive after some algebra at the expression

$$\lim_{\epsilon \to 0^+} \frac{4\sqrt{\alpha'}}{\pi^3} (4\pi^2\alpha')^2 \frac{\log\Lambda}{\tan\pi s_\Lambda} \left[\left(\frac{e}{\epsilon} - \frac{5}{3} c\,\epsilon^3\alpha'^2(\log\Lambda)^2 \right) \right.$$

$$\times (6\,Y_iU^iU^j + 3\,Y^jU_iU^i + Y_i[U^j,U^i] + [U^j,Y_iU^i]$$

$$+ U_i[U^j,Y^i] + 3[U_i,Y^i]U^j + [U^j,U_iY^i] + [U_iU^i,Y^j]$$

$$\left. + [U_i,Y^j]U^i) \right]_{ba}. \tag{4.22}$$

The reproduction of the correct Abelian limit requires the equality

$$e = \tfrac{5}{3} c\,\epsilon^4\alpha'^2(\log\Lambda)^2 \tag{4.23}$$

of the parameters of the logarithmic conformal algebra. As in Eq. (4.19), this restriction forces the entire contribution (4.22) to vanish identically for all N.

Thus, with the parameters of the logarithmic deformations fixed according to Eqs. (4.21) and (4.23), the only contribution to the $n=3$ canonical momentum is from the cubic velocity terms in Eq. (4.17), which we evaluate using Eq. (4.18) and the boundary integrals (C1)–(C19). Using Eq. (3.15) and absorbing the remaining ϵ^{-7} divergence in the total momentum (4.8) using the renormalizations (4.13) and (4.15), we arrive finally at

$$\mathcal{P}_{ab}^{(3)j}(Y,U;0)^{\mathrm{ren}} = -128\pi\alpha'^{3/2} \left(f + \frac{139}{8}c \right) [\bar{U}_i\bar{U}^i\bar{U}^j$$

$$+ \bar{U}_i\bar{U}^j\bar{U}^i + \bar{U}^j\bar{U}_i\bar{U}^i]_{ba}. \tag{4.24}$$

The sum of Eqs. (4.12) and (4.24) now involve three constants a, c and f determined from the logarithmic conformal algebra. We can fix another one of them by requiring that, again in the Abelian limit, one recovers the well-known result predicted from Abelian Born-Infeld theory. One finds (see the next subsection) that the relative coefficient between the $\bar{U}^j$ and $\bar{U}^j\bar{U}_i\bar{U}^i$ terms in the Abelian theory should be $\frac{1}{2}$, which imposes the additional constraint

$$8f + 139c = 64\pi^2a. \tag{4.25}$$

The results above now yield the total canonical momentum (4.8) up to order 3 as

$$\bar\Pi_{ab}^j(\bar{Y},\bar{U}) = \frac{4a\bar{g}_s^2}{\pi\sqrt{\alpha'}} \left[\bar{U}^j + \frac{\bar{g}_s^2}{6}(3\bar{U}_i\bar{U}^i\bar{U}^j + [\bar{U}_i,[\bar{U}^j,\bar{U}^i]]) \right]_{ba}$$

$$+ \mathcal{O}(\bar{g}_s^6). \tag{4.26}$$

The expression (4.26) involves one parameter a determined by the one-point function of the logarithmic D operator. The remaining parameters of the logarithmic conformal algebra that enter into the three-point functions (A11)–(A14) are determined by Eqs. (4.21), (4.23) and (4.25). In this way the matrix D-brane dynamics fixes most of the algebraic information about the logarithmic deformation and localizes the problem to a small region of moduli space. The fact that these parameters are scale-dependent is a general feature of logarithmic conformal field theories [47]. Note that they become scale-independent though with the correlation (3.15).

D. Canonical momentum in non-Abelian Born-Infeld theory

Let us now compare the perturbative result (4.26) to that which comes from the non-Abelian Born-Infeld action (3.5). For this, we expand the spacetime determinant in Eq. (3.5) as a series in powers of $F_{\mu\nu}$ to get

$$\frac{1}{\sqrt{2\pi\alpha'}\,g_s}(\det_{\mu,\nu}[\,\eta_{\mu\nu}I_N+2\pi\alpha'\,g_s^2F_{\mu\nu}])^{1/2}$$

$$=(\sqrt{2\pi\alpha'}\,g_s)^3\Bigg[(2\pi\alpha'\,g_s^2)^{-2}I_N+\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

$$+\frac{1}{12}(2\pi\alpha'g_s^2)[F_{\mu\nu},F^{\nu\lambda}]F_\lambda^\mu$$

$$+\frac{1}{32}(2\pi\alpha'g_s^2)^2((F_{\mu\nu}F^{\mu\nu})^2-4F_{\mu\nu}F^{\nu\lambda}F_{\lambda\rho}F^{\rho\mu})$$

$$+\mathcal{O}((2\pi\alpha'g_s^2)^3)\Bigg]. \qquad (4.27)$$

Since $F_{\mu\nu}=-F_{\nu\mu}$, the symmetrization operation picks out the even powers of the field strength while the antisymmetrized product picks out the odd ones. Using Eq. (3.9) in the gauge $A_0=0$, after some algebra we find that the expansion of the action (3.5) to leading orders in the string coupling constant is

$$\Gamma_{\rm NBI}[Y]=c_0(\sqrt{2\pi\alpha'}\,g_s)^3\int dt\Bigg(N(2\pi\alpha'\,g_s^2)^{-2}$$

$$+\frac{1}{2}\left(\frac{g_s}{2\pi\alpha'}\right)^2{\rm tr}\,\dot{Y}_i\dot{Y}^i$$

$$+\frac{1}{16}\left(\frac{g_s^2}{2\pi\alpha'}\right)^2\Bigg\{{\rm tr}(2\,\dot{Y}_i\dot{Y}^i\dot{Y}_j\dot{Y}^j+\dot{Y}_i\dot{Y}_j\dot{Y}^i\dot{Y}^j)$$

$$+\frac{i\zeta}{3}\frac{1}{2\pi\alpha'}{\rm tr}[Y_i,Y_j][\dot{Y}^i,\dot{Y}^j]\Bigg\}+\mathcal{O}(g_s^6)\Bigg) \qquad (4.28)$$

where $\dot{Y}_i\equiv(d/dt)Y_i$. The perturbative expansion of the canonical momentum in non-Abelian Born-Infeld theory can now be calculated from Eq. (4.28) and after some algebra we find

$$\Pi_{ab}^j(t)\equiv\frac{\delta\left(\dfrac{1}{g_s^3}\Gamma_{\rm NBI}[Y]\right)}{\delta\dot{Y}_j^{ab}(t)}$$

$$=\frac{c_0g_s^2}{\sqrt{2\pi\alpha'}}\Bigg\{\dot{Y}_{ba}^j+\frac{g_s^2}{6}\Bigg([\dot{Y}_i\dot{Y}^i\dot{Y}^j+\dot{Y}_i\dot{Y}^j\dot{Y}^i+\dot{Y}^j\dot{Y}_i\dot{Y}^i]_{ba}$$

$$+\frac{i\zeta}{2\pi\alpha'}[\dot{Y}_i,[Y^j,Y^i]]_{ba}\Bigg)\Bigg\}+\mathcal{O}(g_s^6). \qquad (4.29)$$

In particular, for the case of the D-particle configurations (3.12) corresponding to a Galilean-boosted fat brane, we have

$$\Pi_{ab}^j(t)=\frac{c_0g_s^2}{\sqrt{2\pi\alpha'}}\Bigg(U_{ba}^j+\frac{g_s^2}{6}\Bigg[U_iU^iU^j+U_iU^jU^i+U^jU_iU^i$$

$$+\frac{i\zeta}{2\pi\alpha'}\{[U_i,[Y^j,Y^i]]+([U_i,[U^j,Y^i]]$$

$$+[U_i,[Y^j,U^i]])t+[U_i,[U^j,U^i]]t^2\}\Bigg]_{ba}\Bigg)+\mathcal{O}(g_s^6). \qquad (4.30)$$

We see that the canonical momenta (4.26) and (4.30) agree, up to an overall normalization, when

$$\zeta=0 \qquad (4.31)$$

which corresponds to taking only the symmetrized trace in Eq. (3.5). The possible occurrence of the extra antisymmetrized trace structure in Eq. (3.5) was pointed out in [9] where the worldsheet β-functions (3.10) were computed. As noted there, however, when one properly takes into account the worldsheet fermionic fields for the full superstring theory, it is only the symmetrized trace structure that survives. This feature was elucidated on in [10] where it was shown that the symmetrized action is the only potential generalization for which BPS configurations linearize the non-Abelian Born-Infeld action and minimize its energy. Here we have shown that, within the auxilliary field formalism for the worldsheet matrix σ-model, there exists a particular regularization of the auxilliary quantum field theory which agrees with the results predicted by the symmetrized action, without the need of introducing worldsheet supersymmetry.

There may of course be other regularizations of the auxilliary quantum field theory which reproduce the antisymmetrized trace structure in Eq. (3.5), but we have not been able to find any such one. The renormalization described in Appendix B is the most natural scheme that one can impose and the symmetrized matrix products which occur are natural from the perspective of representing bosonic string amplitudes. It is also that which naturally leads to the correct Abelian reduction of the theory. The full, unrenormalized expression for the canonical momentum in the matrix σ-model is given in Appendix B. To further check the validity of the non-Abelian Born-Infeld action, one would need to extend the calculation of $\mathcal{P}_{ab}^{(n)j}(Y,U;0)^{\rm ren}$ up to $n=5$. This in turn would require explicit knowledge of the five-point correlation functions of the logarithmic operators, which are extremely complicated (see Appendix A), and the calculations at higher orders of perturbation theory become overwhelmingly tedious and difficult to manage.

In any case, the results of this section illustrate a nontrivial application of logarithmic conformal field theory to the study of solitonic states in string theory. We note that the results derived in this section are invariant under T-duality transformations of the string theory. In [23] it was pointed out that an alternative functional integral representation of the quantum D-particle dynamics is given by a σ-model action defined with a non-Abelian Wilson loop operator that

has normal boundary derivatives $\partial_\perp x^i$ for the relevant deformation vertex operators. This model corresponds to the imposition of dynamical Dirichlet boundary conditions, rather than dynamical Neumann ones as in Eq. (3.3) which are equivalent (by T-duality) to the imposition of external Dirichlet boundary conditions. In contrast to the Abelian case, these two models are inequivalent beyond the tree level because of anomalous Jacobian factors in the path integral measure which arise in the non-Abelian case. By careful investigation of the worldsheet β-functions it has been argued in [23] (see also [9]) that the model with dynamical Dirichlet boundary conditions constitutes the appropriate T-dual description of the quantum D-brane dynamics represented by the open string model with free (Neumann) boundary conditions. It is straightforward to see that the perturbative expansion of the canonical momentum in the theory with boundary normal derivatives is equivalent to the one employed in this section, since the boundary correlation functions involved are the same. The results described in this section are therefore independent of which picture one chooses to work in.

V. DYNAMICS ON MODULI SPACE

We can learn more about the fat brane dynamics by studying the structure of the moduli space $\mathcal{M}$ determined by the (dressed) matrix D-brane configurations. Assuming the generic D-brane couplings to admit decompositions (3.21) into pairs of logarithmic operators, this space is the direct sum

$$\mathcal{M}=\mathcal{M}_C \oplus \mathcal{M}_D \tag{5.1}$$

of two subspaces which each have classical dimension $9N^2$. According to the results of the previous section, to lowest

order in the string coupling g_s the decomposition (5.1) can be interpreted as the splitting of the fat-brane collective coordinates into phase space degrees of freedom. However, this will not be true at higher-orders and in general Eq. (5.1) represents a non-trivial mixing between configuration space and phase space variables. As discussed in [14], the logarithmic nature of the deformation makes the geometry on the space (5.1) well-defined.

The Zamolodchikov metric on $\mathcal{M}$ is given by the two-point function of the deformation vertex operators (4.5),

$$G^{ij}_{ab;cd}(s,s')=2N\Lambda^2 e^{-Q\phi/\sqrt{\alpha'}}\langle V^i_{ab}(x;s),V^j_{cd}(x;s')\rangle_{\mathrm{L}}$$

$$L\equiv -2N\Lambda^2\,\mathcal{W}_{U(1)}[\partial\Sigma;A]\frac{\delta}{\delta Y^{ab}_i(x^0(s))}\mathcal{W}_{U(1)}$$

$$\times[\partial\Sigma;A]\frac{\delta}{\delta Y^{cd}_j(x^0(s'))}Z_N[A] \tag{5.2}$$

where we have taken into account the extrinsic curvature term in the Liouville dressing (2.6) which in the case of the disc has $K=2$. With this definition, Eq. (5.2) determines a fiducial metric on moduli space. The $SU(N)$ part of Eq. (5.2) relevant for the constituent D-brane dynamics is given by the perturbative expansion

$$\widetilde{G}^{ij}_{ab;cd}(s,s')=2\Lambda^2\sum_{n=0}^\infty \frac{(-i)^n}{n!}\left(\frac{g_s}{2\pi\alpha'}\right)^{n+2}\mathcal{G}^{(n)ij}_{ab;cd}[Y;s,s'] \tag{5.3}$$

where the $\mathcal{O}(Y(x^0)^n)$ contribution is

$$\mathcal{G}^{(n)ij}_{ab;cd}[Y;s,s']=\lim_{\epsilon\to 0^+}\sum_{c=1}^N\sum_{\substack{a_1,\ldots,a_n\\b_1,\ldots,b_n}}\int_0^1\prod_{k=1}^n ds_k\left\langle\!\!\left\langle\overline{\xi}_c(0)\overline{\xi}_a(s-\epsilon)\xi_b(s)\overline{\xi}_c(s'-\epsilon)\xi_d(s')\left(\prod_{k=1}^n\overline{\xi}_{a_k}(s_k-\epsilon)\xi_{b_k}(s_k)\right)\xi_c(1)\right\rangle\!\!\right\rangle$$

$$\times\left\langle\frac{dx^i(s)}{ds}\frac{dx^j(s')}{ds'}\prod_{k=1}^n Y^{a_k b_k}_{i_k}(x^0(s_k))\frac{d}{ds_k}x^{i_k}(s_k)\right\rangle_0. \tag{5.4}$$

The expression (5.4) can be evaluated as described in Appendix B by writing it as a sum over permutations $P\in S_{n+3}$. However, it is much simpler to note that Eq. (5.3) can be obtained from the canonical momentum (4.7) by functional differentiation,

$$\widetilde{G}^{ij}_{ab;cd}(s,s')=2\Lambda^2\left(-i\frac{\delta}{\delta Y^{ab}_i(x^0(s))}\widetilde{\Pi}^j_{cd}(s')\right). \tag{5.5}$$

This differentiation preserves the renormalization of the auxilliary quantum field theory described in Appendix B, so that Eq. (5.5) also holds for the corresponding renormalized quantities. Equating coefficients of the $\mathcal{O}(Y(x^0)^n)$ terms in the renormalizations of the perturbative expansions (4.8) and (5.3) thus gives for the relevant zero mode contributions

$$\mathcal{G}_{ab;cd}^{(n)ij}[Y;0,0]^{\mathrm{ren}}=\frac{1}{n+1}\sum_{l=1}^{n+1}\int_{0}^{1}\prod_{\substack{k=1\\k\neq l}}^{n+1}ds_{k}\left\langle\frac{dx^{i}(s)}{ds}\bigg|_{s=0}\frac{dx^{j}(s')}{ds'}\bigg|_{s'=0}\right.$$

$$\left.\times\mathrm{Sym}\left[\prod_{k=1}^{l-1}Y_{i_{k}}(x^{0}(s_{k}))\frac{d}{ds_{k}}x^{i_{k}}(s_{k})\otimes\prod_{m=l+1}^{n+1}Y_{i_{m}}(x^{0}(s_{m}))\frac{d}{ds_{m}}x^{i_{m}}(s_{m})\right]_{db;ca}\right\rangle_{0}. \tag{5.6}$$

Let us now compute Eq. (5.6) when the D-brane configuration fields are given by logarithmic deformation operators. Then the expression (5.6) is non-vanishing only when n is even. The leading $n=0$ contribution is the identity operator on $\mathcal{M}$,

$$\mathcal{G}_{ab;cd}^{(0)ij}(Y,U;0,0)^{\mathrm{ren}}=\frac{8\pi^{2}\alpha'}{\Lambda^{2}}\eta^{ij}\delta_{da}\delta_{bc} \tag{5.7}$$

while the next contribution is at $n=2$ which gives

$$\mathcal{G}_{ab;cd}^{(2)ij}(Y,U;0,0)^{\mathrm{ren}}=\frac{1}{3}\lim_{\epsilon\to0^{+}}\int_{0}^{1}ds_{1}\,ds_{2}\left\langle\frac{d}{ds}x^{i}(s)\bigg|_{s=0}\frac{d}{ds'}x^{j}(s')\bigg|_{s'=0}\frac{d}{ds_{1}}x^{i_{1}}(s_{1})\frac{d}{ds_{2}}x^{i_{2}}(s_{2})\right\rangle_{0}$$

$$\times\mathrm{Sym}[\sqrt{\alpha'}\langle C(s_{1};\epsilon)D(s_{2};\epsilon)\rangle_{0}\{I_{N}\otimes(Y_{i_{1}}U_{i_{2}}+U_{i_{2}}Y_{i_{1}})+Y_{i_{1}}\otimes U_{i_{2}}+U_{i_{2}}\otimes Y_{i_{1}}$$

$$+(Y_{i_{1}}U_{i_{2}}+U_{i_{2}}Y_{i_{1}})\otimes I_{N}\}+\langle D(s_{1};\epsilon)D(s_{2};\epsilon)\rangle_{0}\{I_{N}\otimes U_{i_{1}}U_{i_{2}}+U_{i_{1}}\otimes U_{i_{2}}+U_{i_{1}}U_{i_{2}}\otimes I_{N}\}]_{db;ca}.$$

$$\tag{5.8}$$

Using Wick's theorem, and substituting the boundary string propagators (A17) and the two-point correlation functions (A7),(A8) of the logarithmic operators into Eq. (5.8) yields

$$\mathcal{G}_{ab;cd}^{(2)ij}(Y,U;0,0)^{\mathrm{ren}}=\lim_{\epsilon\to0^{+}}\frac{(4\pi^{2}\alpha')^{2}b}{3}\left[\frac{2\sqrt{\alpha'}}{\Lambda^{2}}I_{g}^{(1)}\eta^{ij}\{I_{N}\otimes(Y_{k}U^{k}+U^{k}Y_{k})+Y_{k}\otimes U^{k}+U^{k}\otimes Y_{k}+(Y_{k}U^{k}+U^{k}Y_{k})\otimes I_{N}\}\right.$$

$$+\sqrt{\alpha'}\,I_{g}^{(2)}\{I_{N}\otimes(Y^{i}U^{j}+Y^{j}U^{i}+U^{j}Y^{i}+U^{i}Y^{j})+Y^{i}\otimes U^{j}+Y^{j}\otimes U^{i}+U^{j}\otimes Y^{i}+U^{i}\otimes Y^{j}+(Y^{i}U^{j}+Y^{j}U^{i}$$

$$+U^{j}Y^{i}+U^{i}Y^{j})\otimes I_{N}\}-\frac{2\alpha'}{\Lambda^{2}}I_{g}^{(3)}\eta^{ij}\{I_{N}\otimes U_{k}U^{k}+U_{k}\otimes U^{k}+U_{k}U^{k}\otimes I_{N}\}-\alpha'I_{g}^{(4)}\{I_{N}\otimes(U^{i}U^{j}+U^{j}U^{i})+U^{i}$$

$$\left.\otimes U^{j}+U^{j}\otimes U^{i}+(U^{i}U^{j}+U^{j}U^{i})\otimes I_{N}\}\right]_{db;ca} \tag{5.9}$$

where we have used Eq. (3.15) and the boundary integrals $I_{g}^{(l)}$ are given in Eqs. (C20)–(C23).

We see that in the limit $\epsilon\to0^{+}$ the most dominant contribution to Eq. (5.9) comes from the integral $I_{g}^{(4)}$ which yields

the only non-vanishing contributions with the renormalizations (4.13) and (4.15), and the bulk-boundary scaling relation (4.18). Then the total Zamolodchikov metric up to second order in the perturbative expansion is

$$\widetilde{G}^{ij}_{ab;cd}(\bar{Y},\bar{U}) = \frac{4\bar{g}_s^2}{\alpha'}\left[\eta^{ij}\,I_N\otimes I_N - \frac{\bar{g}_s^2 b}{9\pi^2}\right.$$
$$\times\{I_N\otimes(\bar{U}^i\bar{U}^j+\bar{U}^j\bar{U}^i)+\bar{U}^i\otimes\bar{U}^j+\bar{U}^j\otimes\bar{U}^i$$
$$\left.\left.+(\bar{U}^i\bar{U}^j+\bar{U}^j\bar{U}^i)\otimes I_N\}\right]_{db;ca}+\mathcal{O}(\bar{g}_s^6).$$

$$(5.10)$$

Now the logarithmic conformal algebra comes into play again and implies an important property. If we renormalize the position of the fat brane as

$$Y_i=\sqrt{\alpha'}\,\epsilon\,\bar{Y}_i \qquad (5.11)$$

then the β-function equations (3.23) imply that

$$\frac{d\bar{Y}_i}{dt}=\bar{U}_i\,. \qquad (5.12)$$

The pair of renormalization group equations (4.14) and (5.12) are just the Galilean evolution equations for (renormalized) velocities. If we now further adjust the parameters of the logarithmic conformal algebra as

$$a=\pi,\quad b=-\frac{\pi^2}{4} \qquad (5.13)$$

then the canonical momentum (4.26) can be written as

$$\widetilde{\Pi}^j_{ab}(\bar{Y},\bar{U})=\sqrt{\alpha'}\sum_{c,d=1}^N\widetilde{G}^{ji}_{ab;cd}(\bar{Y},\bar{U})\dot{\bar{Y}}^{cd}_i \qquad (5.14)$$

and so we recover the canonical moduli space dynamics [see Eq. (2.28)]. Note that the fixing of the coefficients (5.13) does not completely determine all parameters of the logarithmic operator correlators, as there is still some freedom coming from the relation (4.25).

The corresponding Liouville problem satisfies the Helmholtz conditions of Sec. II and the associated action (2.27) in the limit $\epsilon\to0^+$ coincides at leading orders with the (symmetrized) non-Abelian Born-Infeld action described in Sec. IV D. The Zamolodchikov C-function is given by the C-theorem (2.10) which in the present case can be expressed as

$$\lim_{t\to\infty}\frac{\partial C(\bar{Y},\bar{U};t)}{\partial t}=\sqrt{\alpha'}\mathrm{e}^{-Q^2t/\sqrt{\alpha'}}\sum_{a,b,c,d}\bar{U}^{ab}_i\,\widetilde{G}^{ij}_{ab;cd}(\bar{Y},\bar{U})\bar{U}^{cd}_j$$

$$=\frac{4\bar{g}_s^2}{\sqrt{\alpha'}}\mathcal{F}(\bar{U})\mathrm{e}^{-Q^2t/\sqrt{\alpha'}}+\mathcal{O}(\bar{g}_s^6)$$

$$(5.15)$$

where we have introduced the velocity dependent invariant function

$$\mathcal{F}(\bar{U})\equiv\mathrm{tr}\,\bar{U}_i\bar{U}^i+\frac{\bar{g}_s^2}{36}\mathrm{tr}(2\,\bar{U}_i\bar{U}^i\bar{U}_j\bar{U}^j+\bar{U}_i\bar{U}_j\bar{U}^i\bar{U}^j).$$

$$(5.16)$$

Note that the expression (5.10) for the Zamolodchikov metric is explicitly time independent and, strictly speaking, only valid for $t\to\infty$ because of the scaling property (3.15). Notice also that in Eq. (5.15) we have reintroduced the appropriate scaling factors required for the monotonic decreasing property of the C-function and also the expansion property (2.30) which is crucial to the validity of the Helmholtz conditions.

The above results show that the geometry of the moduli space, determined by the Zamolodchikov metric (5.10), is a complicated function of the fat brane dynamical parameters, which will be the key to its use in examining the short-distance spacetime structures probed by D-particles. In the next section we shall examine the genus expansion of the matrix σ-model which will lead to a canonical quantization of the moduli space dynamics described above. In particular, the velocity matrix $\bar{U}_i$ will become a quantum operator. The same is true of the central charge deficit Q which, neglecting irrelevant terms that can be removed by a change of renormalization scheme, is given by

$$Q(\bar{Y},\bar{U};t)\equiv\sqrt{C(\bar{Y},\bar{U};t)}. \qquad (5.17)$$

The quantity (5.17) defines the ''physical'' target space time in the Liouville framework via [25,43]

$$T(\bar{Y},\bar{U};t)\equiv\phi=Q(\bar{Y},\bar{U};t)t \qquad (5.18)$$

where $t=-\sqrt{\alpha'}\log\Lambda$ is the rescaling (flat worldsheet) time variable and ϕ is the zero mode of the Liouville field. Then the time evolution of the Liouville dressed couplings with respect to the target space time variable are governed by conventional worldsheet β-functions upon replacing bare coupling constants with dressed ones. The definition (5.18) comes from the normalization of the Liouville field kinetic term $\partial\varphi\bar{\partial}\varphi$ appropriate to the Robertson-Walker metric on spacetime [43,48]. The physical time (5.18) becomes a quantum operator upon summing over worldsheet genera [49]. In general, the expression (5.15) which determines it as a function of t is a complicated highly non-linear first order differential equation. If we assume, however, that C varies slowly with time, then Eq. (5.15) can be solved at linear order in the string tension by quadratures to give

$$T(\bar{Y},\bar{U};t)\simeq\frac{2\bar{g}_st}{\alpha'^{1/4}}\sqrt{\mathcal{F}(\bar{U})}\left(\int_0^td\tau\,\mathrm{e}^{2(\tau^2-t^2)\bar{g}_s^2\mathcal{F}(\bar{U})/\alpha'}\right)^{1/2}.$$

$$(5.19)$$

The limit of slowly varying C-function holds near any fixed point in moduli space. This assumption is consistent with the assumptions of small σ-model and string couplings and also of a slowly-moving (non-relativistic) fat-brane which is the kinematical region of interest here. We note that, in contrast to the Abelian case, the time variable (5.19) is a complicated

function of the various fat-brane velocities because of the trace structure of the invariant function (5.16).

VI. QUANTIZATION AND SPACETIME UNCERTAINTY RELATIONS

In this section we will apply the formalism of [25] to sum over worldsheet genera of the partition function (3.3). The pertinent deformation couplings represented by the logarithmic operators have vanishing conformal dimension in the limit $\epsilon \to 0^+$ [see Eq. (3.22)], and as a result extra logarithmic divergences appear in pinched annulus diagrams. This will amount to a quantization of the couplings $Y_i(x^0)^{ab}$ from which we will be able to derive a set of stringy uncertainty relations.

A. Resummation of the genus expansion

We consider the partition function (3.3) defined on a genus h surface Σ_h. This surface has h ''holes'' in it and for all h its boundary has the topology of a circle, so that, in the notation above, $\Sigma_0 \equiv \Sigma$. The genus expansion is

$$\sum_{h=0}^{\infty} Z_N^h[A] = \sum_{h=0}^{\infty} \langle W[\partial\Sigma_h; A] \rangle_0^h \qquad (6.1)$$

where the average is taken in the free σ-model (3.4) defined on Σ_h. Since we assume that $\partial\Sigma_h$ has the topology of a disjoint union of $h+1$ circles, the sum over genera commutes with the representation (4.2) of the Wilson loop operator in terms of auxilliary fields and we can write

$$\sum_{h=0}^{\infty} Z_N^h[A] = \left\langle\!\!\left\langle \sum_{c=1}^{N} \bar{\xi}_c(0) \sum_{h=0}^{\infty} \left\langle \exp\!\left(\sum_{a,b=1}^{N} \sum_{k=0}^{h} \right.\right.\right.$$
$$\left.\left.\left. \times \int_0^1 ds_k\, Y_i^{ab}(x^0(s_k)) V_{ab}^i(x;s_k) \right)\!\right\rangle_0^h \xi_c(1) \right\rangle\!\!\right\rangle \qquad (6.2)$$

where for simplicity we have set $N\mathcal{W}_{U(1)}[\partial\Sigma_h; A] = 1$ and we work in the temporal gauge $A_0 = 0$ as usual. The double brackets in Eq. (6.2) denote, as before, the average over the auxilliary fields as in Eq. (4.1) and the boundary vertex operators V_{ab}^i are defined in Eq. (4.5).

For the recoil operators (3.21) we can insert a temporal delta-function $1 = \int_0^\infty dt\, \delta(t - x^0(s))$ into Eq. (6.2) to get

$$\sum_{h=0}^{\infty} Z_N^h[A] = \lim_{\epsilon\to 0^+} \left\langle\!\!\left\langle \sum_{c=1}^{N} \bar{\xi}_c(0) \sum_{h=0}^{\infty} \left\langle \exp\!\left(\sum_{a,b=1}^{N} \int_{-\infty}^{\infty} d\omega \right.\right.\right.$$
$$\times \int_0^\infty dt\, Y_i^{ab}(t;\epsilon) e^{i\omega t} \sum_{k=0}^{h} \int_0^1 ds_k\, e^{-i\omega x^0(s_k)}$$
$$\left.\left.\left. \times \Theta(x^0(s_k);\epsilon) V_{ab}^i(x;s_k) \right)\!\right\rangle_0^h \xi_c(1) \right\rangle\!\!\right\rangle \qquad (6.3)$$

where $Y_i^{ab}(t;\epsilon) = \sqrt{\alpha'}\, Y_i^{ab}\epsilon + U_i^{ab}t$. If we introduce the Fourier transform

$$\check{Y}_i^{ab}(\omega) = \lim_{\epsilon\to 0^+} \int_0^\infty dt\, e^{i\omega t}\, Y_i^{ab}(t;\epsilon) \qquad (6.4)$$

and the new boundary vertex operators

$$\oint_{\partial\Sigma_h} V_{ab}^i(x;\omega) \equiv \lim_{\epsilon\to 0^+} \frac{ig_s}{2\pi\alpha'}$$
$$\times \sum_{k=0}^{h} \int_0^1 ds_k\, e^{-i\omega x^0(s_k)}\, \Theta(x^0(s_k);\epsilon)$$
$$\times \bar{\xi}_a(s_k - \epsilon)\xi_b(s_k)\frac{d}{ds_k}x^i(s_k) \qquad (6.5)$$

then the sum over genera in Eq. (6.3) takes the usual form of a set of σ-model couplings $\check{Y}_i^{ab}(\omega)$ associated with the deformation vertex operators (6.5),

$$\sum_{h=0}^{\infty} Z_N^h[A] = \left\langle\!\!\left\langle \sum_{c=1}^{N} \bar{\xi}_c(0) \right.\right.$$
$$\times \sum_{h=0}^{\infty} \left\langle \exp\!\left(\sum_{a,b=1}^{N} \int_{-\infty}^{\infty} d\omega\, \check{Y}_i^{ab}(\omega) \right.\right.$$
$$\left.\left.\left. \times \oint_{\partial\Sigma_h} V_{ab}^i(x;\omega) \right)\!\right\rangle_0^h \xi_c(1) \right\rangle\!\!\right\rangle. \qquad (6.6)$$

The representation (6.6), along with Eq. (3.14), justifies the identification of the Liouville field φ with the fundamental temporal embedding field x^0, in the limit $\epsilon\to 0^+$. The latter field appears in the tachyon operator part of Eq. (6.5), thereby dressing the boundary theory analogously to that by two-dimensional quantum gravity. Some further aspects of this correspondence, such as the properties of the induced target space geometry, are discussed in [19].

We now focus on the properties of the (Abelianized) average over fundamental string fields in Eq. (6.6). As we will show, the resummation of Eq. (6.6) over *pinched* genera yield the dominant worldsheet divergences, thereby spoiling the conformal symmetry. Conformal invariance requires absorbing such singularities into renormalized quantities at lower genera, leading to a generalized version of the Fischler-Susskind mechanism [50]. Such degenerate Riemann surfaces involve a string propagator over thin long worldsheet strips of thickness $\delta\to 0$ that are attached to a disc. These strips can be thought of as two-dimensional quantum gravity wormholes. Consider first the resummation of one-loop worldsheets, i.e. those with an annular topology, in the pinched approximation (Fig. 3). String propagation on
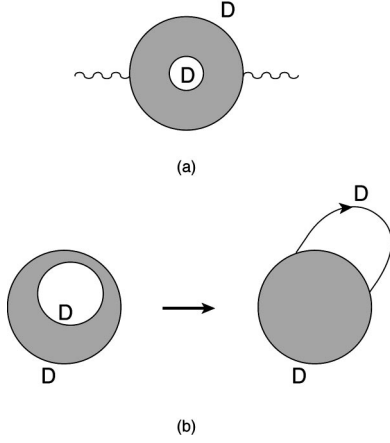
FIG. 3. (a) World-sheet annulus diagram for the leading quantum correction to the propagation of a string state V in a D-brane background, and (b) the pinched annulus configuration which is the dominant divergent contribution to the quantum recoil.

such a worldsheet can be described formally by adding bilocal worldsheet operators $\mathcal{B}$ [51] which in the present case are defined by

$$\mathcal{B}(\omega,\omega') = \sum_{a,b,c,d} \oint_{\partial\Sigma} \oint_{\partial\Sigma'} V_{ab}^i(x;\omega)$$

$$\times \frac{G_{ij}^{ab;cd}(\omega,\omega')}{L_0-1} V_{cd}^j(x;\omega') \qquad (6.7)$$

where the Zamolodchikov metric in Eq. (6.7) is the two-point correlation function of the vertex operators defined in Eq. (6.5) and L_0 denotes the usual Virasoro generator. The operator insertion $(L_0-1)^{-1}$ in Eq. (6.7) represents the string propagator $\triangle_s$ on the thin strip of the pinched annulus.

Inserting a complete set of intermediate string states $\mathcal{E}_I$, we can rewrite Eq. (6.7) as an integral over the variable $q \equiv e^{2\pi i\tau}$, where τ is the complex modular parameter characterizing the worldsheet strip. The string propagator over the strip then reads

$$\triangle_s(z,z') = \sum_I \int dq\, q^{\Delta_I-1} \{\mathcal{E}_I(z)\otimes(\text{ghosts})\otimes\mathcal{E}_I(z')\}_{\Sigma\#\Sigma'}$$
$$(6.8)$$

where Δ_I are the conformal dimensions of the states $\mathcal{E}_I$. The sum in Eq. (6.8) is over all states which propagate along the long thin strip connecting the discs Σ and Σ' (in the degenerating annulus handle case of interest here, $\Sigma'=\Sigma$). As indicated in Eq. (6.8), the sum over states must include ghosts, whose central charge cancels that of the worldsheet matter fields in any critical string model.

In Eq. (6.8) we have assumed that the Virasoro operator L_0 can be diagonalized in the basis of string states with eigenvalues their conformal dimensions Δ_I, i.e.

$$L_0|\mathcal{E}_I\rangle = \Delta_I|\mathcal{E}_I\rangle, \qquad q^{L_0-1}|\mathcal{E}_I\rangle = q^{\Delta_I-1}|\mathcal{E}_I\rangle. \qquad (6.9)$$

However, this simple diagonalization fails in the presence of the logarithmic pair of operators [14], due to the non-trivial

mixing between C and D in the Jordan cell of L_0. Generally, states with $\Delta_I=0$ may lead to extra logarithmic divergences in Eq. (6.8), because such states make contributions to the integral of the form $\int dq/q \sim \log \delta$, in the limit $q \sim \delta \to 0$ representing a long thin strip of thickness δ. We assume that such states are discrete in the space of all string states, i.e. that they are separated from other states by a gap. In that case, there are factorizable logarithmic divergences in Eq. (6.8) which depend on the background surfaces Σ and Σ'. These are precisely the states corresponding to the logarithmic recoil operators (3.13) and (3.16), with vanishing conformal dimension (3.22) as $\epsilon \to 0^+$.

In the case of mixed logarithmic states, the pinched topologies are characterized by divergences of a double logarithmic type which arise from the form of the string propagator in Eq. (6.7) in the presence of generic logarithmic operators C and D,

$$\int dq\, q^{\Delta_\epsilon-1} \langle C,D| \begin{pmatrix} 1 & \log q \\ 0 & 1 \end{pmatrix} |C,D\rangle. \qquad (6.10)$$

As shown in [14], the mixing between C and D states along degenerate handles leads formally to divergent string propagators in physical amplitudes, whose integrations have leading divergences of the form

$$\int \frac{dq}{q} \log q \int d^2z\, D(z;\epsilon) \int d^2z'\, C(z';\epsilon)$$

$$\simeq (\log \delta)^2 \int d^2z\, D(z;\epsilon) \int d^2z'\, C(z';\epsilon). \qquad (6.11)$$

These $(\log \delta)^2$ divergences can be cancelled by imposing momentum conservation in the scattering process of the light string states off the D-brane background [28]. This cancellation of leading divergences of the genus expansion in the non-Abelian case is demonstrated explicitly in Appendix D. It is shown there that this renormalization requires that the change in (renormalized) velocity of the fat brane due to the recoil from the scattering of string states be

$$\bar{U}_i^{ab} = -\frac{1}{M_s}(k_1+k_2)_i\, \delta^{ab} = \frac{d\bar{Y}_i^{ab}}{dt} \qquad (6.12)$$

where $k_{1,2}$ are the initial and final momenta in the scattering process and $M_s=1/\sqrt{\alpha'}\,\bar{g}_s$ is the BPS mass of the string soliton [1].[6] This means that, to leading order, the constituent D-branes move parallel to one another with a common velocity and there are no interactions among them. Thus the leading recoil effects imply a commutative structure and the fat brane behaves as a single D-particle. Note that the relation (6.12) also shows directly that $d\bar{g}_s/dt=0$.

———————

[6]Note that this differs from the mass normalization of the derived canonical momentum (4.26). In Eq. (6.12), the $k_{1,2}$ are true physical momenta so that M_s represents the actual BPS mass of the D-particles.
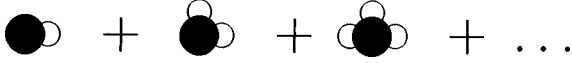
FIG. 4. Resummation of the genus expansion in the pinched approximation. The solid circles are the worldsheet discs and the thin lines are strips attached to them with infinitesimal pinching size δ. Each strip corresponds to an insertion of a bilocal operator (6.7) on the worldsheet.

In addition to this divergence, there are sub-leading log δ singularities, corresponding to the diagonal terms $\int d^2z\, D(z;\epsilon)\int d^2z'\, D(z';\epsilon)$ and $\int d^2z\, C(z;\epsilon)\int d^2z'\, C(z';\epsilon)$. With our choice of basis (5.1) on the moduli space of D-brane configurations, these latter terms are the ones we should concentrate upon for the purposes of deriving the quantum fluctuations of the collective D-particle coordinates. As we will see, it is these sub-leading divergences in the genus expansion which lead to interactions between the constituent D-branes and provide the appropriate noncommutative quantum extension of the leading dynamics (6.12).

In the weak-coupling case, we can truncate the genus expansion (6.6) to a sum over pinched annuli (Fig. 4). This truncation corresponds to a semi-classical approximation to the full quantum string theory in which we treat the D-particles as heavy non-relativistic objects in target space. Then the dominant contributions to the sum are given by the log δ modular divergences described above, and the effects of the dilute gas of wormholes on the disc are to exponentiate the bilocal operator (6.7). In the pinched approximation, the genus expansion thus leads to an effective change in the matrix σ-model action in Eq. (6.6) by

$$
\Delta S \simeq \frac{g_s^2}{2} \log \delta \sum_{a,b,c,d} \int_{-\infty}^{\infty} d\omega\, d\omega'
$$

$$
\times \oint_{\partial\Sigma} \oint_{\partial\Sigma'} V_{ab}^i(x;\omega) G_{ij}^{ab;cd}(\omega,\omega') V_{cd}^j(x;\omega').
$$

$$(6.13)$$

The bilocal action (6.13) can be cast into the form of a local worldsheet effective action by using standard tricks of wormhole calculus [52] and rewriting it as a functional Gaussian integral

$$
e^{\Delta S} = \int [d\check{\rho}] \exp\left[-\frac{1}{2} \sum_{a,b,c,d} \int_{-\infty}^{\infty} d\omega\, d\omega'\, \check{\rho}_i^{ab}(\omega) \right.
$$

$$
\times \oint_{\partial\Sigma} \oint_{\partial\Sigma'} G_{ab;cd}^{ij}(\omega,\omega') \check{\rho}_j^{cd}(\omega')
$$

$$
\left. + g_s \sqrt{\log \delta} \sum_{a,b=1}^{N} \int_{-\infty}^{\infty} d\omega\, \check{\rho}_i^{ab}(\omega) \oint_{\partial\Sigma} V_{ab}^i(x;\omega) \right]
$$

$$(6.14)$$

where $\check{\rho}_i^{ab}(\omega)$ are quantum coupling constants of the worldsheet matrix σ-model. Thus the effect of the resumma-

over pinched genera is to induce quantum fluctuations of the collective D-brane background, leading to a set of effective quantum coordinates

$$
\check{Y}_i^{ab}(\omega) \to \hat{\mathcal{Y}}_i^{ab}(\omega) = \check{Y}_i^{ab}(\omega) + g_s \sqrt{\log \delta}\, \check{\rho}_i^{ab}(\omega)
$$

$$(6.15)$$

viewed as position operators in a co-moving target space frame.

Transforming the quantum couplings to the temporal field representation using the inverse transformations which led to Eq. (6.6), we find that the genus expansion (6.1) in the pinched approximation is

$$
\sum_{h^{(p)}} Z_N^{h^{(p)}}[A] \simeq \left\langle \left(\int_{\mathcal{M}} [d\rho] \wp[\rho] W\left[\partial\Sigma; A - \frac{1}{2\pi\alpha'}\rho \right] \right) \right\rangle_0
$$

$$(6.16)$$

where the sum is over all pinched genera of infinitesimal pinching size, and

$$
\wp[\rho] = \exp\left[-\frac{1}{2\Gamma^2} \sum_{a,b,c,d} \int_0^1 ds\, ds'\, \rho_i^{ab}(x^0(s)) \right.
$$

$$
\left. \times G_{ab;cd}^{ij}(s,s') \rho_j^{cd}(x^0(s')) \right]
$$

$$(6.17)$$

is a functional Gaussian distribution on moduli space of width

$$
\Gamma = g_s \sqrt{\log \delta}.
$$

$$(6.18)$$

In Eq. (6.16) we have normalized the functional Haar integration measure $[d\rho]$ appropriately. We see therefore that the diagonal sub-leading logarithmic divergences in the modular cutoff scale δ, associated with degenerate strips in the genus expansion of the matrix σ-model, can be treated by absorbing these scaling violations into the width Γ of the probablity distribution characterizing the quantum fluctuations of the (classical) D-brane configurations $Y_i^{ab}(x^0(s))$. In this way the interpolation among families of D-brane field theories corresponds to a quantization of the worldsheet renormalization group flows. Note that the worldsheet wormhole parameters, being functions on the moduli space (5.1), can be decomposed as

$$
\rho_i^{ab}(x^0(s)) = \lim_{\epsilon \to 0^+} ([\rho_C]_i^{ab} C(x^0;\epsilon) + [\rho_D]_i^{ab} D(x^0;\epsilon)).
$$

$$(6.19)$$

The fields $\rho_{C,D}$ are then renormalized in the same way as the D-brane couplings, so that the corresponding renormalized wormhole parameters generate the same type of (Galilean) β-function equations (3.23) [28]. This will be implicitly assumed in the following.

According to the standard Fischler-Susskind mechanism for cancelling string loop divergences [50], modular infinities should be identified with worldsheet divergences at lower genera. Thus the strip divergence log δ should be as-

sociated with a worldsheet ultraviolet cutoff scale which in turn is identified with the Liouville field as described earlier. We may in effect take δ independent from Λ, in which case we can first let $\epsilon \to 0^+$ in the above and then take the limit $\delta \to 0$. Interpreting $\log \delta$ in this way as a renormalization group time parameter (interpolating among D-brane field theories), the time dependence of the renormalized width (6.18) expresses the usual properties of the distribution function describing the time evolution of a wavepacket in moduli space [42]. The inducing of a statistical Gaussian spread of the D-brane couplings is the essence of the quantization procedure.

B. String interactions and diagonalization of moduli space

The Gaussian distribution functional (6.17) can be used to determine the quantum fluctuations $\Delta \bar{Y}_i^{ab}$ in the initial D-brane positions to leading order in the string coupling constant expansion. For this, we first need to diagonalize the Zamolodchikov metric (5.10). As we will see, the parameters of the diagonalization of the geometry of moduli space expose the precise nature of the string interactions inherent in the multi D-brane system. This eigenvalue problem is somewhat intractable in general, but in the limit $g_s \ll 1$ of weakly coupled strings it can be carried through with some work.

In the free string limit, the interactions between the constituent D-branes are negligible to lowest order and their position matrices commute. In the temporal gauge that we are working in, the configuration fields can then be simultaneously diagonalized by a time independent gauge transformation

$$\bar{Y}^i = \Omega \, \mathrm{diag}(y_1^i, \ldots, y_N^i) \Omega^{-1}, \quad \Omega \in U(N). \quad (6.20)$$

The eigenvalues $y_a^i \in \mathbb{R}$ represent the positions of the constituent D-branes themselves which move at velocities $u_a^i = dy_a^i/dt$. The noncommutativity of spacetime is encoded through the unitary matrix Ω which represents the string interactions between the D-particles. In this way we will study the coordinate fluctuations both as a quantum mechanical effect and geometrically as the perturbations around classical (commutative) spacetime represented by the diagonal matrix configurations in Eq. (6.20). This limit corresponds to a configuration of well-separated branes and it represents a Born-Oppenheimer approximation to the D-particle interactions, which is valid for small velocities [35], whereby the diagonal D-particle coordinates are separated from the off-diagonal parts of the adjoint Higgs fields representing the short open string excitations connecting them.

Using Eq. (6.20) the Zamolodchikov metric (5.10) can be written as

$$\tilde{G}^{ij}(\bar{Y}, \bar{U}) = \frac{4 \bar{g}_s^2}{\alpha'} (\Omega \otimes \Omega) \left(\eta^{ij} I_N \otimes I_N + \frac{\bar{g}_s^2}{36} \mathcal{U}^{ij} + \mathcal{O}(\bar{g}_s^4) \right)$$
$$\times (\Omega \otimes \Omega)^{-1} \quad (6.21)$$

where $\mathcal{U}_{ab;cd}^{ij} = \mathcal{U}_{ab}^{ij} \delta_{ad} \delta_{bc}$ is the $u(N) \otimes u(N)$ diagonal matrix with entries

$$\mathcal{U}_{ab}^{ij} = 2 u_a^i u_a^j + 2 u_b^i u_b^j + u_a^i u_b^j + u_a^j u_b^i. \quad (6.22)$$

We now need to diagonalize the symmetric matrix (6.22) with respect to the 9×9 spacetime indices i, j. For this, we assume that $\eta_{ij} = \delta_{ij}$ and consider separately the two cases $a = b$ and $a \neq b$.

Consider first the case $a = b$. Upon examination of the characteristic equation for the matrix $\mathcal{U}_{aa}^{ij} = 6 u_a^i u_a^j$ one easily sees that there are two eigenvalues $\lambda = 6 \|u_a\|^2$ and $\lambda = 0$, where $\|u_a\| = \sqrt{\sum_i u_a^i u_a^i}$ is the Euclidean norm of the vector $u_a \in \mathbb{R}^9$. The dimension of the kernel of $\mathcal{U}_{aa}^{ij}$ is 8 because there are precisely eight linearly independent vectors in $\mathbb{R}^9$ which are orthogonal to u_a^i. Thus the eigenvalues are

$$\lambda_{aa}^1 = 6 \|u_a\|^2, \quad \lambda_{aa}^2 = \cdots = \lambda_{aa}^9 = 0. \quad (6.23)$$

The normalized eigenvector corresponding to λ_{aa}^1 is just $u_a / \|u_a\|$ and the remaining ones span the eight-dimensional space transverse to this line, which we refer to as the "string frame" because it represents the coordinate system relative to the fundamental open string excitations which start and end on the same D-particle a. Upon rotation to the one-dimensional string frame, the 9×9 orthogonal matrix Ξ_{aa} which diagonalizes Eq. (6.22) for $a = b$ is just the identity matrix,

$$\Xi_{aa} = I_9. \quad (6.24)$$

The situation for $a \neq b$ is similar but a bit more technically involved. We assume that the velocity vectors u_a and u_b are linearly independent. There are then seven linearly independent vectors which are orthogonal to both u_a and u_b, and therefore there is a zero eigenvalue of multiplicity 7. The remaining two eigenvectors are linear combinations of the velocity vectors,

$$\psi_{ab}^{(1,2)i} = u_a^i + B^{(1,2)} u_b^i \quad (6.25)$$

up to an overall normalization. Solving the eigenvalue equations for the eigenvectors (6.25) gives after some tedious algebra the two non-zero eigenvalues,

$$\lambda_{ab}^{1,2} \equiv \lambda_{\pm} = \|u_a\|^2 + \|u_b\|^2 + u_a \cdot u_b \pm \left\{ (\|u_a\|^2 + \|u_b\|^2 + u_a \cdot u_b)^2 + \frac{[(u_a \cdot u_b)^2 + \|u_a\|^2 \|u_b\|^2 + 2 u_a \cdot u_b (\|u_a\|^2 + \|u_b\|^2)]^2}{\|u_a\|^2 \|u_b\|^2 - (u_a \cdot u_b)^2} \right\}^{1/2}$$

$$\lambda_{ab}^3 = \cdots = \lambda_{ab}^9 = 0 \quad (6.26)$$

and the coefficients

$$B^{(1,2)} = \frac{(u_a \cdot u_b)^2 + \|u_a\|^2 \|u_b\|^2 + 2u_a \cdot u_b(\|u_a\|^2 + \|u_b\|^2) - \lambda_{ab}^{1,2} u_a \cdot u_b}{2\|u_a\|^2 u_a \cdot u_b + 2(u_a \cdot u_b)^2 + 2\|u_a\|^4 - \lambda_{ab}^{1,2}\|u_a\|^2} \tag{6.27}$$

where the dot between vectors denotes the usual Euclidean inner product on $\mathbb{R}^9$. The remaining seven orthonormal eigenvectors are those which span the space transverse to the plane in $\mathbb{R}^9$ generated by the vectors (6.25), which defines the two-dimensional string frame representing the fundamental open string which starts on D-brane a and ends on D-brane b. Note that for $a \neq b$ the dimension of this coordinate system increases by one because of the increase in degrees of freedom of the string which now stretches between two different branes. Once again the orthogonal diagonalization transformation matrix Ξ_{ab} is particularly simple in the string frame. We parametrize the plane spanned by u_a and u_b via $u_a^i = \|u_a\| \, \delta^{i,1}$ and the angle θ_{ab} between the two vectors. Then upon rotation to the two-dimensional string frame we have

$$\Xi_{ab} = \begin{pmatrix} \mathcal{N}(\|u_a\| + B^{(1)}\|u_b\|\cos\theta_{ab}) & -\mathcal{N}B^{(1)}\|u_b\|\sin\theta_{ab} & 0 & \dots & 0 \\ \mathcal{N}B^{(1)}\|u_b\|\sin\theta_{ab} & \mathcal{N}(\|u_a\| + B^{(1)}\|u_b\|\cos\theta_{ab}) & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix} \tag{6.28}$$

where we have orthogonalized the 2×2 block matrix corresponding to the string frame and

$$\mathcal{N} = \|u_a + B^{(1)} u_b\|^{-1} \tag{6.29}$$

is the appropriate normalization constant.

With the above constructions, the Zamolodchikov metric can now be written as a unitary transformation of a diagonal metric on $\mathcal{M}$,

$$\tilde{G}_{ab;cd}^{ij}(\bar{Y},\bar{U}) = \frac{4\bar{g}_s^2}{\alpha'} \eta_{kl} \sum_{e,f=1}^{N} \Omega_{af}(\bar{Y})$$

$$\times \Omega_{be}(\bar{Y}) \Xi_{ef}^{ik}(u) \Xi_{fe}^{jl}(u) \Omega_{ce}^*(\bar{Y}) \Omega_{df}^*(\bar{Y})$$

$$\times \left(1 + \frac{\bar{g}_s^2}{36} \lambda_{ef}^k(u) + \mathcal{O}(\bar{g}_s^4) \right). \tag{6.30}$$

We see therefore that the diagonalization of the Zamolodchikov metric on moduli space $\mathcal{M}$ naturally encodes within it the geometry of the string interactions among the D-branes. In particular, we see the enormous complexity involved in going from the dynamics for a single D-particle ($a = b$) to the interactions between constituent D-branes ($a \neq b$). These properties will have important ramifications for the physical consequences of the stringy spacetime uncertainty relations which we now proceed to derive.

C. Quantum fluctuations of collective D-brane configurations

Given the diagonalization (6.30) of the bilinear form of the Gaussian distribution functional (6.17), we can now write down the quantum fluctuations of the D-brane coordinates. Substituting Eq. (6.30) into Eq. (6.17) and redefining the matrix-valued wormhole parameters ρ_i^{ab} leads to a complex bilinear form in a new set of complex-valued wormhole parameters. Since the metric of the bilinear form is diagonal, one can associate a width to each direction $i = 1, \dots, 9$ and D-brane configuration $a, b = 1, \dots, N$. The coordinate transformation

$$\tilde{Y}_{ab}^i = \Xi_{ab}^{ji} [\Omega^{*-1} \bar{Y}_j \Omega]_{ba} \equiv \Xi_{ab}^{ji} X_j^{ab}(\bar{Y}) \tag{6.31}$$

is precisely the one which achieves the desired diagonalization and leads to the statistical variances

$$(\Delta \tilde{Y}_{ab}^i)(\Delta \tilde{Y}_{ab}^i)^\dagger = \frac{\alpha' \Gamma^2}{2\bar{g}_s^2} \left(1 - \frac{\bar{g}_s^2}{36} \lambda_{ab}^i(u) + \mathcal{O}(\bar{g}_s^4) \right). \tag{6.32}$$

Note that, as a result of Eq. (3.15), the renormalized string coupling $\bar{g}_s$ is imaginary, i.e. $\bar{g}_s^2 < 0$, owing to the Minkowskian signature of the spacetime.

The time dependence in the width (6.18) can be absorbed into the usual renormalization of the string coupling constant by taking the correlation

$$\log \delta = 2|\bar{g}_s|^\chi \epsilon^{-2} \tag{6.33}$$

between the modular worldsheet and target space scale parameters. The exponent $\chi \geq 0$ is left arbitrary for the moment. Later on we shall fix it by demanding consistency of certain results with conventional D-particle mechanics. The variances (6.32) are therefore time-independent and represent not the spread in time of a wavepacket on $\mathcal{M}$, but rather the true quantum fluctuations of the D-brane configurations. The collective D-particle coordinates $X_i(\bar{Y})$ naturally encode the effects of the open string excitations. Their uncertainties may be computed using the formula

$$(\Delta \tilde{Y}^i_{ab})(\Delta \tilde{Y}^i_{ab})^\dagger \equiv ((\tilde{Y}^i_{ab}|[\tilde{Y}^i_{ab}]^\dagger)) - |((\tilde{Y}^i_{ab}))|^2$$

$$= \Xi^{ji}_{ab} \Xi^{ki}_{ab} ((X^{ab}_j|[X^{ab}_k]^\dagger))_{\text{conn}} \quad (6.34)$$

where the brackets denote statistical averages with respect to the wormhole probability distribution (6.17) and the average of the X fields in Eq. (6.34) is a connected correlation function. In this subsection we shall always work in string coordinates, but, by covariance, the qualitative features are the same in any reference frame.

Let us first consider the relation (6.32) in the case $a = b$, which corresponds to a single D-particle. Using Eqs. (6.23) and (6.24) it is straightforward to see that the variances (6.32) and (6.34) lead to the position uncertainties

$$|\Delta X^{aa}_i| = |\bar{g}_s|^{\chi/2}\sqrt{\alpha'}\left(1 + \frac{|\bar{g}_s|^2}{12}\|u_a\|^2 \, \delta_{i,1} + \mathcal{O}(|\bar{g}_s|^4)\right)$$

$$\geq |\bar{g}_s|^{\chi/2}\sqrt{\alpha'} \quad (6.35)$$

for the individual D-particle coordinates. For $\chi = 0$ the minimal length in Eq. (6.35) coincides with the standard smearing [29] due to the finite size of the string, while for $\chi = \frac{2}{3}$ it matches the 11-dimensional Planck length $l_P^{(11)}$ which arises from the kinematical properties of D-particles [35]. A choice of $\chi \neq 0$ is more natural since the modular strip divergences should be small for weakly interacting strings. Note that the uncertainty (6.35) is always larger in the string frame, representing the additional energetic smearing that arises from the open string excitations on the D-particles. Outside of this frame we obtain exactly the standard stringy smearings directly from the worldsheet formalism, without the need of postulating an auxilliary uncertainty relation as is done in [29,35]. With the present normalization of the mass of the D-particles [see Eq. (4.26)], we see that the velocity-dependent shift in Eq. (6.35) is just the kinetic energy of D-particle a.

The coordinate uncertainties for $a \neq b$ are responsible for the emergence of a true noncommutative quantum spacetime

and represent the genuine non-Abelian characteristics of the D-particle dynamics. From Eqs. (6.26) and (6.28) it follows that, outside the string frame, the uncertainties $|\Delta X^{ab}_i|$, $i > 2$, are given by the same minimal length (6.35) as for the individual D-particles. In string coordinates, we may assume, by symmetry, that $|\Delta X^{ab}_1| \sim |\Delta X^{ab}_2|$. Then Eqs. (6.32) and (6.34) lead to a system of two linear equations in two unknowns,

$$|\bar{g}_s|^\chi \alpha'\left(1 + \frac{|\bar{g}_s|^2}{36}\lambda_\pm(u)\right)$$

$$= |\Delta X^{ab}_1|^2 \pm 4\mathcal{N}^2 B^{(1)}\|u_b\|\sin\theta_{ab}$$

$$\times (\|u_a\| + B^{(1)}\|u_b\|\cos\theta_{ab}) \, \text{Re}((X^{ab}_1|[X^{ab}_2]^\dagger))_{\text{conn}}$$

$$(6.36)$$

which hold up to $\mathcal{O}(|\bar{g}_s|^2)$. Adding the two equations (6.36) gives the smearings

$$|\Delta X^{ab}_1| = |\bar{g}_s|^{\chi/2}\sqrt{\alpha'}\left[1 + \frac{|\bar{g}_s|^2}{144}(3s_{ab} + t_{ab}) + \mathcal{O}(|\bar{g}_s|^4)\right]$$

$$(6.37)$$

where we have introduced the kinematical invariants $s_{ab} = \|u_a + u_b\|^2$ and $t_{ab} = \|u_a - u_b\|^2$ representing, respectively, the center of mass kinetic energy and momentum transfer of the scattering of D-particles a and b. The uncertainty in measurement of an open string coordinate thus depends on both the center of mass and relative energies of the two D-particles to which it is attached. Its minimum coincides with that of Eq. (6.35). Note that when D-particles a and b move orthogonally to one another, i.e. their scattering angle is $\theta_{ab} = \pi/2$, the uncertainty (6.37) depends only on the total kinetic energy of the two particles. This is the case that is discussed in [37].

Subtracting the two equations (6.36) gives the connected correlation function

$$\text{Re}((X^{ab}_1|[X^{ab}_2]^\dagger))_{\text{conn}} = \frac{|\bar{g}_s|^{2+\chi}\alpha'\|u_a + B^{(1)}u_b\|^2 \mathcal{X}_{ab}(u)}{144 B^{(1)}\|u_b\|\sin\theta_{ab}(\|u_a\| + B^{(1)}\|u_b\|\cos\theta_{ab})}, \quad a \neq b \quad (6.38)$$

to $\mathcal{O}(|\bar{g}_s|^2)$, with

$$B^{(1)} = \frac{\|u_a\|^2\|u_b\|^2 + u_a \cdot u_b[\|u_a\|^2 + \|u_b\|^2 - \mathcal{X}_{ab}(u)]}{2(u_a \cdot u_b)^2 + \|u_a\|^2[u_a \cdot u_b + \|u_a\|^2 - \|u_b\|^2 - \mathcal{X}_{ab}(u)]}$$

$$\mathcal{X}_{ab}(u) = \frac{1}{4}\sqrt{(3s_{ab} + t_{ab})^2 + \frac{16\|u_a\|^2\|u_b\|^2}{\sin^2\theta_{ab}}\left[1 + \cos^2\theta_{ab} + \left(\frac{\|u_a\|}{\|u_b\|} + \frac{\|u_b\|}{\|u_a\|}\right)\cos\theta_{ab}\right]^2}.$$

$$(6.39)$$

The result (6.38) shows that for the scattering of D-particles, the position operators of the open strings which mediate the interactions are not independent random variables and have a non-trivial quantum mechanical correlation. This is a new form of quantum spacetime uncertainty relations between different *spatial* directions of target space. When $\chi = \frac{2}{3}$ the right-hand side of Eq. (6.38) can be written in terms of $(l_P^{(11)})^2$ and an additional complicated function of the D-particle kinetic energies. For the transverse scattering of two D-particles of equal speed, this function is just the total kinetic energy of the D-particles [37]. In general though, the right-hand side of Eq. (6.38) is a horrendously complicated function of the scattering parameters. It demonstrates the complexity of the open string interactions between D-branes, in that the smearing of the string coordinates is a highly non-trivial function of the kinematical invariants of the D-particles to which they are attached.

The energy dependence of Eqs. (6.35), (6.37) and (6.38) is a quantum decoherence effect which can be understood from a generalization of the Heisenberg microscope whereby we scatter a low-energy probe, represented by a closed string state with definite energy and momentum, off the D-particle configuration. As the closed string state hits a D-particle, it splits into two open string states, represented by the recoil of the particle upon impact with the detector, which absorb energy from the scattering. Formally, such a splitting is described by means of the conformal field theory formalism developed in [53]. When a closed string state, represented as a bulk deformation by a closed string matter excitation O on Σ of scaling dimension Δ_O, approaches the boundary $\partial\Sigma$, then one can infer the operator product expansion [53,54]

$$O(z,\bar{z};s) \sim \sum_I (2s)^{\Delta_I - \Delta_O} C^A_{O,\mathcal{E}_I} \mathcal{E}_I(s) \qquad (6.40)$$

provided that the set of boundary conditions A does not break the conformal symmetry. The splitting amplitudes $C^A_{O,\mathcal{E}_I}$ can be expressed [53] in terms of bulk operator product expansion coefficients c^k_{ij}. In the context of recoiling D-particles, the splitting coefficients for a closed string state to split into a pair of open string excitations, with their ends attached to the D-particles, have been shown [19] to be non-zero by expressing them in terms of the bulk amplitude $c^D_{O,O}$ for an ''in'' closed string state to scatter off the D-brane into an ''out'' string state, including the recoil operator D, the latter being represented as a worldsheet bulk operator [15,16],

$$(C^A_{O,D})^2 \simeq \frac{1}{\sqrt{\log \Lambda}} c^D_{O,O} . \qquad (6.41)$$

In Eq. (6.41) we have concentrated for simplicity on the leading divergent contributions as $\epsilon \to 0^+$ which are associated with the D operator. This allows for closed-to-open string state transitions within the present framework.

For an isolated D-particle, these open string excitations have their ends attached to the same point. For two D-particles the ends of the open string can attach to different points. Since the recoil of the constituent D-particles causes the fat brane as a whole to recoil as well, the interactions mediated by the open strings cause a non-trivial correlation between different coordinate degrees of freedom stretched between the two particles. Only when there is no recoil ($u_a = u_b = 0$) can one measure independently the positions of the two D-particles. In this way the uncertainties in length measurements and the position correlations between two D-branes depend on the energy content of the scattering process and grow with increasing recoil energies.

Notice that the correlation (6.38) we have derived is not simply a product of uncertainties $\Delta X_1 \Delta X_2$, as is the usual case in axiomatic approaches to spacetime quantization based on noncommutative geometry [32] or as one would have naively expected from the Lie algebraic noncommutativity of the multiple D-brane matrix coordinates X_i^{ab}. The Schwarz inequality

$$|((X_1^{ab}|X_2^{ab}))_{\text{conn}}| \leqslant \Delta X_1^{ab} \Delta X_2^{ab} \qquad (6.42)$$

leads to a spacetime uncertainty relation in the spirit of [32]. However, the quantum mechanical correlation (6.38) is much stronger than this uncertainty relation, because two random variables can be independent yet have non-vanishing variances, and as such it probes much deeper into the short distance structure of spacetime. The present worldsheet approach associates the Lie algebraic noncommutativity to a spacetime noncommutativity only rather subtly through the relation (6.38). This differs from the approach of [34] which identifies the two types of noncommutative algebras using the Schild formalism of string theory, in which case the uncertainties in the D-particle positions are given by

$$(\Delta y_i^a)^2 \equiv [(Y_i - Y_i^{aa})^2]^{aa} = \sum_{b \neq a} |Y_i^{ab}|^2. \qquad (6.43)$$

In contrast to our uncertainties, the smearing (6.43) is a direct result of the open string interactions between particle a and all of the other D-branes. The inequalities (6.38),(6.42) essentially summarize the implications of the noncommutative nature of spacetime on the measurability of lengths. Their energy dependence distinguishes them from the usual inequalities which arise in axiomatic noncommutative field theories (which involve only the spacetime Planck length), and moreover the present uncertainties are derived from Lagrangian dynamics for the system of D-particles.

D. Quantum phase space

The quantum phase space of the multi-D-brane system is determined by the canonical momentum (4.26) which, according to Eq. (2.31), upon quantization becomes an operator $\hat{\Pi}^j_{ab}$ obeying the Heisenberg commutation relations

$$[[\hat{Y}_i^{ab}, \hat{\Pi}^j_{cd}]] = i\hbar_{\mathcal{M}} \delta_i^j \delta_c^a \delta_d^b \qquad (6.44)$$

on $\mathcal{M}$. The relation (6.44) leads to the moduli space Heisenberg uncertainty principle

$$\Delta \bar{Y}_i^{ab} \Delta \tilde{\Pi}_{cd}^j \geq \tfrac{1}{2} \hbar_{\mathcal{M}} \delta_i^j \delta_c^a \delta_d^b . \tag{6.45}$$

The Planck constant $\hbar_{\mathcal{M}}$ can be determined by noting that, in the present context, the partition function (6.16) is identified with the wave function of the multi-D-brane system. The lower bound in Eq. (6.45) is then saturated if one interprets Eq. (6.16) as a minimum uncertainty wavepacket on moduli space. In the single D-particle case, such an assumption is consistent with the solution of a generalized Schrödinger equation [19], stemming from an application of a worldsheet Wilsonian renormalization group equation, under the identification of the Liouville field with target time.

Since we have effectively been representing the canonical momentum $\tilde{\Pi}_{ab}^j$ as an operator in coupling constant space [see Eq. (2.32)], the effects of the summation over worldsheet topologies on it are implicitly already taken into account. This means that the variance $(\Delta \tilde{\Pi}_{ab}^j)^2$ can be computed in the worldsheet σ-model on a tree-level disc topology. In this way, using the two-point and one-point functions (5.10) and (4.26) we find

$$(\Delta \tilde{\Pi}_{ab}^j)^2 = \tilde{G}_{ab;ab}^{jj}(\bar{Y},\bar{U}) - (\Pi_{ab}^j(\bar{Y},\bar{U}))^2$$

$$= \frac{4\bar{g}_s^2}{\alpha'} \delta_{ab} + \frac{2\bar{g}_s^4}{9\alpha'} (2\delta_{ab}[(\bar{U}^j)^2]_{ba}$$

$$- 287(\bar{U}_{ba}^j)^2) + \mathcal{O}(\bar{g}_s^6) \tag{6.46}$$

to lowest orders in $\bar{g}_s$. $\hbar_{\mathcal{M}}$ can then be found by performing a Galilean boost to a co-moving target space frame in which the recoil velocities vanish. For example, setting $a = b = c = d$ in Eq. (6.45) with the inequality saturated and substituting in Eqs. (6.35) and (6.46) for $\bar{U} = 0$, we can solve for the moduli space Planck constant to get

$$\hbar_{\mathcal{M}} = 4|\bar{g}_s|^{1+\chi/2} \tag{6.47}$$

which we note is time independent. Thus the basic constant $\hbar_{\mathcal{M}}$ of the resulting quantum phase space is proportional to the string coupling $|\bar{g}_s|$, which owes to the fact that in the present case quantum mechanics is induced by string interactions.

The velocity-dependent terms in Eq. (6.46) correspond to stringy corrections. As mentioned at the beginning of Sec. V, to lowest order in the string coupling constant expansion, the moduli space coincides with the phase space of the D-particle system. This means that, with the appropriate mass normalization, we can identify the canonical momentum with the velocity $\bar{U}$, so that to lowest orders the position and velocity operators have a canonical quantum commutator of the form (6.44). We can therefore compute the commutator $[[\hat{Y}_i^{ab}, \hat{p}_{cd}^j]]$ iteratively, using Eq. (4.26), by assuming a position-velocity commutator of the form (6.44) and identifying the velocity-squared terms which arise from the

commutators involving the $\bar{U}^3$ terms in Eq. (4.26) with squares of the momentum operator $\hat{p}$. After some algebra, this leads to the string-modified Heisenberg commutation relations

$$[[\hat{Y}_i^{ab}, \hat{p}_{cd}^j]] = i\hbar_{\mathcal{M}}(\delta_i^j \delta_c^a \delta_d^b + \tfrac{1}{96} |\bar{g}_s|^2 \bar{\alpha}_s'[\delta_i^j(\delta_c^a[\hat{p}_k \hat{p}^k]_d^b$$

$$+ \delta_d^b[\hat{p}_k \hat{p}^k]_c^a + [\hat{p}_k]_c^a[\hat{p}^k]_d^b) + \delta_c^a\{\hat{p}_i, \hat{p}^j\}_d^b$$

$$+ \delta_d^b\{\hat{p}_i, \hat{p}^j\}_c^a + [\hat{p}_i]_d^b[\hat{p}^j]_c^a + [\hat{p}_i]_c^a[\hat{p}^j]_d^b]$$

$$+ \cdots) \tag{6.48}$$

to leading orders, where $\bar{\alpha}_s' = \alpha'/|\bar{g}_s|^4$ is the (time independent) 0-brane scale with the present mass normalization.

The commutation relation (6.48) represents the appropriate generalization of the string-modified phase space relations (1.7),(1.8) to the multi-D-particle case. For $a = b = c = d$ and $i = j$ it reproduces the standard string-modified phase space uncertainty principle [29] for a single recoiling D-particle [17,28]. However, it also takes into account of the various string interactions among D-particles [the off-diagonal parts of Eq. (6.48)]. Minimizing the off-diagonal components (in both Lorentz and color indices) of the uncertainty relations corresponding to Eq. (6.48) leads to nontrivial kinetic energy dependent uncertainties among the various open string excitations, and also along different spatial directions. The relation (6.48) represents the phase space version of the noncommutative quantum uncertainties that were derived in the previous subsection. We note that, even for a single D-particle, at higher orders in $\bar{g}_s$ the phase space uncertainty relations here are different from the ones derived in [29] in that the modifications depend on the recoil velocities and not only on the uncertainties in the momenta. In fact, the present approach gives a formal prescription for evaluating the higher-order stringy corrections to the Heisenberg uncertainty relations in string perturbation theory, in principle to arbitrary order in the (weak) string coupling constant.

E. Space-time uncertainty principles

Upon summation over worldsheet genera the physical target space time coordinate (5.19) becomes a quantum operator $\hat{T}$ [49], unlike the situation in conventional quantum mechanics. Within the present Born-Oppenheimer approximation, we can expand the function (5.16) as a power series in $\|\bar{U}_{ab}\|/\|u_c\| \ll 1$, $a \neq b$, using the identity

$$\text{tr} \bar{U}_i \bar{U}^i = \frac{E_{\text{tot}}}{|\bar{g}_s|^2} \left(1 + |\bar{g}_s|^2 \sum_{a \neq b} \frac{\bar{U}_i^{ab} \bar{U}_{ba}^i}{E_{\text{tot}}}\right) \tag{6.49}$$

where $E_{\text{tot}} = |\bar{g}_s|^2 \Sigma_{a=1}^N \|\bar{U}_{aa}\|^2$ is the total kinetic energy (per unit string length) of the individual D-particles. We substitute Eq. (6.49) into Eq. (5.19), expand the square root to lowest order in the off-diagonal velocities, and average over

the worldsheet renormalization group time parameter $\log \Lambda$. Identifying the velocity operators with $\hat{\Pi}_{ab}^{j}$ as described in the previous subsection and using the Heisenberg commutation relations (6.44) we arrive at the space-time quantum commutation relations

$$[[\hat{Y}_i^{ab}, \hat{T}]] \simeq \frac{i \alpha' \hbar_{\mathcal{M}}}{2|\bar{g}_s|} \left(\delta^{ab} + (1 - \delta^{ab}) \frac{\sqrt{\alpha'}}{4|\bar{g}_s|} \frac{\hat{\Pi}_i^{ab}}{\sqrt{E_{\text{tot}}}} \right) \tag{6.50}$$

to leading order in $\bar{g}_s$ (or equivalently in the off-diagonal velocity expansion).

From Eqs. (6.47) and (6.50) we infer the space-time uncertainty relation

$$\Delta \bar{Y}_i^{aa} \Delta T \geqslant |\bar{g}_s|^{\chi/2} \alpha' \tag{6.51}$$

for the individual D-particle coordinates. For $\chi = 0$, Eq. (6.51) yields the standard lower bound (1.9) which is independent of the string coupling, as argued in [33–35] from basic string ideas. But then the minimal distance (6.35) does not probe scales down to the 11-dimensional Planck length. This fact can be understood by noting that the physical target space (Liouville) time coordinate T is not the same as the longitudinal worldline coordinate of a D-particle, as is assumed in the arguments leading to the hypothesis (1.9), but is rather a collective time coordinate of the D-particle system which is induced by all of the string interactions among the particles. However, we can adjust the uncertainty relations to match the dynamical properties of 11-dimensional supergravity by multiplying the definition (5.19) by an overall factor of $|\bar{g}_s|^{-\chi/2}$. This redefinition will be assumed below, and it implies that with weak string interactions the target space propagation time for the D-particles is very long.

To see the effects of the string interactions between D-particles in this space-time framework, we again use the canonical (minimal) smearing Eq. (6.45) between $\hat{Y}_i^{ab}$ and $\hat{\Pi}_{ab}^i$ for $a \neq b$ in Eq. (6.50) to arrive at a triple uncertainty relation

$$(\Delta \bar{Y}_i^{ab})^2 \Delta T \geqslant \frac{|\bar{g}_s|^{\chi/2} \alpha'^{3/2}}{2 \sqrt{E_{\text{tot}}}}, \quad a \neq b. \tag{6.52}$$

The uncertainty principle (6.52) depends on the total kinetic energy of the constituent D-branes. It implies that the system of D-particles, through their open string interactions, can probe distances much smaller than the characteristic distance scale in Eq. (6.52), which for $\chi = \frac{2}{3}$ is $l_P^{(11)} l_s^2$, provided that their kinetic energies are large enough. In the fully relativistic case the existence of a limiting speed $\|u_a\| < 1$ implies a lower bound on Eq. (6.52). With the minimum spatial extensions obtained in Sec. VI C, this bound yields, for $\chi = \frac{2}{3}$, the characteristic temporal length

$$\Delta T \geqslant |\bar{g}_s|^{-1/3} \sqrt{\alpha'} \tag{6.53}$$

for D-particles [35] [see Eq. (1.10)]. Triple uncertainty relations involving only the 11-dimensional Planck length have been suggested in [35] based on the holographic principle of M theory.

Again the present approach formally gives a prescription for evaluating higher-order contributions to the space-time quantum commutator (6.50) in the string coupling $\bar{g}_s$ (or in the velocity expansion). A characteristic feature of the uncertainty relations we have derived in this section, which distinguishes D-particle dynamics from ordinary quantum mechanics, is their dependences on the recoil momenta. The dependence of quantum uncertainties in the measurement of certain quantities on the magnitude of the quantities themselves (here the kinetic energies of the D-branes) is characteristic of decoherence effects which are induced by quantum gravity [55]. It was argued in [19] that the quantum recoil degrees of freedom are responsible for inducing decoherence in low-energy systems. In the case of a single D-particle, the analysis of [56] demonstrates explicitly the induced decoherence by exhibiting particle creation in the direction of the recoiling velocities for the scattering of a spectator light mode in the presence of a D-particle due to the scattering of another closed string state off the defect. The analysis of this section thus shows that multiple D-particle field theory in flat target spaces naturally incorporates quantum gravity effects into the sub-Planckian spacetime structure. It therefore illuminates the manner in which D-particle interactions probe very short distances where the effects of quantum gravity are significant.

VII. CONCLUSIONS

In this paper we have employed a worldsheet approach to the study of the collective dynamics of N parallel D-branes, interacting through the exchange of open (or closed) strings, which are scattered off them. This is the simplest model of multi-brane dynamics, where the branes do not intersect. Working with Neumann boundary conditions, in which the coupling constants of the pertinent σ-model are $U(N)$ gauge potentials, we have developed a formalism for describing recoil of the multi-brane system after scattering with low-energy string states. This formalism utilizes generic properties of logarithmic conformal field theories on the worldsheet. In this way we have shown that the recoil deformations define a system of collective coordinates and momenta which are consistent with the corresponding ones derived from a (symmetrized) non-Abelian Born-Infeld effective action. We have argued that worldsheet genus expansion produces quantum fluctuations (in target space) of these σ-model couplings. For a specific choice of consistent gauge field backgrounds, therefore, a quantum phase space arises, which however involves noncommutativity among all coordinate directions as a result of the interactions of the branes. We also derived new coordinate uncertainty relations, among different components of the coordinate matrices of the interacting D-branes, consistent with generic expectations from noncommutative geometry analyses. These relations justify properly the association of Lie algebraic noncommutativity with quantum mechanical noncommutativity, and as we have

discussed this is a non-trivial fact. We have also discussed the definition of target time in the context of the Liouville approach, and shown that it becomes an operator in this formalism, which exhibits unconventional uncertainty relations with the collective coordinates.

There are many aspects of the approach of this paper that still require examination. The most glaring one is the arbitrariness of the exponent χ in Eq. (6.33). In the present approach, which considers only string interactions, we have not found any way to fix its value, but it may be fixed upon considering brane exchanges between the system of D-branes. Another aspect that needs to be worked out is the explicit calculation of the perturbation expansion to some higher-orders which will begin to involve not just the velocities of the D-particles, but also their collective coordinates. The resulting moduli space geometry, which as we have shown naturally describes the structure of spacetime at sub-Planckian scales, will then contain information not only about the kinematics of the D-particles, but also of their dynamics which are governed by terms such as the Yang-Mills potential (1.1). This would then lead to spacetime noncommutativity from the quantum phase space structure itself, and presumably new forms of spacetime uncertainty relations. Of course, the present results only strictly apply to the simplest physical system whose motion is governed by fat brane dynamics. It would be interesting to consider more complicated matrix D-brane couplings involving, for example, higher-rank Jordan blocks in the spectrum of the underlying logarithmic conformal field theory. Such generalizations may probe deeper into the nature of the string interactions among the branes, and hence into the small-scale structure of spacetime. Another generalization involves the incorporation of intersecting D-branes in this formalism. It would be interesting to see whether there exists an appropriate generalization of logarithmic operators that describes quantum fluctuations of such systems. Such constructions are crucial to the understanding of the stringy quantum spacetime at sub-Planckian scales. They may also shed further light on the short-distance structure, fundamental degrees of freedom and dynamics of M theory within the geometrical framework of moduli space dynamics.

It would be interesting to see if the present worldsheet approach, which exhibits unconventional properties of string spacetimes, is amenable in some way to experimental verification. The presence of multi-D-brane domain wall structures, like the ones considered in this paper, may act as traps of low-energy string states, thereby resulting in a decoherent medium nature of quantum gravity spacetime foam. In the present case the quantum coordinate fluctuations, due to the open string excitations between the D-particles, can lead to quantum decoherence for a low-energy observer who cannot detect such recoil fluctuations in the sub-Planckian spacetime structure. These foamy properties of the noncommutative structure of the D-particle spacetime might require a reformulation of the phenomenological analyses of length measurements as probes of quantum gravity. If one accepts the generic l_P maximal suppression effects by the gravitational (Planck) mass scales, then, as described in some recent literature, there may be sensitive probes such as neutral kaon

sytems [57] or cosmological gamma-ray burst spectroscopy [58]. However, such approaches do not incorporate length measurements in the transverse directions to the probe, so that it is unclear how to incorporate noncommutative uncertainty relations such as Eq. (6.38) into these analyses.

APPENDIX A: CORRELATION FUNCTIONS OF LOGARITHMIC OPERATORS

In this appendix we will describe some properties and compute the first few correlation functions of the C and D logarithmic operators that were introduced in Sec. III. They are calculated using fundamental string averages which are evaluated with the propagator

$$\langle x^\mu(z_1,\bar{z}_1)x^\nu(z_2,\bar{z}_2)\rangle_0 = 2\alpha'\,\eta^{\mu\nu}\log|z_1-z_2| \quad \text{(A1)}$$

associated with the action (3.4). The coincidence limit of the two-point function (A1) is defined using the short-distance cutoff Λ as

$$\langle x^\mu(z,\bar{z})x^\nu(z,\bar{z})\rangle_0 = 2\alpha'\,\eta^{\mu\nu}\log\Lambda. \quad \text{(A2)}$$

The correlators of the logarithmic operators (3.13) and (3.16) can now be evaluated using the regulated step function (3.14). Note that upon integrating by parts the D operator can be written as

$$D(x^0;\epsilon) = -\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{dq}{(q-i\epsilon)^2}e^{iqx^0} = -\frac{1}{\epsilon}\frac{\partial}{\partial\epsilon}C(x^0;\epsilon). \quad \text{(A3)}$$

The second equality in Eq. (A3) also follows from the general property $D = \alpha'\,\partial C/\partial\Delta_\epsilon$ of logarithmic conformal field theories [59]. This property enables one to deduce expressions for many of the correlators once the expectation values of the C operator are known [60]. Using these identities one can compute explicitly the one-point correlation functions in the correlated limit $\epsilon\to 0^+$ with the relation (3.15) to get [17]

$$\langle C(x^0;\epsilon)\rangle_0 = \mathcal{O}(\epsilon), \quad \langle D(x^0;\epsilon)\rangle_0 = a/\epsilon \quad \text{(A4)}$$

where here and in the following $a,b,\ldots$ denote (in principle arbitrary) dimensionless constants.

The higher-point correlators can be computed using the Koba-Nielsen formula

$$\left\langle \prod_{i=1}^{n} e^{iq_i x^0(z_i,\bar{z}_i)}\right\rangle_0 = \prod_{i,j} e^{-q_i q_j\langle x^0(z_i,\bar{z}_i)x^0(z_j,\bar{z}_j)\rangle_0/2}. \quad \text{(A5)}$$

For the two-point functions one finds, always in the correlated limit $\epsilon \rightarrow 0^+$, the expressions [17]

$$\langle C(z,\bar{z};\epsilon)C(w,\bar{w};\epsilon)\rangle_0 = \mathcal{O}(\epsilon^2) \tag{A6}$$

$$\langle C(z,\bar{z};\epsilon)D(w,\bar{w};\epsilon)\rangle_0 = \frac{b}{|z-w|^{2\Delta_\epsilon}} \tag{A7}$$

$$\langle D(z,\bar{z};\epsilon)D(w,\bar{w};\epsilon)\rangle_0 = \frac{1}{\epsilon^2}\langle C(z,\bar{z};\epsilon)D(w,\bar{w};\epsilon)\rangle_0$$

$$= -\frac{b\alpha'}{|z-w|^{2\Delta_\epsilon}}\left(\frac{1}{2\Delta_\epsilon}+\log\left|\frac{z-w}{\Lambda}\right|^2\right). \tag{A8}$$

From Eq. (3.15) it follows that Eqs. (A6)–(A8) have the canonical form of the two-point correlation functions of a generic logarithmic conformal field theory. The constant in Eq. (A8) which depends on the anomalous dimension Δ_ϵ can be made arbitrary by shifting the D operator according to Eq. (3.19) (i.e. by a worldsheet scale transformation), whereas the coefficient b is fixed by the leading logarithmic terms in the conformal blocks. Note that the correlators in Eqs. (A4) and (A6)–(A8) involving solely the C field vanish while those involving only the D field diverge as $\epsilon \rightarrow 0^+$.

The three-point functions of the logarithmic pair can be calculated using the canonical forms derived for general logarithmic conformal field theories in [60]. As in Eqs. (A6)–(A8), these correlators involve some arbitrary (integration) constants, while the coefficients of the logarithmically divergent terms are fixed by the leading logarithmic behaviors of the conformal blocks. We can therefore apply the results of [60] to the present case [using the behaviors of Eqs. (A6)–(A8)] provided we know the leading behaviors of the three-point functions as $\epsilon \rightarrow 0^+$. For example, consider the three-point function of the C fields, which using Eqs. (A5), (A1) and (A2) is given by

$$\langle C(z_1,\bar{z}_1;\epsilon)C(z_2,\bar{z}_2;\epsilon)C(z_3,\bar{z}_3;\epsilon)\rangle_0$$

$$= \frac{\epsilon^3}{(2\pi i)^3}\int_{-\infty}^{\infty}\prod_{k=1}^{3}\frac{dq_k}{q_k-i\epsilon}e^{-\alpha' q_k^2 \log \Lambda}$$

$$\times \prod_{k<j} e^{-2\alpha' q_k q_j \log|z_{kj}|}. \tag{A9}$$

Using Eq. (3.15) and rescaling the integration variables in Eq. (A9) as $q_k = \epsilon \tilde{q}_k$, we have

$$\langle C(z_1,\bar{z}_1;\epsilon)C(z_2,\bar{z}_2;\epsilon)C(z_3,\bar{z}_3;\epsilon)\rangle_0$$

$$= \frac{\epsilon^3}{(2\pi i)^3}\int_{-\infty}^{\infty}\prod_{k=1}^{3}\frac{d\tilde{q}_k}{\tilde{q}_k-i}e^{-\tilde{q}_k^2/2}\prod_{k<j} e^{-2\alpha' \epsilon^2 \tilde{q}_k \tilde{q}_j \log|z_{kj}|}. \tag{A10}$$

The last product in Eq. (A10) has the form $\prod_{k<j}e^{-2\alpha'\epsilon^2\tilde{q}_k\tilde{q}_j\log|z_{kj}|}\sim 1+\mathcal{O}(\epsilon^2)$, so that the three-point

function has leading constant term which vanishes as ϵ^3, while the remaining z-dependent terms coming from the final product in Eq. (A10) are sub-leading in ϵ. Thus $\langle CCC\rangle_0 \sim \epsilon^3$. Using exactly the same method one shows that $\langle CCD\rangle_0 \sim \epsilon$, $\langle CDD\rangle_0 \sim 1/\epsilon$ and $\langle DDD\rangle_0 \sim 1/\epsilon^3$.

From these leading behaviors in ϵ we can now read off from [60] the three-point correlation functions,[7]

$$\langle C(z_1,\bar{z}_1;\epsilon)C(z_2,\bar{z}_2;\epsilon)C(z_3,\bar{z}_3;\epsilon)\rangle_0$$

$$= \frac{c\epsilon^3}{|z_{12}z_{23}z_{31}|^{2\Delta_\epsilon}} \tag{A11}$$

$$\langle D(z_1,\bar{z}_1;\epsilon)C(z_2,\bar{z}_2;\epsilon)C(z_3,\bar{z}_3;\epsilon)\rangle_0$$

$$= \frac{1}{|z_{12}z_{23}z_{31}|^{2\Delta_\epsilon}}\left(d\epsilon+\frac{c}{2}\epsilon^3\alpha'\log\left|\frac{z_{23}\Lambda}{z_{12}z_{31}}\right|^2\right) \tag{A12}$$

$$\langle D(z_1,\bar{z}_1;\epsilon)D(z_2,\bar{z}_2;\epsilon)C(z_3,\bar{z}_3;\epsilon)\rangle_0$$

$$= \frac{1}{|z_{12}z_{23}z_{31}|^{2\Delta_\epsilon}}\left(\frac{e}{\epsilon}-d\epsilon\alpha'\log\left|\frac{z_{12}}{\Lambda}\right|^2\right.$$

$$\left.+c\epsilon^3\alpha'^2\left[\left(\frac{\log\left|\frac{z_{23}}{\Lambda}\right|^2}{\log\left|\frac{z_{31}}{\Lambda}\right|^2}\right)^2+\frac{1}{4}\left(\log\left|\frac{z_{12}}{\Lambda}\right|^2\right)^2\right]\right) \tag{A13}$$

$$\langle D(z_1,\bar{z}_1;\epsilon)D(z_2,\bar{z}_2;\epsilon)D(z_3,\bar{z}_3;\epsilon)\rangle_0$$

$$= \frac{1}{|z_{12}z_{23}z_{31}|^{2\Delta_\epsilon}}\left\{\frac{f}{\epsilon^3}-\frac{e}{2\epsilon}\alpha'\log\left|\frac{z_{12}z_{23}z_{31}}{\Lambda^3}\right|^2\right.$$

$$+d\epsilon\alpha'^2\left[\log\left|\frac{z_{12}}{\Lambda}\right|^2\log\left|\frac{z_{23}}{\Lambda}\right|^2+\log\left|\frac{z_{12}}{\Lambda}\right|^2\log\left|\frac{z_{31}}{\Lambda}\right|^2\right.$$

$$\left.+\log\left|\frac{z_{23}}{\Lambda}\right|^2\log\left|\frac{z_{31}}{\Lambda}\right|^2\right]-\frac{d\epsilon}{4}\alpha'^2\left(\log\left|\frac{z_{12}z_{23}z_{31}}{\Lambda^3}\right|^2\right)^2$$

$$+c\epsilon^3\alpha'^3\log\left|\frac{z_{12}}{\Lambda}\right|^2\log\left|\frac{z_{23}}{\Lambda}\right|^2\log\left|\frac{z_{31}}{\Lambda}\right|^2$$

$$-\frac{c}{2}\epsilon^3\alpha'^3\log\left|\frac{z_{12}z_{23}z_{31}}{\Lambda^3}\right|^2\left[\log\left|\frac{z_{12}}{\Lambda}\right|^2\log\left|\frac{z_{23}}{\Lambda}\right|^2\right.$$

$$\left.\left.+\log\left|\frac{z_{12}}{\Lambda}\right|^2\log\left|\frac{z_{31}}{\Lambda}\right|^2+\log\left|\frac{z_{23}}{\Lambda}\right|^2\log\left|\frac{z_{31}}{\Lambda}\right|^2\right]\right\}$$

---

[7]In the perturbative calculations of Secs. IV and V we neglected throughout the parts of the correlators which involve exponents of the scaling dimension Δ_ϵ, as these terms do not contribute to the leading divergences as $\epsilon \rightarrow 0^+$.

$$+\frac{c}{8}\epsilon^3\alpha'^3\left(\log\left|\frac{z_{12}z_{23}z_{31}}{\Lambda^3}\right|^2\right)^3\right\}. \tag{A14}$$

Note that on the boundary of the worldsheet Σ where $z_i = e^{2\pi i s_i}$, $s_i \in [0,1]$, the propagator (A1) becomes

$$\langle x^\mu(s_1)x^\nu(s_2)\rangle_0 = \alpha'\eta^{\mu\nu}\log[2-2\cos 2\pi(s_1-s_2)]. \tag{A15}$$

This can be used to express all correlators above in terms of the boundary variables. Comparing Eqs. (A15) with (A2) we see that the short-distance cutoff on the boundary variables is

$$s_\Lambda = \frac{1}{2\pi}\arccos\left(1-\frac{\Lambda^2}{2}\right) = \frac{\Lambda^2}{2\pi}+\frac{\Lambda^4}{48\pi}+\mathcal{O}(\Lambda^6). \tag{A16}$$

Furthermore, differentiating Eq. (A15) we arrive at the correlator

$$\left\langle\frac{d}{ds_1}x^i(s_1)\frac{d}{ds_2}x^j(s_2)\right\rangle_0 = \frac{4\pi^2\alpha'\eta^{ij}}{1-\cos 2\pi(s_1-s_2)}. \tag{A17}$$

The calculation of n-point functions with $n\geq 4$ is quite cumbersome. As described in [60], they can be evaluated in principle by noting that the C operators are primary fields and hence have standard conformal field theoretical correlation functions, from which all other correlators of the logarithmic pair may be found via differentiation using the identity (A3). Their behaviors as $\epsilon\to 0^+$ can be deduced rather directly using relations analogous to Eq. (2.16) between the three-point functions and the operator product expansion coefficients, which remain valid in the presence of logarithmic deformations [47]. The logarithmic pair C,D form a complete set of states in the 2×2 Jordan cell of the Virasoro generator L_0. From Eqs. (A6)–(A8) it follows that the Zamolodchikov metric in the C,D basis behaves as [28,47]

$$G_{CC}\sim\epsilon^2,\quad G_{DD}\sim\epsilon^{-2},\quad G_{CD}=G_{DC}\sim\text{const}. \tag{A18}$$

Then Eq. (2.17) yields, for example, the scaling behavior

$$\langle CC\rangle_0\sim G^{CC}\langle CCC\rangle_0\langle C\rangle_0 + G^{DD}\langle CCD\rangle_0\langle D\rangle_0$$
$$+G^{CD}(\langle CCC\rangle_0\langle D\rangle_0 + \langle CCD\rangle_0\langle C\rangle_0). \tag{A19}$$

From Eq. (A6) we see that the left-hand side of Eq. (A19) is $\mathcal{O}(\epsilon^2)$. Then using Eqs. (A4) and (A18) we can immediately deduce the anticipated small ϵ behaviors of $\langle CCC\rangle_0$ and $\langle CCD\rangle_0$. The general result is

$$\left\langle\prod_{i=1}^n C(z_i,\bar{z}_i;\epsilon)\prod_{j=1}^m D(w_j,\bar{w}_j;\epsilon)\right\rangle_0 \sim\mathcal{O}(\epsilon^{n-m}). \tag{A20}$$

This relation does not, however, yield any information about the logarithmic scaling violations present in the correlation functions, i.e. their dependences on the worldsheet renormalization group scale $\log\Lambda$.

APPENDIX B: RENORMALIZATION OF THE CANONICAL MOMENTUM

In this appendix we shall derive the expression (4.11) for the renormalized canonical momentum. From Eq. (4.10) it follows that the momentum contribution (4.9) can be written as a sum over permutations $P\in S_{n+2}$. This sum can be decomposed into a sum over permutations $P\in S_n\times S_2$ which permute only contractions among the $\Pi_{k=1}^n\bar{\xi}_{a_k}(s_k) -\epsilon)\xi_{b_k}(s_k)$ part of the auxilliary field expectation value in Eq. (4.9) among themselves, and the remaining (abc) part of this correlator among themselves, plus a sum over the remaining ones $P\in S_{n+2}-(S_n\times S_2)$. Let us first introduce some short-hand notation. For each positive integer m we define an m-dimensional integration measure $d\mu_m$ on $[0,1]^m$ by

$$\int_{[0,1]^m}d\mu_m(s_1,\ldots,s_m)$$

$$\equiv\int_0^1\prod_{k=1}^m ds_k\left(\prod_{l=1}^{m-1}\Theta(s_{l+1}-s_l)\right)\Theta(s_1-s_m)$$

$$=\int_0^1 ds_m\int_0^1\prod_{k=2}^{2[m/2]-2}ds_k\int_{\alpha(m)}^{s_m}ds_{m-1}$$

$$\times\left(\prod_{k=2[m/2]-3}^2\int_{s_k}^{s_{k+2}}ds_{k+1}\right)\int_{s_m}^{s_2}ds_1 \tag{B1}$$

where $[m/2]$ is the integer part of $m/2$, and $\alpha(m)=s_{m-2}$ for m even and $\alpha(m)=0$ for m odd. We define the initial value $\int d\mu_{m=0}\equiv 1$. We also define an $N\times N$ Hermitian matrix $\mathcal{T}_m$ by

$$\mathcal{T}_m[Y,x;s_1,\ldots,s_m]_{ab}\equiv\left[\prod_{k=1}^m Y_{j_k}(x^0(s_k))\frac{d}{ds_k}x^{j_k}(s_k)\right]_{ab} \tag{B2}$$

with the initial value $[\mathcal{T}_{m=0}]_{ab}\equiv\delta_{ab}$.

We begin by evaluating the contribution to Eq. (4.9) from $P\in S_n\times S_2$, which give

$$\mathcal{P}_{ab}^{(n)j}[Y;s]|_{S_n\times S_2}$$

$$=\lim_{\epsilon\to 0^+}\sum_{c=1}^N\sum_{b_1,\ldots,b_n}\sum_{P\in S_n}\int_0^1\prod_{k=1}^n ds_k\Theta(s_{P(k)}-s_k)$$

$$\times(\Theta(\epsilon)\delta_{ab}\delta_{cc}+\delta_{ac}\delta_{cb})$$

$$\times\left\langle\frac{d}{ds}x^j(s)\prod_{k=1}^n Y_{i_k}^{b_{P(k)},b_k}(x^0(s_k))\frac{d}{ds_k}x^{i_k}(s_k)\right\rangle_0 \tag{B3}$$

where we have explicitly summed over the S_2 part. To express Eq. (B3) in a more succinct form, we decompose each permutation $P \in S_n$ into a product of disjoint cycles $C_i(P)$,

$$P = \prod_{i=1}^{n} C_i(P) \tag{B4}$$

and let $L_i(P) \geq 0$ denote the length of the cycle $C_i(P)$, so that the set of integers $\{L_i(P)\}$ form a partition of n,

$$\sum_{i=1}^{n} L_i(P) = n. \tag{B5}$$

It is then possible to express the boundary measure and Y-matrix products in Eq. (B3) in a more explicit form by writing products $\Pi_{k=1}^{n}$ in terms of this cyclic decomposition as $\Pi_{i=1}^{n}\Pi_{m=1}^{L_i(P)}$ for each $P \in S_n$. We can explicitly combine the products in the correlation functions in Eq. (B3) into matrix products, using the cyclicity of each $C_i(P)$ and summing over the b_i's. We can also label the boundary integrations $s_k \equiv s_{C_i(P)_{(k)}}$ in terms of the components of the cycles in Eq. (B4), giving the integration measure (B1) for each $i = 1, \ldots, n$. In this way the sum over permutations in Eq. (B3) can be written as a sum over partitions (B5), and the result after some algebra is

$$
\mathcal{P}_{ab}^{(n)j}[Y;s]\big|_{S_n \times S_2} = \lim_{\epsilon \to 0^+} (N\Theta(\epsilon)+1)\delta_{ab} \sum_{\substack{0 \leq L_1, \ldots, L_n \leq n \\ \sum_i L_i = n}} \Theta(\epsilon)^{\sum_i \delta_{L_i,1}} \prod_{i=1}^{n} \int_{[0,1]^{L_i}} d\mu_{L_i}(s_1^{(i)}, \ldots, s_{L_i}^{(i)})
$$

$$
\times \left\langle \frac{d}{ds}x^j(s) \prod_{i=1}^{n} \mathrm{tr}(\mathcal{T}_{L_i}[Y,x;s_1^{(i)}, \ldots, s_{L_i}^{(i)}]) \right\rangle_0 . \tag{B6}
$$

The contributions from the remaining permutations $P \in S_{n+2} - (S_n \times S_2)$ are somewhat more involved. We decompose this sum into three disjoint sums of permutations. In the first class, whose contributions we denote by $\mathcal{P}_{ab}^{(n)j}[Y;s]^{[1]}$, for each P there is a unique integer $k_0 \in \{1, \ldots, n\}$ for which $P(k_0) \notin \{1, \ldots, n\}$ with $P(k_0) = n+1$, while the second class of permutations, whose contributions we denote by $\mathcal{P}_{ab}^{(n)j}[Y;s]^{[2]}$, are those for which there is a unique integer $k_0 \in \{1, \ldots, n\}$ with $P(k_0) \notin \{1, \ldots, n\}$ and $P(k_0) = n+2$. The final contributions $\mathcal{P}_{ab}^{(n)j}[Y;s]^{[3]}$ come from permutations for which there are two integers $k_1, k_2 \in \{1, \ldots, n\}$ with $P(k_1), P(k_2) \notin \{1, \ldots, n\}$ and $P(k_1) = n+1, P(k_2) = n+2$.

We have

$$
\mathcal{P}_{ab}^{(n)j}[Y;s]^{[1]} = \sum_{c=1}^{N} \sum_{b_1, \ldots, b_n} \sum_{k_0=1}^{n} \sum_{\substack{P \in S_{n+2}-(S_n \times S_2) \\ P(k_0)=n+1}} \int_0^1 \prod_{\substack{k=1 \\ k \neq k_0}}^{n} ds_k \Theta(s_{P(k)} - s_k) \int_0^1 ds_{k_0} \Theta(s - s_{k_0})
$$

$$
\times (\delta_{c,b_{P(n+2)}}\delta_{ac} + \Theta(s_{P(n+1)} - s)\delta_{cc}\delta_{a,b_{P(n+1)}})
$$

$$
\times \left\langle \frac{d}{ds}x^j(s) \prod_{\substack{k=1 \\ k \neq k_0}}^{n} Y_{i_k}^{b_{P(k)},b_k}(x^0(s_k)) \frac{d}{ds_k}x^{i_k}(s_k) Y_{i_{k_0}}^{b,b_{k_0}}(x^0(s_{k_0})) \frac{d}{ds_{k_0}}x^{i_{k_0}}(s_{k_0}) \right\rangle_0 . \tag{B7}
$$

In Eq. (B7) the terms with $\delta_{c,b_{P(n+2)}}$ correspond to permutations with $P(n+1) = n+2$, $P(n+2) \in \{1, \ldots, n\}$, while the δ_{cc} terms come from those with $P(n+2) = n+2$, $P(n+1) \in \{1, \ldots, n\}$. In the former terms, we consider the orbit of the integer $n+2$ under a given permutation P, and let $l(P)+2 \geq 3$ be the order of the orbit of k_0 under P, i.e. $P^{l(P)}(n+2) = P^{l(P)+2}(k_0) = k_0$. The sum over $b_{k_0}, b_{P(k_0)}, \ldots, b_{P^{l(P)+2}(k_0)=P^{l(P)-1}(n+2)}$ then yields $\mathcal{T}_{l(P)}[Y,x;s_1, \ldots, s_{l(P)}]_{ba}$ for the corresponding Y-matrix products in Eq. (B7). The corresponding boundary integration measure is $d\mu_{l(P)}(s_1, \ldots, s_{l(P)})$, as before (after appropriate relabellings of the s-indices), with additional step function restrictions as given in Eq. (B7) which must be carefully incorporated into the integration measure (B1). The remaining part of P that does not act on this particular orbit is an element of $S_{n-l(P)}$, so that the remaining sums and products can be decomposed into cycles exactly as in Eq. (B6). For each $l \geq 1$ there are $(n-1)!/(n-l)!$ permutations P under which k_0 has an orbit of order l.

For the latter δ_{cc} terms in Eq. (B7), the integer $l(P)+1 \geq 2$ is the order of the orbit of k_0 under P, i.e. $P^{l(P)+1}(n+1) = P^{l(P)+1}(k_0) = k_0$. The sums over b_i's and all products in Eq. (B7) give the same contribution as for the former c-dependent terms. It follows that the sum over permutations in Eq. (B7) can be written as a sum over the orbit integers $l(P)$ and, for each such integer, a sum over partitions of $n-l(P)$. After some algebra the result is finally

$$\mathcal{P}_{ab}^{(n)j}[Y;s]^{[1]} = \lim_{\epsilon \to 0^+} \sum_{l=1}^{n} \frac{n!}{(n-l)!} \sum_{\substack{0 \leqslant L_1, \dots, L_{n-l} \leqslant n-l \\ \sum_i L_i = n-l}} \Theta(\epsilon) \sum_i \delta_{L_i,1} \prod_{i=1}^{n-l} \int_{[0,1]^{L_i}} d\mu_{L_i}(s_1^{(i)}, \dots, s_{L_i}^{(i)}) \int_{[0,1]^l} d\mu_l(r_1, \dots, r_l)$$

$$\times \Theta(s-r_1)(1+N\Theta(r_l-s)) \left\langle \frac{d}{ds} x^j(s) \prod_{i=1}^{n-l} \mathrm{tr}(\mathcal{T}_{L_i}[Y,x;s_1^{(i)}, \dots, s_{L_i}^{(i)}]) \mathcal{T}_l[Y,x;r_1, \dots, r_l]_{ba} \right\rangle_0. \tag{B8}$$

The resummation of $\mathcal{P}_{ab}^{(n)j}[Y;s]^{[2]}$ carries through in an identical fashion, with the roles of the integers $n+1$ and $n+2$ interchanged. The result is identical to Eq. (B8), except for some changes in the combinatorics of the indices. We find

$$\mathcal{P}_{ab}^{(n)j}[Y;s]^{[2]} = \lim_{\epsilon \to 0^+} \sum_{c=1}^{N} \sum_{b_1, \dots, b_n} \sum_{k_0=1}^{n} \sum_{\substack{P \in S_{n+2}-(S_n \times S_2) \\ P(k_0)=n+2}} \int_0^1 \prod_{\substack{k=1 \\ k \neq k_0}}^{n} ds_k \Theta(s_{P(k)}-s_k) \int_0^1 ds_{k_0} (\Theta(s_{P(n+1)}-s)\delta_{cb}\delta_{a,b_{P(n+1)}}$$

$$+ \Theta(\epsilon)\delta_{c,b_{P(n+2)}}\delta_{ab}) \left\langle \frac{d}{ds} x^j(s) \prod_{\substack{k=1 \\ k \neq k_0}}^{n} Y_{i_k}^{b_{P(k)},b_k}(x^0(s_k)) \frac{d}{ds_k} x^{i_k}(s_k) Y_{i_{k_0}}^{c,b_{k_0}}(x^0(s_{k_0})) \frac{d}{ds_{k_0}} x^{i_{k_0}}(s_{k_0}) \right\rangle_0$$

$$= \lim_{\epsilon \to 0^+} \sum_{l=1}^{n} \frac{n!}{(n-l)!} \sum_{\substack{0 \leqslant L_1, \dots, L_{n-l} \leqslant n-l \\ \sum_i L_i = n-l}} \Theta(\epsilon) \sum_i \delta_{L_i,1} \prod_{i=1}^{n-l} \int_{[0,1]^{L_i}} d\mu_{L_i}(s_1^{(i)}, \dots, s_{L_i}^{(i)}) \int_{[0,1]^l} d\mu_l(r_1, \dots, r_l)$$

$$\times \Theta(s-r_1) \left\langle \frac{d}{ds} x^j(s) \prod_{i=1}^{n-l} \mathrm{tr}(\mathcal{T}_{L_i}[Y,x;s_1^{(i)}, \dots, s_{L_i}^{(i)}]) \{\Theta(r_l-s)\mathcal{T}_l[Y,x;r_1, \dots, r_l]_{ba} \right.$$

$$\left. + \Theta(\epsilon)\mathrm{tr}(\mathcal{T}_l[Y,x;r_1, \dots, r_l])\delta_{ab}\} \right\rangle_0. \tag{B9}$$

Finally, the combinatorics of the resummation of $\mathcal{P}_{ab}^{(n)j}[Y;s]^{[3]}$ now involve tracing the orbits of both $P(n+1), P(n+2)$ $\in \{1, \dots, n\}$, i.e. we introduce two integers $l_1(P)$ and $l_2(P)$ representing the orders of the orbits of k_1 and k_2, respectively, under a given permutation P. The evaluation is then identical to that above with these two orbits taken into account, and we find

$$\mathcal{P}_{ab}^{(n)j}[Y;s]^{[3]} = \sum_{c=1}^{N} \sum_{b_1, \dots, b_n} \sum_{1 \leqslant k_1 \neq k_2 \leqslant n} \sum_{\substack{P \in S_{n+2}-(S_n \times S_2) \\ P(k_1)=n+1, P(k_2)=n+2}} \int_0^1 \prod_{\substack{k=1 \\ k \neq k_1,k_2}}^{n} ds_k \Theta(s_{P(k)}-s_k) \int_0^1 ds_{k_1} \Theta(s-s_{k_1})$$

$$\times \int_0^1 ds_{k_2} \Theta(s_{P(n+1)}-s)\delta_{c,b_{P(n+2)}}\delta_{a,b_{P(n+1)}}$$

$$\times \left\langle \frac{d}{ds} x^j(s) \prod_{\substack{k=1 \\ k \neq k_1,k_2}}^{n} Y_{i_k}^{b_{P(k)},b_k}(x^0(s_k)) \frac{d}{ds_k} x^{i_k}(s_k) Y_{i_{k_1}}^{b,b_{k_1}}(x^0(s_{k_1})) \frac{d}{ds_{k_1}} x^{i_{k_1}}(s_{k_1}) Y_{i_{k_2}}^{c,b_{k_2}}(x^0(s_{k_2})) \frac{d}{ds_{k_2}} x^{i_{k_2}}(s_{k_2}) \right\rangle_0$$

$$= \lim_{\epsilon \to 0^+} \sum_{l_1=1}^{n-1} \sum_{l_2=1}^{n-l_1} \frac{n!}{(n-l_1)(n-l_1-l_2)!} \sum_{\substack{0 \leqslant L_1, \dots, L_{n-l_1-l_2} \leqslant n-l_1-l_2 \\ \sum_i L_i = n-l_1-l_2}} \Theta(\epsilon) \sum_i \delta_{L_i,1} \prod_{i=1}^{n-l_1-l_2}$$

$$\times \int_{[0,1]^{L_i}} d\mu_{L_i}(s_1^{(i)}, \dots, s_{L_i}^{(i)}) \int_{[0,1]^{l_2}} d\mu_{l_2}(t_1, \dots, t_{l_2}) \int_{[0,1]^{l_1}} d\mu_{l_1}(r_1, \dots, r_{l_1}) \Theta(r_{l_1}-s)\Theta(s-r_1)$$

$$\times \left\langle \frac{d}{ds} x^j(s) \prod_{i=1}^{n-l_1-l_2} \mathrm{tr}(\mathcal{T}_{L_i}[Y,x;s_1^{(i)}, \dots, s_{L_i}^{(i)}]) (\mathcal{T}_{l_1}[Y,x;r_1, \dots, r_{l_1}] \cdot \mathcal{T}_{l_2}[Y,x;t_1, \dots, t_{l_2}])_{ba} \right\rangle_0. \tag{B10}$$

The total order n contribution Eq. (4.9) is now the sum of Eqs. (B6) and (B8)–(B10), and after some algebra we arrive at the final expression for the terms in the momentum expansion (4.8),

$$\mathcal{P}_{ab}^{(n)j}[Y;s]= \lim_{\epsilon\to 0^+}\sum_{l_1=0}^{n-1}\sum_{l=l_1+1-\delta_{l_1,0}}^{n}\frac{n!}{(n-l)!}\sum_{\substack{0\leqslant L_1,\ldots,L_{n-l}\leqslant n-l\\ \sum_i L_i=n-l}}\Theta(\epsilon)^{\sum_i \delta_{L_i,1}}\prod_{i=1}^{n-l}\int_{[0,1]^{L_i}}d\mu_{L_i}(s_1^{(i)},\ldots,s_{L_i}^{(i)})$$

$$\times \int_{[0,1]^{l-l_1}}d\mu_{l-l_1}(r_1,\ldots,r_{l-l_1})\int_{[0,1]^{l_1}}d\mu_{l_1}(t_1,\ldots,t_{l_1})\Theta(t_{l_1}-s)\Theta(s-t_1)$$

$$\times\left\langle\frac{d}{ds}x^j(s)\prod_{i=1}^{n-l}\text{tr}(\mathcal{T}_{L_i}[Y,x;s_1^{(i)},\ldots,s_{L_i}^{(i)}])\left\{\left(\frac{1}{n-l_1}+\delta_{l_1,0}\left[\Theta(s-r_1)(1+(N+1)\Theta(r_l-s))-\frac{1}{n}\right]\right)\right.\right.$$

$$\left.\left.\times(\mathcal{T}_{l_1}[Y,x;t_1,\ldots,t_{l_1}]\cdot\mathcal{T}_{l-l_1}[Y,x;r_1,\ldots,r_{l-l_1}])_{ba}+\delta_{l_1,0}\Theta(\epsilon)\text{tr}(\mathcal{T}_l[Y,x;r_1,\ldots,r_l])\delta_{ab}\right\}\right\rangle_0. \quad \text{(B11)}$$

This expression contains ambiguous factors of $\Theta(\epsilon),\epsilon\to 0^+$ and products such as $\Theta(s)\Theta(-s)$ which depend on the particular choice of regularization of the step function. The auxilliary quantum field theory contains most of the information about the non-Abelian dynamics, and, to obtain an expression which is explicitly independent of such regularizations, we need to choose an appropriate renormalization scheme for it.[8] The removal of these ambiguous factors is also required in order that Eq. (4.2) be a proper representation of the Wilson loop operator. This renormalization has been discussed in [22]. In terms of the Feynman diagram representation of the $\bar{\xi}\xi$ field averages in Eq. (4.9), we keep only those graphs corresponding to Wick contractions in which there is a single continuous line connecting the (same) boundary points $s=0$ and $s=1$, i.e. we restrict to connected Feynman graphs. This will also ensure that the final result is independent of the s-dependence of the auxilliary field representation, as it should be. From Eq. (4.10) this means that we restrict the sum over permutations $P\in S_{n+2}$ to those whose cyclic decomposition contains only a single cycle $C_{n+2}(P)$ of length $L_{n+2}(P)=n+2$. This is achieved essentially by normalizing the functional integration measure $D\bar{\xi}D\xi$ in Eq. (4.1) so that $\langle\langle 1\rangle\rangle=1$.

The renormalized canonical momentum is thus calculated by restricting to cyclic permutations of length $n+2$. By definition, this immediately eliminates the contributions $\mathcal{P}_{ab}^{(n)j}[Y;s]|_{S_n\times S_2}$ and $\mathcal{P}_{ab}^{(n)j}[Y;s]^{[3]}$ above. This scheme removes the δ_{cc} terms in Eq. (B7) and the $\delta_{c,b_{P(n+2)}}$ terms in the first equality in Eq. (B9). Then, we keep only the orbits of length $l=n$ in Eq. (B8) and in the second equality of Eq. (B9). The sum $\mathcal{P}_{ab}^{(n)j}[Y;s]^{\text{ren}}$ of these two terms contains no ambiguities from the step functions involved. Furthermore, after some careful algebra one can rewrite the resulting integration measure from this sum as $\int_0^1\prod_{k=1}^n ds_k$, and the corresponding integrand with the appropriate relabelling of indices is readily seen to form a symmetrized matrix product. The result is finally

$$\mathcal{P}_{ab}^{(n)j}[Y;s]^{\text{ren}}= \int_0^1\prod_{k=1}^n ds_k\left\langle\frac{dx^j(s)}{ds}\text{Sym }\mathcal{T}_n[Y,x;s_1,\ldots,s_n]_{ba}\right\rangle_0 \quad \text{(B12)}$$

which yields the expression (4.11).

APPENDIX C: BOUNDARY CORRELATION FUNCTIONS

In this appendix we will present the results of the boundary integrations which are used in the perturbative calculations of Secs. IV and V. In general, the integrals are divergent, and difficult to do analytically. However, we need only determine their most divergent parts as $\Lambda\to 0$, dropping sub-divergent pieces which vanish upon taking the limit $\epsilon\to 0^+$ with the correlation (3.15). To see how these calculations proceed, let us consider as an example the boundary integral

$$I_c^{(1)}\equiv \int_0^1 ds_1 ds_2 ds_3\frac{\log[2-2\cos 2\pi(s_1-s_2)]}{[1-\cos 2\pi s_1][1-\cos 2\pi(s_2-s_3)]} \quad \text{(C1)}$$

which arises in the evaluation of the Y^2U contributions to the canonical momentum of Sec. IV C. The integral over s_3 can be done as in Eq. (4.12) to give

---

[8]Note that with the regularization (3.14) we have $\Theta(s)\Theta(-s)=-\Theta(\epsilon)^2-\Theta(\epsilon)$, so that such a renormalization scheme can be understood as removing all powers of the ambiguous term $\Theta(\epsilon)$.

$$I_c^{(1)} = \frac{1}{\pi} \int_0^1 ds_1 \, ds_2 \frac{1}{\tan \pi s_2} \frac{\log(2|\sin \pi s_1 \cos \pi s_2 - \sin \pi s_2 \cos \pi s_1|)}{\sin^2 \pi s_1}. \tag{C2}$$

The divergent contributions to the integral over s_1 come from the short distance boundary behavior $s_1 - s_2 \sim s_\Lambda$, i.e. $\sin \pi s_1 \sim \sin \pi s_2$. Expanding the integrand of Eq. (C2) about this point gives the most divergent contribution leading to

$$I_c^{(1)} \simeq \frac{1}{\pi^2} \log(2 \sin \pi s_\Lambda) \int_0^1 ds_2 \frac{\cos \pi s_2}{\sin^3 \pi s_2} \tag{C3}$$

where here and in the following $\simeq$ denotes the most divergent contribution as $\Lambda \to 0$. Using the boundary cutoff (A16) and evaluating the final integration over s_2 using this cutoff we arrive finally at

$$I_c^{(1)} \simeq -\frac{2}{\pi^3} \frac{\log \Lambda}{\tan^2 \pi s_\Lambda}. \tag{C4}$$

All other boundary integrations are evaluated using similar sorts of asymptotic approximation techniques. Below we list their leading divergent behaviors as $\Lambda \to 0$. For the $Y^2 U$ terms of the canonical momentum calculation of Sec. IV C, which come from the correlation function (A12), in addition to Eqs. (C1), (C4) we used the integrals

$$I_0 \equiv \int_0^1 ds_1 \, ds_2 \, ds_3 \frac{1}{[1 - \cos 2\pi s_1][1 - \cos 2\pi(s_2 - s_3)]}$$

$$\simeq \frac{4}{\pi^3} \frac{\log \Lambda}{\tan \pi s_\Lambda} \tag{C5}$$

$$I_u^{(1)} \equiv \int_0^1 ds_1 ds_2 ds_3 \frac{\log[2 - 2 \cos 2\pi(s_2 - s_3)]}{[1 - \cos 2\pi s_1][1 - \cos 2\pi(s_2 - s_3)]}$$

$$\simeq -\frac{8}{\pi^3} \frac{(\log \Lambda)^2}{\tan \pi s_\Lambda}. \tag{C6}$$

The additional integrals involved in the calculation of the $Y U^2$ part in Sec. IV C, which come from the correlators (A13), are

$$I_u^{(2)} \equiv \int_0^1 ds_1 \, ds_2 \, ds_3 \frac{(\log[2 - 2 \cos 2\pi(s_2 - s_3)])^2}{[1 - \cos 2\pi s_1][1 - \cos 2\pi(s_2 - s_3)]}$$

$$\simeq -\frac{16}{3\pi^3} \frac{(\log \Lambda)^3}{\tan \pi s_\Lambda} \tag{C7}$$

$$I_c^{(2)} \equiv \int_0^1 ds_1 \, ds_2 \, ds_3 \frac{(\log[2 - 2 \cos 2\pi(s_1 - s_3)])^2}{[1 - \cos 2\pi s_1][1 - \cos 2\pi(s_2 - s_3)]}$$

$$\simeq -\frac{4}{3\pi^3} \frac{(\log \Lambda)^2}{\tan^2 \pi s_\Lambda} \tag{C8}$$

$$I_q^{(1)} \equiv \int_0^1 ds_1 \, ds_2 \, ds_3 \frac{1}{[1 - \cos 2\pi s_1][1 - \cos 2\pi(s_2 - s_3)]} \left(\frac{\log \frac{1}{\Lambda^2}[2 - 2 \cos 2\pi(s_1 - s_3)]}{\log \frac{1}{\Lambda^2}[2 - 2 \cos 2\pi(s_1 - s_2)]} \right)^2$$

$$\simeq -\frac{2}{\pi^3} \frac{(\log \Lambda)^2}{\tan \pi s_\Lambda} \log \log \Lambda \tag{C9}$$

$$I_q^{(2)} \equiv \int_0^1 ds_1 \, ds_2 \, ds_3 \frac{1}{[1-\cos 2\pi s_1][1-\cos 2\pi(s_2-s_3)]} \left(\frac{\log \frac{1}{\Lambda^2}[2-2\cos 2\pi(s_2-s_3)]}{\log \frac{1}{\Lambda^2}[2-2\cos 2\pi(s_1-s_2)]} \right)^2$$

$$\simeq \frac{1}{8\pi^3} \frac{1}{\tan^2 \pi s_\Lambda} \tag{C10}$$

$$I_q^{(3)} \equiv \int_0^1 ds_1 \, ds_2 \, ds_3 \frac{1}{[1-\cos 2\pi s_1][1-\cos 2\pi(s_2-s_3)]} \left(\frac{\log \frac{1}{\Lambda^2}[2-2\cos 2\pi(s_1-s_3)]}{\log \frac{1}{\Lambda^2}[2-2\cos 2\pi(s_2-s_3)]} \right)^2$$

$$\simeq -\frac{2}{\pi^3} \frac{(\log \Lambda)^2}{\tan \pi s_\Lambda} \log \log \Lambda. \tag{C11}$$

For the U^3 terms of Sec. IV C, which come from the correlation function (A14), we use the integrals

$$I_m^{(1)} \equiv \int_0^1 ds_1 \, ds_2 \, ds_3 \frac{\log[2-2\cos 2\pi(s_1-s_2)]\log[2-2\cos 2\pi(s_2-s_3)]}{[1-\cos 2\pi s_1][1-\cos 2\pi(s_2-s_3)]}$$

$$\simeq \frac{8}{\pi^3} \frac{(\log \Lambda)^2}{\tan^2 \pi s_\Lambda} \tag{C12}$$

$$I_m^{(2)} \equiv \int_0^1 ds_1 \, ds_2 \, ds_3 \frac{\log[2-2\cos 2\pi(s_1-s_2)]\log[2-2\cos 2\pi(s_1-s_3)]}{[1-\cos 2\pi s_1][1-\cos 2\pi(s_2-s_3)]}$$

$$\simeq \frac{16}{\pi^3} \frac{(\log \Lambda)^2}{\tan^2 \pi s_\Lambda} \tag{C13}$$

$$I_t^{(1)} \equiv \int_0^1 ds_1 \, ds_2 \, ds_3 \frac{\log[2-2\cos 2\pi(s_1-s_2)]\log[2-2\cos 2\pi(s_2-s_3)]}{[1-\cos 2\pi s_1][1-\cos 2\pi(s_2-s_3)]} \log[2-2\cos 2\pi(s_1-s_3)]$$

$$\simeq \frac{64}{\pi^3} \frac{(\log \Lambda)^3}{\tan^2 \pi s_\Lambda} \tag{C14}$$

$$I_t^{(2)} \equiv \int_0^1 ds_1 \, ds_2 \, ds_3 \frac{(\log[2-2\cos 2\pi(s_1-s_2)])^2 \, \log[2-2\cos 2\pi(s_2-s_3)]}{[1-\cos 2\pi s_1][1-\cos 2\pi(s_2-s_3)]}$$

$$\simeq \frac{32}{\pi^3} \frac{(\log \Lambda)^3}{\tan^2 \pi s_\Lambda} \tag{C15}$$

$$I_t^{(3)} \equiv \int_0^1 ds_1 \, ds_2 \, ds_3 \frac{(\log[2-2\cos 2\pi(s_1-s_2)])^2 \log[2-2\cos 2\pi(s_1-s_3)]}{[1-\cos 2\pi s_1][1-\cos 2\pi(s_2-s_3)]}$$

$$\simeq \frac{24}{\pi^3} \frac{(\log \Lambda)^3}{\tan^2 \pi s_\Lambda} \tag{C16}$$

$$I_t^{(4)} \equiv \int_0^1 ds_1 \, ds_2 \, ds_3 \frac{\log[2 - 2\cos 2\pi(s_1 - s_2)](\log[2 - 2\cos 2\pi(s_2 - s_3)])^2}{[1 - \cos 2\pi s_1][1 - \cos 2\pi(s_2 - s_3)]}$$

$$\simeq \frac{16}{\pi^3} \frac{(\log \Lambda)^3}{\tan^2 \pi s_\Lambda} \tag{C17}$$

$$I_t^{(5)} \equiv \int_0^1 ds_1 \, ds_2 \, ds_3 \frac{(\log[2 - 2\cos 2\pi(s_1 - s_2)])^3}{[1 - \cos 2\pi s_1][1 - \cos 2\pi(s_2 - s_3)]}$$

$$\simeq -\frac{16}{\pi^3} \frac{(\log \Lambda)^3}{\tan^2 \pi s_\Lambda} \tag{C18}$$

$$I_t^{(6)} \equiv \int_0^1 ds_1 \, ds_2 \, ds_3 \frac{(\log[2 - 2\cos 2\pi(s_2 - s_3)])^3}{[1 - \cos 2\pi s_1][1 - \cos 2\pi(s_2 - s_3)]}$$

$$\simeq -\frac{32}{\pi^3} \frac{(\log \Lambda)^4}{\tan \pi s_\Lambda}. \tag{C19}$$

Finally, the boundary integrals arising in the evaluation of the Zamolodchikov metric of Sec. V, which come from the two-point correlation functions (A7) and (A8) of the logarithmic operators, are

$$I_g^{(1)} \equiv \int_0^1 ds_1 \, ds_2 \frac{1}{1 - \cos 2\pi(s_1 - s_2)}$$

$$\simeq \frac{4}{\pi^2} \log \Lambda \tag{C20}$$

$$I_g^{(2)} \equiv \int_0^1 ds_1 \, ds_2 \frac{1}{(1 - \cos 2\pi s_1)(1 - \cos 2\pi s_2)}$$

$$\simeq \frac{2}{\pi^2 \Lambda^2} \frac{1}{\tan \pi s_\Lambda} \tag{C21}$$

$$I_g^{(3)} \equiv \int_0^1 ds_1 \, ds_2 \frac{\log[2 - 2\cos 2\pi(s_1 - s_2)]}{1 - \cos 2\pi(s_1 - s_2)}$$

$$\simeq -\frac{8}{\pi^2} (\log \Lambda)^2 \tag{C22}$$

$$I_g^{(4)} \equiv \int_0^1 ds_1 \, ds_2 \frac{\log[2 - 2\cos 2\pi(s_1 - s_2)]}{(1 - \cos 2\pi s_1)(1 - \cos 2\pi s_2)}$$

$$\simeq -\frac{2}{3\pi^2 \Lambda^2} \frac{\log \Lambda}{\tan^2 \pi s_\Lambda}. \tag{C23}$$

APPENDIX D: WARD IDENTITIES AND LEADING DIVERGENCES IN THE GENUS EXPANSION

In this appendix we shall show how the leading $(\log \delta)^2$ modular divergences which appear in Eq. (6.11) can be removed by invoking an appropriate Ward identity for the fundamental string fields of the matrix σ-model. As we shall show, this is equivalent to imposing momentum conservation for scattering processes in the matrix D-brane background. This has been demonstrated explicitly for the single D-particle case in [28]. Within the framework of the auxilliary field representation of the Wilson loop operator, the effective Abelianization of the matrix σ-model leads to a relatively straightforward generalization of this proof, as we now demonstrate.

The pertinent bilocal term induced by Eq. (6.11), which exponentiates upon summing over pinched topologies, can be written as a local worldsheet effective action using the wormhole parameters $[\rho_{C,D}]_i^{ab}$ to give

$$
e^{\Delta S^{CD}} = \lim_{\epsilon \to 0^+} \int d\rho_C \, d\rho_D \exp\left[\sum_{a,b=1}^N \left(-\frac{1}{2g_s^2(\log \delta)^2} G^{LM} \sum_{c,d=1}^N G^{ij}_{ab;cd}[\rho_L]_i^{ab}[\rho_M]_j^{cd} + \frac{ig_s[\rho_C]_i^{ab}}{2\pi\alpha'} \right. \right.
$$
$$
\left. \left. \times \int_0^1 ds\, C(x^0(s);\epsilon)\bar{\xi}_a(s-\epsilon)\xi_b(s)\frac{d}{ds}x^i(s) + \frac{ig_s[\rho_D]_i^{ab}}{2\pi\alpha'}\int_0^1 ds\, D(x^0(s);\epsilon)\bar{\xi}_a(s-\epsilon)\xi_b(s)\frac{d}{ds}x^i(s) \right) \right]. \quad \text{(D1)}
$$

Here we have for simplicity considered only the zero frequency modes of the fields involved with respect to the Fourier transformations defined at the beginning of Sec. VI A. They will be sufficient to describe the relevant cancellations. In Eq. (D1) the (dimensionless) moduli space metric G^{LM} (where $L, M = C, D$) is an appropriate off-diagonal 2×2 matrix with respect to the decomposition (5.1) [see Eq. (A18)] which is required to reproduce the initial bilocal operator with the CD-mixing of the logarithmic operators. This off-diagonal metric includes all the appropriate normalization factors $\mathcal{N}_L$ for the zero mode states. These factors are essentially the inverse of the CD two-point function (A7) which is finite.

We consider the propagation of two (closed string) matter tachyon states $T_{1,2} = e^{i(k_{1,2})_i x^i}$ in the background of Eq. (D1) at the tree level. In what follows the effects of the C operator are sub-leading and can be ignored. Then, we are interested in the amplitude

$$
\mathcal{A}_{CD} \equiv \left\langle\!\left\langle\!\left\langle \sum_{c'=1}^N \bar{\xi}_{c'}(0) T_1 T_2 e^{\Delta S^{CD}} \xi_{c'}(1) \right\rangle\!\right\rangle\!\right\rangle_0
$$

$$
= \lim_{\epsilon \to 0^+} \sum_{c'=1}^N \int d\rho_C \, d\rho_D \int Dx\, D\bar{\xi}\, D\xi\, \bar{\xi}_{c'}(0) \exp\left(-N^2 S_0[x] - \sum_{c=1}^N \int_0^1 ds\, \bar{\xi}_c(s-\epsilon)\frac{d}{ds}\xi_c(s) \right) T_1[x]\, T_2[x]
$$

$$
\times \exp\left[\sum_{a,b=1}^N \left(-\frac{1}{2g_s^2(\log \delta)^2} G^{LM} \sum_{c,d=1}^N G^{ij}_{ab;cd}[\rho_L]_i^{ab}[\rho_M]_j^{cd} + \frac{ig_s[\rho_D]_i^{ab}}{2\pi\alpha'}\int_0^1 ds\, D(x^0(s);\epsilon)\bar{\xi}_a(s-\epsilon) \right. \right.
$$

$$
\left. \left. \times \xi_b(s)\frac{d}{ds}x^i(s) \right) \right] \xi_{c'}(1) + \cdots \quad \text{(D2)}
$$

where $\cdots$ represent sub-leading terms. The scaling property (3.19) of the logarithmic operators must be taken into account. Under a scale transformation (3.17) on the worldsheet the C operator emerges from D due to mixing with a scale-dependent coefficient $\sqrt{\alpha'}\, t$. This will contribute to the scaling infinities we are considering here.

The composite D operator insertion in Eq. (D2) needs to be normal-ordered on the disc. Normal ordering in the present case amounts to subtracting scaling infinities originating from divergent contributions of $D(x^0(s);\epsilon)$ as $\epsilon \to 0^+$. To determine these infinities, we first note that the one-point function of the composite D operators, computed with respect to the free σ-model and auxilliary field actions, can be written as

$$
\left\langle\!\left\langle\!\left\langle \sum_{c'=1}^N \bar{\xi}_{c'}(0)\exp\left(\sum_{a,b=1}^N \frac{ig_s[\rho_D]_i^{ab}}{2\pi\alpha'}\int_0^1 ds\, D(x^0(s);\epsilon)\bar{\xi}_a(s-\epsilon)\xi_b(s)\frac{d}{ds}x^i(s) \right) \xi_{c'}(1) \right\rangle\!\right\rangle\!\right\rangle_0
$$

$$
= \left\langle\!\left\langle \sum_{c'=1}^N \bar{\xi}_{c'}(0)\exp\left(-\sum_{a,b,c,d} \frac{g_s^2[\rho_D]_i^{ab}[\rho_D]_j^{cd}}{2(2\pi\alpha')^2}\int_0^1 ds\, ds'\, \langle D(x^0(s);\epsilon)D(x^0(s');\epsilon)\rangle_0 \bar{\xi}_a(s-\epsilon)\xi_b(s)\bar{\xi}_c(s'-\epsilon)\xi_d(s') \right. \right.
$$

$$
\left. \left. \times \left\langle \frac{d}{ds}x^i(s)\frac{d}{ds'}x^j(s') \right\rangle_0 \right) \xi_{c'}(1) \right\rangle\!\right\rangle
$$

$$
= \exp\left(-\sum_{a,b=1}^N \frac{g_s^2[\rho_D]_i^{ab}[\rho_D]_j^{ba}}{2(2\pi\alpha')^2}\int_0^1 ds\, ds'\, \langle D(x^0(s);\epsilon)D(x^0(s');\epsilon)\rangle_0 \left\langle \frac{d}{ds}x^i(s)\frac{d}{ds'}x^j(s') \right\rangle_0 \right) \quad \text{(D3)}
$$

where we have used Wick's theorem. The second equality in Eq. (D3) follows after removing ambiguous $\Theta(\epsilon)$ type terms from the Wick expansion in the auxilliary fields using the renormalization scheme described in Appendix B. One finds that this procedure has the overall effect of replacing the product of auxilliary fields in the first equality in Eq. (D3) by the delta-functions $\delta_{ad}\delta_{bc}$.

In what follows we shall ignore, for simplicity, the basic divergences that come from the fundamental string propagator in Eq. (D3). Such divergences will appear globally in all correlators below and will not affect the final result. As a consequence of the logarithmic algebra (A7) and the scale transformation (3.17),(3.19), there are leading (scaling) divergences in Eq. (D3) for $\epsilon \to 0^+$ which behave as

$$g_s^2 b \, \alpha'^{-1/2} t \, \mathrm{tr}[\rho_D]_i [\rho_D]^i. \tag{D4}$$

Thus, normal ordering of the D operator amounts to adding a term of opposite sign to Eq. (D4) into the argument of the exponential in Eq. (D2) in order to cancel such divergences.

Let us now introduce a complete set of states $|\mathcal{E}_I\rangle$ into the two-point function of string matter fields on the disc,

$$\langle T_1 T_2 \rangle_0 = \sum_I |\mathcal{N}_I|^2 \langle T_1 | \mathcal{E}_I \rangle_0 \langle \mathcal{E}_I | T_2 \rangle_0 \tag{D5}$$

where $\mathcal{N}_I$ is a normalization factor for the fundamental string states (determined by the Zamolodchikov metric). Taking into account the effects of the C operator included in D under the scaling (3.17), we see that the leading divergent contributions to Eq. (D5) are of the form

$$\langle T_1 T_2 \rangle_0 \simeq -\sqrt{\alpha'}\, t \, \langle T_1 | C \rangle_0 \langle C | T_2 \rangle_0 + \cdots \tag{D6}$$

where we have used Eqs. (A18) and (3.15). We now notice that the C deformation vertex operator plays the role of the Goldstone mode for the translation symmetry of the fundamental string coordinates x^i, and as such we can apply the corresponding Ward identity in the matrix σ-model path integral to represent the action of the C deformation on physical states by $-i\delta/\delta x^i$ [14,15]. The leading contribution to Eq. (D5) can thus be exponentiated to yield

$$\langle T_1 T_2 \rangle_0 \simeq \lim_{\epsilon \to 0^+} \sum_{c'=1}^N \int Dx \, D\bar{\xi} D\xi \, \bar{\xi}_{c'}(0) \exp\left(-N^2 S_0[x] - \sum_{c=1}^N \int_0^1 ds \, \bar{\xi}_c(s-\epsilon) \frac{d}{ds} \xi_c(s) \right) T_1[x]$$

$$\times \exp\left(-\frac{g_s^2 \sqrt{\alpha'}\, t}{2} \sum_{a,b=1}^N \int_0^1 ds \, ds' \, \bar{\xi}_a(s-\epsilon)\xi_b(s)\bar{\xi}_b(s'-\epsilon)\xi_a(s') \frac{\bar{\delta}}{\delta x_i(s)} \frac{\bar{\delta}}{\delta x^i(s')} \right) T_2[x]\xi_{c'}(1) \tag{D7}$$

where we have used the on-shell condition $T_j[(\delta/\delta x_i)(\delta/\delta x^i)]T_k = 0$ for the tachyon fields. Equation (D7) expresses the non-Abelian version of the Ward identity in the presence of logarithmic deformations.

Using Eqs. (D4), (D7) and normalizing the parameters of the logarithmic conformal algebra appropriately, it follows that Eq. (D2) can be written as

$$\mathcal{A}_{CD} \simeq \lim_{\epsilon \to 0^+} \sum_{c'=1}^N \int d\rho_C \, d\rho_D \int Dx \, D\bar{\xi} D\xi \, \bar{\xi}_{c'}(0) \exp\left(-N^2 S_0[x] - \sum_{c=1}^N \int_0^1 ds \, \bar{\xi}_c(s-\epsilon) \frac{d}{ds} \xi_c(s) \right) T_1[x]$$

$$\times \exp\left[\sum_{a,b=1}^N \left(-\frac{1}{2g_s^2 (\log \delta)^2} G^{LM} \sum_{c,d=1}^N G^{ij}_{ab;cd} [\rho_L]_i^{ab} [\rho_M]_j^{cd} - \frac{g_s^2 \alpha'^{-1/2} t}{2} \eta^{ij}[\rho_D]_i^{ab}[\rho_D]_j^{ba} + i g_s t \, [\rho_D]_i^{ab} \right.\right.$$

$$\left.\left. \times \int_0^1 ds \, \bar{\xi}_a(s-\epsilon)\xi_b(s) \frac{\bar{\delta}}{\delta x_i(s)} - \frac{g_s^2 \sqrt{\alpha'}\, t}{2} \int_0^1 ds \, ds' \, \bar{\xi}_a(s-\epsilon)\xi_b(s)\bar{\xi}_b(s'-\epsilon)\xi_a(s') \frac{\bar{\delta}}{\delta x_i(s)} \frac{\bar{\delta}}{\delta x^i(s')} \right) \right] T_2[x]\xi_{c'}(1)$$

$$+ \cdots$$

$$
= \lim_{\epsilon \to 0^+} \sum_{c'=1}^{N} \int d\rho_C \, d\rho_D \int Dx \, D\bar{\xi} D\xi \, \bar{\xi}_{c'}(0) \exp\!\left(-N^2 S_0[x] - \sum_{c=1}^{N} \int_0^1 ds \, \bar{\xi}_c(s-\epsilon) \frac{d}{ds} \xi_c(s) \right)
$$

$$
\times T_1[x] \exp\!\left[\sum_{a,b=1}^{N} \left\{ -\frac{1}{2g_s^2(\log\delta)^2} G^{LM} \sum_{c,d=1}^{N} G_{ab;cd}^{ij} [\rho_L]_i^{ab} [\rho_M]_j^{cd} - \frac{g_s^2 \alpha'^{-1/2} t}{2} \eta^{ij} \left([\rho_D]_i^{ab} - \frac{i\sqrt{\alpha'}}{g_s} \right.\right.\right.
$$

$$
\times \left.\left.\left. \int_0^1 ds \, \bar{\xi}_a(s-\epsilon)\xi_b(s) \frac{\overleftarrow{\delta}}{\delta x_i(s)} \right) \left([\rho_D]_j^{ba} - \frac{i\sqrt{\alpha'}}{g_s} \int_0^1 ds \, \bar{\xi}_b(s-\epsilon)\xi_a(s) \frac{\overleftrightarrow{\delta}}{\delta x_j(s)} \right) \right\} \right] T_2[x]\xi_{c'}(1) + \cdots . \tag{D8}
$$

From Eq. (D8) it follows that the limit $t \to \infty$ localizes the worldsheet wormhole parameter integrations with delta-function support

$$
\prod_{a,b=1}^{N} \prod_{i=1}^{9} \delta\!\left([\rho_D]_i^{ab} - \frac{\sqrt{\alpha'}}{g_s}(k_1+k_2)_i \int_0^1 ds \, \bar{\xi}_a(s-\epsilon)\xi_b(s) \right) \tag{D9}
$$

where $(k_{1,2})_i$ are the momenta of the closed string matter states. This result shows that the leading modular divergences in the genus expansion are cancelled by the scattering of (closed) string states off the matrix D-brane background. Upon rescaling ρ_D by g_s^2, averaging over the auxilliary boundary fields, and incorporating Eq. (D9) as an effective shift in the velocity recoil operator [see Eq. (6.15)], we can identify this renormalization as fixing the velocity matrix

$$
U_i^{ab} = -\sqrt{\alpha'}\, g_s \, (k_1+k_2)_i \, \delta^{ab} \tag{D10}
$$

of the fat brane background. Thus momentum conservation for the D-brane dynamics guarantees conformal invariance of the matrix σ-model as far as leading divergences are concerned.

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