Quantum scalar field on the massless $(2+1)$-dimensional black hole background

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The behavior of a quantum scalar field is studied in the metric ground state of the $(2+1)$-dimensional black hole of Bañados, Teitelboim, and Zanelli which contains a naked singularity. The one-loop BTZ partition function, the associate black hole effective entropy, and the expectation value of the quantum fluctuation, as well as the renormalized expectation value of the stress tensor are explicitly computed in the framework of the ζ-function procedure. This is done for all values of the coupling with the curvature, the mass of the field, and the temperature of the quantum state. In the massless conformally coupled case, the found stress tensor is used for determining the quantum back reaction on the metric due to the scalar field in the quantum vacuum state, by solving the semiclassical Einstein equations. It is finally argued that, within the framework of the $1/N$ expansion, the cosmic censorship hypothesis is implemented since the naked singularity of the ground state metric is shielded by an event horizon created by the back reaction. [S0556-2821(99)01410-1]

I. INTRODUCTION

Recently, the three-dimensional gravity theory has been studied in detail. Despite the simplicity of the three-dimensional case (absence of propagating gravitons), it is a common belief that it deserves attention as a useful laboratory in order to understand several fundamental issues associated with black hole entropy, such as its statistical origin and horizon divergence problems (see, for example, [1–3]). In fact, a black hole solution has been found by Bañados, Teitelboim, and Zanelli [4], the so-called BTZ black hole; in particular, the simple geometrical structure of this black hole solution allows exact computations, since its Euclidean counterpart is locally isometric to the constant curvature three-dimensional hyperbolic space $\mathbb{H}^3$. Furthermore, investigations in the three-dimensional case seem to be relevant for several other reasons, among which we would like to recall the conformal-field-theory–anti–de Sitter (CFT)(AdS) correspondence [5], and the fact that higher dimensional black holes can be related to the BTZ black hole (namely, the near-horizon geometry of these higher dimensional black holes is essentially the BTZ one). With regard to this, the BTZ entropy issue has been recently reviewed in [6] (where a complete list of references can also be found) and in [7]; the quantum evolution of the BTZ black hole within a Kaluza-Klein reduction has instead been investigated in [8].

In this paper we shall discuss the behavior of a quantum scalar field propagating in the gravitational ground state of the BTZ black hole (i.e., the BTZ solution in the limit of a

vanishing black hole mass), generalizing to the nonconformally invariant case previous results obtained in [9–14]. We shall also attempt to explore the possible relevance of the quantum fluctuations with regard to the issue of the cosmic censorship hypothesis, since the BTZ ground state solution shows a naked singularity and, presumably, it might be the final state at the end of the black hole evaporation process. It is worthwhile stressing that the global topology of this ground state is completely different from the topology of the BTZ black hole, and thus it could be dangerous, in order to investigate the one-loop effective potential of a quantum scalar field in this background, considering the results for a massive BTZ black hole and taking the limit $M \to 0$ naively; as a consequence, we shall compute all the quantities directly, employing the ζ-function procedure. This is true also for the expectation value of the stress tensor, since no good reasons were found for considering the zero temperature thermal state as the only physically sensible one.

The content of the paper is organized as follows. In Sec. II we briefly review the geometry of the Euclidean BTZ black hole and its ground state. In Sec. III we present an elementary derivation of the heat kernel and the ζ function related to a Laplace-like operator necessary for the computation of the ζ-function regularized functional determinant. In Sec. IV, the one-loop relative partition function associated with the BTZ background and its ground state is computed and some comments on the effective black hole entropy are presented. In Sec. V the computation of the quadratic fluctuations of the scalar field is performed, and the expectation value of the associated stress tensor is evaluated in the framework of the local ζ-function approach. In Sec. VI, the back reaction due to the quantum fluctuations is computed. The paper ends with some concluding remarks in Sec. VII,

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and with an appendix, where some computational technicalities are presented.

II. EUCLIDEAN BTZ BLACK HOLE AND ITS GROUND STATE

Here, following [15], we summarize the geometrical aspects of the nonrotating BTZ black hole [4] and its gravitational ground state, which are relevant for our discussion.

In local coordinates (t,r,φ), with

$$r \in (\sqrt{8GM}\,l, +\infty), \quad t \in (-\infty, +\infty), \quad \varphi \in [0, 2\pi), \tag{2.1}$$

and $\varphi = 0$ identified with $\varphi = 2\pi$, the static Lorentzian metric of the (nonrotating) BTZ black hole reads

$$ds_L^2 = -\left(\frac{r^2}{l^2} - 8GM\right)dt^2 + \left(\frac{r^2}{l^2} - 8GM\right)^{-1}dr^2 + r^2 d\varphi^2, \tag{2.2}$$

where M is the standard Arnowitt-Deser-Misner (ADM) mass and l is a dimensional constant. Notice the couple of Killing fields ∂_t and ∂_φ which are, respectively, time like and space like. A direct calculation shows that the metric above is a solution of the three-dimensional vacuum Einstein's equations with negative cosmological constant, i.e.,

$$\mathcal{R}_{\mu\nu} = 2\Lambda g_{\mu\nu}, \quad \mathcal{R} = 6\Lambda = -\frac{6}{l^2}. \tag{2.3}$$

Thus, the sectional curvature k is constant and negative, namely, $k = \Lambda = -1/l^2$. The metric (2.2) has a horizon radius given by

$$r_+ = \sqrt{8GM}\,l, \tag{2.4}$$

and it describes a space-time locally isometric to AdS^3.

A Euclidean section related to this choice of coordinates is obtained by the Wick rotation $t \to i\tau$ ($\tau \in \mathbb{R}$) and reads

$$ds^2 = \left(\frac{r^2}{l^2} - \frac{r_+^2}{l^2}\right)d\tau^2 + \left(\frac{r^2}{l^2} - \frac{r_+^2}{l^2}\right)^{-1}dr^2 + r^2 d\varphi^2. \tag{2.5}$$

Changing the coordinates (τ, r, φ) into the (y, x_1, x_2) ones, by means of the transformation

$$y = \frac{r_+}{r}e^{(r_+\varphi)/l}, \tag{2.6}$$

$$x_1 + ix_2 = \frac{1}{r}\sqrt{r^2 - r_+^2}\exp\left(i\frac{r_+\tau}{l^2} + \frac{r_+\varphi}{l}\right), \tag{2.7}$$

the metric becomes that of the upper-half space representation of $\mathbb{H}^3$, i.e.,

$$ds^2 = \frac{l^2}{y^2}(dy^2 + dx_1^2 + dx_2^2). \tag{2.8}$$

Anyhow, the range of the coordinates is not the maximal one for $\mathbb{H}^3$, since y is bounded above because of the upper bound $\varphi < 2\pi$ and the lower bound $r > r_+$. Nevertheless, we can maximally extend the range of the new coordinates into $x_1, x_2 \in \mathbb{R}$ and $y \in \mathbb{R}_+$, obtaining the whole hyperbolic three-space. As a consequence, it is now obvious that, barring the identification $0 \sim 2\pi$ in φ, the Euclidean section (2.5) describes a manifold isometric to a submanifold of the hyperbolic space $\mathbb{H}^3$. Actually we can say much more employing the theory of Lie's groups of isometries.

Recalling that the group of isometries of $\mathbb{H}^3$ is SL(2,$\mathbb{C}$), we shall consider a discrete subgroup $\Gamma \subset \mathrm{PSL}(2,\mathbb{C}) \equiv \mathrm{SL}(2,\mathbb{C})/\{\pm\,\mathrm{Id}\}$ (Id is the identity element), which acts discontinuously at the point z belonging to the extended complex plane $\mathbb{C} \cup \{\infty\}$. We also recall that a transformation $\gamma \in \Gamma$, with $\gamma \neq \mathrm{Id}$ and

$$\gamma z = \frac{az + b}{cz + d}, \quad ad - bc = 1, \quad a,b,c,d \in \mathbb{C}, \tag{2.9}$$

is called elliptic if $(\mathrm{Tr}\,\gamma)^2 = (a+d)^2$ satisfies $0 \leq (\mathrm{Tr}\,\gamma)^2 < 4$, hyperbolic if $(\mathrm{Tr}\,\gamma)^2 > 4$, parabolic if $(\mathrm{Tr}\,\gamma)^2 = 4$ and loxodromic if $(\mathrm{Tr}\,\gamma)^2 \in \mathbb{C}/[0,4]$. The element $\gamma \in \mathrm{SL}(2,\mathbb{C})$ acts on $\mathbf{x} = (y, w) \in \mathbb{H}^3$, with $w = x_1 + ix_2$, by means of the following linear-fractional transformation:

$$\gamma\mathbf{x} = \left(\frac{y}{|cw+d|^2 + |c|^2y^2}, \frac{(aw+b)(\bar{c}\bar{w}+\bar{d}) + a\bar{c}y^2}{|cw+d|^2 + |c|^2y^2}\right). \tag{2.10}$$

The periodicity of the angular coordinate φ in Eq. (2.6), which corresponds to a one-parameter group of isometries, allows one to describe the BTZ black hole manifold (2.5) as the quotient $\mathcal{H}^3 \equiv \mathbb{H}^3/\Gamma$, Γ being a discrete group of isometry possessing a primitive element $\gamma_h \in \Gamma$ defined by the identification

$$\gamma_h(y,w) = (e^{2\pi r_+/l}y, e^{2\pi r_+/l}w) \sim (y,w), \tag{2.11}$$

induced by $0 \sim 2\pi$ in Eq. (2.6). According to (2.10), this corresponds to the matrix

$$\gamma_h = \begin{pmatrix} e^{\pi r_+/l} & 0 \\ 0 & e^{-\pi r_+/l} \end{pmatrix}, \tag{2.12}$$

namely, to a hyperbolic element consisting of a pure dilation. Furthermore, since the Euclidean time τ becomes an angular-type variable with period β, one is led also to the identification

$$\gamma_e(y,w) = (y, e^{i\beta r_+/l^2}w) \sim (y,w), \tag{2.13}$$

which is generated by an elliptic element $\gamma_e \in \Gamma$, given by

$$\gamma_e = \begin{pmatrix} -e^{i\beta r_+/l^2} & 0 \\ 0 & -e^{-i\beta r_+/l^2} \end{pmatrix}. \tag{2.14}$$

Anyway, requiring the absence of the conical singularity, we get the relation

$$\frac{\beta r_+}{l^2} = 2\pi, \tag{2.15}$$

so that $\gamma_e = \mathrm{Id}$, and the period β, interpreted now as the inverse of the Hawking temperature [16], is determined to be

$$\beta_H = \frac{2\pi l^2}{r_+}. \tag{2.16}$$

The tree-level Bekenstein-Hawking entropy S_H can also be evaluated, and is given by

$$S_H = \sqrt{\frac{2M}{G}}\, l = \frac{1}{4}\frac{2\pi r_+}{G}, \tag{2.17}$$

which is the well-known area law for the black hole entropy.

The space-time we are particularly interested in is the ground state of the BTZ black hole, namely, the BTZ black hole in the limit of a vanishing mass; this space-time is thus described by the line element

$$ds_0^2 = \frac{r^2}{l^2}d\tau^2 + \frac{l^2}{r^2}dr^2 + r^2 d\varphi^2. \tag{2.18}$$

This ground state corresponds also to the zero-temperature, zero-entropy, and zero-energy state; moreover, $r=0$ is a naked singularity in its Lorentzian section [9,17] and corresponds to a point out of the Euclidean manifold (its geodesical distance from the remaining points is infinite).

By setting

$$r = \frac{l^2}{y}, \quad \tau = x_1, \quad \varphi = \frac{x_2}{l}, \tag{2.19}$$

we get again the metric of the upper-half model of the hyperbolic space:

$$ds_0^2 = \frac{l^2}{y^2}(dy^2 + dx_1^2 + dx_2^2). \tag{2.20}$$

From Eq. (2.18) and the comment following that equation, it is clear that the coordinate τ can be compactified in a circle with any period $\beta > 0$ (in particular $\beta = \infty$), preserving the smoothness of the manifold; moreover, φ has the usual 2π period. In this way, the ground state solution corresponds to the identification

$$(y, w + \beta + 2\pi i l) \sim (y, w), \tag{2.21}$$

which is generated by the two parabolic elements

$$\gamma_{p_1} = \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix}, \quad \gamma_{p_2} = \begin{pmatrix} 1 & 2\pi i l \\ 0 & 1 \end{pmatrix}. \tag{2.22}$$

Thus, our ground state space can be regarded as the quotient $\mathcal{H}_0^3 = \mathbb{H}^3/\Gamma_0$, with Γ_0 generated by the two primitive para-

bolic elements γ_{p_1} and γ_{p_2}; one should further notice that, in the limit $M \to 0$, the topology of the solution changes and thus the ground state case must be considered separately.

We finally remind that for negative masses, one gets solutions with a naked conical singularity [18] unless one arrives at $M = -1$, namely, $\mathbb{H}^3$, the Euclidean counterpart of AdS^3; this solution is a permissible one, and can be regarded as a bound state [4].

III. EFFECTIVE ACTION FOR A SCALAR FIELD IN THE BTZ GROUND STATE

In this section we investigate the spectral properties of a Laplace-like operator acting on scalar functions on the noncompact hyperbolic manifold $\mathcal{H}_0^3$, in order to evaluate the related functional determinant and, so, the effective action. The BTZ massive case has been considered in [13,19]. For simplicity, from now on, we put $l=1$; thus $|k|=1/l^2=1$ and all the quantities are dimensionless (the physical dimensions can be restored by dimensional analysis at the end of the calculations).

The heat kernel related to the Laplace-like operator (see also the Appendix)

$$L = -\Delta - 1 \tag{3.1}$$

is well known, and reads

$$K_t^{\mathbb{H}^3}(\mathbf{x},\mathbf{x}'|L) = \frac{1}{(4\pi t)^{3/2}} \frac{\sigma(\mathbf{x},\mathbf{x}')}{\sinh \sigma(\mathbf{x},\mathbf{x}')} \exp\left[-\frac{\sigma^2(\mathbf{x},\mathbf{x}')}{4t}\right], \tag{3.2}$$

where the geodesic distance of $\mathbf{x}$ from $\mathbf{x}'$ in $\mathbb{H}^3$ is

$$\sigma(\mathbf{x},\mathbf{x}') = \cosh^{-1}\left[1 + \frac{(y-y')^2 + (x_1-x_1')^2 + (x_2-x_2')^2}{2yy'}\right], \tag{3.3}$$

and is usually given in terms of the fundamental invariant of any pair of points:

$$u(\mathbf{x},\mathbf{x}') = \frac{1}{2}[\cosh \sigma(\mathbf{x},\mathbf{x}') - 1], \quad u(\mathbf{x},\mathbf{x}) = 0. \tag{3.4}$$

Since we are interested in scalar fields propagating in the classical BTZ background which are described by the action

$$I[\phi] = -\frac{1}{2}\int d^3x \sqrt{g}(\nabla_\alpha \phi \nabla^\alpha \phi + m^2\phi^2 + \xi \mathcal{R}\phi^2), \tag{3.5}$$

we have to deal with the motion operator

$$L_b = L + b, \tag{3.6}$$

where b is a constant given in terms of the mass and the gravitational coupling of the field:

$$b = 1 + m^2 + \xi \mathcal{R}. \tag{3.7}$$

It should be noticed that, in this way, the massless conformally invariant case corresponds to the choice $b = 1/4$.

Now, the result (3.2) is trivially generalized for such an operator, and gives

$$K_t^{\mathbb{H}^3}(\mathbf{x}, \mathbf{x}' | L_b) = \frac{1}{(4\pi t)^{3/2}} \frac{\sigma(\mathbf{x}, \mathbf{x}')}{\sinh \sigma(\mathbf{x}, \mathbf{x}')} \exp\left[-tb - \frac{\sigma^2(\mathbf{x}, \mathbf{x}')}{4t} \right].$$

(3.8)

This Euclidean expression has a Lorentzian counterpart associated with Dirichlet boundary conditions at spatial infinity, which have to be imposed because AdS^3 is not globally hyperbolic.

With regard to the heat kernel on $\mathcal{H}_0^3$, we can apply the method of images; namely, we can write

$$K_t^{\mathcal{H}_0^3}(\mathbf{x}, \mathbf{x}' | L_b) = \sum_{\gamma_p} K_t^{\mathbb{H}^3}(\mathbf{x}, \gamma_p \mathbf{x}' | L_b)$$

$$= K_t^{\mathbb{H}^3}(\mathbf{x}, \mathbf{x}' | L_b) + \sum_{\gamma_p \neq \mathrm{Id}} K_t^{\mathbb{H}^3}(\mathbf{x}, \gamma_p \mathbf{x}' | L_b),$$

(3.9)

where the separation between the identity and the nontrivial topological contribution has been done, and we have defined

$$\gamma_p = \gamma_{p_1} \cdot \gamma_{p_2}.$$

(3.10)

Moreover, notice that the isometry group generated by γ_p is Abelian, so that the corresponding transformation law for a scalar field reads as

$$\phi(\gamma \mathbf{x}) = \chi \phi(\mathbf{x}),$$

(3.11)

where χ is a finite-dimensional unitary representation (a character) of Γ.

So on the diagonal part ($\mathbf{x} = \mathbf{x}'$), the heat kernel depends only on y, and turns out to be

$$K_t^{\mathcal{H}_0^3}(\mathbf{x}, \mathbf{x} | L_b) = K_t^{\mathcal{H}_0^3}(y | L_b)$$

$$= \frac{e^{-tb}}{(4\pi t)^{3/2}} + \frac{1}{(4\pi t)^{3/2}} \sum_{\mathbf{n} \neq \mathbf{0}} \frac{\chi_{\mathbf{n}} \sigma_{\mathbf{n}}(y)}{\sinh \sigma_{\mathbf{n}}(y)}$$

$$\times \exp\left[-tb - \frac{\sigma_{\mathbf{n}}^2(y)}{4t} \right],$$

(3.12)

with

$$\sigma_{\mathbf{n}}(y) = \cosh^{-1}\left[1 + \frac{\beta^2 n_1^2 + 4\pi^2 n_2^2}{2y^2} \right].$$

(3.13)

It is worth noticing that the Euclidean method selects for the quantization of a scalar field in the BTZ ground state the only boundary condition (Dirichlet) leading to a finite sum over images. Within the Lorentzian methods, since AdS^3 is not globally hyperbolic, also the Neumann and transparent boundary conditions can be used (see, for example, [9]).

However, in [9] it has been shown that when the mass of the BTZ black hole goes to zero, only Dirichlet boundary conditions give a regular and smooth renormalized vacuum expectation value for a scalar field. We will recover the same result making use of the ζ-function regularization.

One can now compute the local ζ function by means of the Mellin transform of the heat kernel (3.12) and then analytically continue it to the whole complex plane, obtaining

$$\zeta^{\mathcal{H}_0^3}(s, y | L_b) = \frac{b^{3/2-s}}{(4\pi)^{3/2}} \frac{\Gamma\left(s - \frac{3}{2}\right)}{\Gamma(s)} + \frac{b^{3/4-s/2} 2^{5/2-s}}{(4\pi)^{3/2} \Gamma(s)}$$

$$\times \sum_{\mathbf{n} \neq \mathbf{0}} \frac{\chi_{\mathbf{n}} \sigma_{\mathbf{n}}^{s-1/2}(y)}{\sinh \sigma_{\mathbf{n}}(y)} K_{3/2-s}[\sqrt{b}\, \sigma_{\mathbf{n}}(y)].$$

(3.14)

The first term in the equation above is the local ζ function for L_b acting on $\mathbb{H}^3$, which turns out to be coordinate independent, as it should since $\mathbb{H}^3$ is a symmetric space. For future reference we also report the local ζ function of the BTZ Euclidean section [19]:

$$\zeta^{\mathcal{H}^3}(s, r | L_b) = \frac{b^{3/2-s}}{(4\pi)^{3/2}} \frac{\Gamma\left(s - \frac{3}{2}\right)}{\Gamma(s)} + \frac{b^{3/4-s/2} 2^{5/2-s}}{(4\pi)^{3/2} \Gamma(s)}$$

$$\times \sum_{n \neq 0} \frac{\chi_n \sigma_n^{s-1/2}(r)}{\sinh \sigma_n(r)} K_{3/2-s}[\sqrt{b}\, \sigma_n(r)],$$

(3.15)

where now

$$\sigma_n(r) = \cosh^{-1}\left[1 + \frac{2r^2}{r_+^2}(\sinh^2 \pi n r_+) \right].$$

(3.16)

With regard to the computation of the effective action, one needs the analytical continuation of the global ζ function, obtained by performing the integration over the fundamental domain of the diagonal part of the related local quantity. It is easy to show that the fundamental domain $\mathcal{F}_0$ of $\mathcal{H}_0^3$ is noncompact, and that it is given as follows

$$\mathcal{F}_0 = \{ 0 \leq y < \infty, \ 0 \leq \tau < \beta, \ 0 < \varphi < 2\pi \}. \quad (3.17)$$

This means that the volume $V(\mathcal{F}_0) = V_0$ of the fundamental domain is divergent and we must introduce a regularization; the simplest one consists of limiting the integration in y between $1/R_0 < y < \infty$, with R_0 large enough.

Thus we have

$$V_0(R_0) = \int_{1/R_0}^{\infty} \frac{dy}{y^3} \int_0^{2\pi} d\varphi \int_0^{\beta} d\tau = \pi \beta R_0^2 \quad (3.18)$$

or, in the original coordinates,

$$V_0(R_0) = \int_0^\beta d\tau \int_0^{2\pi} d\varphi \int_0^{R_0} r\,dr = \pi\beta R_0^2. \quad (3.19)$$

In this way, starting from the heat kernel associated with the Laplace-like operator L_b, one has

$$K^{\mathcal{H}_0^3}(t|L_b) = \frac{V_0(R_0)e^{-tb}}{(4\pi t)^{3/2}} + \frac{2\pi\beta e^{-tb}}{(4\pi t)^{3/2}}$$

$$\times \sum_{\mathbf{n}\neq\mathbf{0}} \int_0^\infty \frac{dy}{y^{3-\varepsilon}} \frac{\sigma_{\mathbf{n}}(y)}{\sinh\sigma_{\mathbf{n}}(y)} \exp\left[-\frac{\sigma_{\mathbf{n}}^2(y)}{4t}\right],$$

$$(3.20)$$

where, as previously remarked, R is the cutoff of the identity volume element, and ε is the parabolic regularization parameter, necessary to regularize the divergence associated with the cusp (and which goes to zero at the end of the calculation). It should be noticed that in Eq. (3.20) (and from now on), it is assumed that our scalar field obeys Bose-Einstein statistics (i.e., $\chi_{\mathbf{n}} = 1 \quad \forall \mathbf{n}$).

Making the change of variable

$$u = \cosh^{-1}\left[1 + \frac{\beta^2 n_1^2 + 4\pi^2 n_2^2}{y^2}\right], \quad (3.21)$$

one has

$$K^{\mathcal{H}_0^3}(t|L_b) = \frac{V_0(R_0)e^{-tb}}{(4\pi t)^{3/2}} + 2^{\varepsilon/2}\pi\beta E_2$$

$$\times\left(1 - \frac{\varepsilon}{2}\middle|\frac{\beta^2}{4}, \pi^2\right) I_{t,b}(\varepsilon), \quad (3.22)$$

where

$$I_{t,b}(\varepsilon) = \frac{e^{-tb}}{2(4\pi t)^{3/2}} \int_0^\infty du\, u\, e^{-u^2/4t} (\cosh u - 1)^{-\varepsilon/2}$$

$$(3.23)$$

and

$$E_2(s|a_1, a_2) = \sum_{\mathbf{n}\neq\mathbf{0}} (a_1 n_1^2 + a_2 n_2^2)^{-s}, \quad (3.24)$$

is the Epstein ζ function, which is defined for $\text{Re}\, s > 1$ and can be analytically continued into the whole complex plane, its meromorphic continuation having a simple pole at $s = 1$ and being regular at $s = 0$. In particular, one has [20]

$$E_2(0|a_1, a_2) = -1,$$

$$E_2'(0|a_1, a_2) = \frac{1}{2}\ln\frac{a_2}{4\pi^2} - 2\pi\sqrt{\frac{a_1}{a_2}}\zeta(-1) - 2H\left(2\pi\sqrt{\frac{a_1}{a_2}}\right),$$

$$(3.25)$$

where $H(t)$ is the Hardy-Ramanujan modular function, which is given by

$$H(t) = \sum_{n=1}^\infty \ln(1 - e^{-tn}), \quad (3.26)$$

and satisfies the functional equation

$$H(t) = -\frac{\pi^2}{6t} - \frac{1}{2}\ln\left(\frac{t}{2\pi}\right) + \frac{t}{24} + H\left(\frac{4\pi^2}{t}\right). \quad (3.27)$$

Making use of the Epstein functional equation with $a_1 = (\beta/2)^2$, $a_2 = \pi^2$, one has

$$E_2\left(1 - \frac{\varepsilon}{2}\middle|\frac{\beta^2}{4}, \pi^2\right) = \frac{2}{\beta\pi^\varepsilon} \frac{2\Gamma\left(\dfrac{\varepsilon}{2}\right)}{\Gamma\left(1 - \dfrac{\varepsilon}{2}\right)}$$

$$\times E_2\left(\frac{\varepsilon}{2}\middle|\frac{4}{\beta^2}, \frac{1}{\pi^2}\right) + O(\varepsilon),$$

$$(3.28)$$

so that, after a first order expansion,

$$K^{\mathcal{H}_0^3}(t|L_b) = \frac{V_0(R_0)e^{-tb}}{(4\pi t)^{3/2}} - \frac{4\pi}{\varepsilon} I_{t,b}(0)$$

$$- 4\pi[I_{t,b}'(0) + I_{t,b}(0)G(\beta)] + O(\varepsilon),$$

$$(3.29)$$

where

$$I_t'(0) = -\frac{e^{-t}}{16\sqrt{\pi t}} \int_0^\infty du\, u\, e^{-u^2/4t} \ln(\cosh u - 1)$$

$$(3.30)$$

and, finally,

$$G(\beta) = \frac{3}{2}\ln 2 + \ln\pi - C + \frac{4\pi^2}{\beta}\zeta(-1) + 2H\left(\frac{4\pi^2}{\beta}\right). \quad (3.31)$$

(C is the Euler-Mascheroni constant).

So, besides the divergence of the volume (noncompact manifold) controlled by R, one has another divergence due to the continuum spectrum associated with the cusp, namely, the pole at $\varepsilon = 0$. It turns out that this singularity appears also in the spectral representation of the heat-kernel trace and it may be removed by means of suitable definition of the trace, as in the case of noncompact hyperbolic manifolds with a finite volume (see, for example [21] and references quoted therein). Thus, one has

$$K^{\mathcal{H}_0^3}(t|L_b) = \frac{V_0(R_0)e^{-tb}}{(4\pi t)^{3/2}} - 4\pi[I_{t,b}'(0) + I_{t,b}(0)G(\beta)]$$

$$+ O(\varepsilon). \quad (3.32)$$

As a consequence of the results obtained, one can now compute the global ζ function associated with our operator L_b, finding

$$\zeta(s|L_b) = \frac{V_0(R_0)b^{3/2-s}}{(4\pi)^{3/2}} \frac{\Gamma\left(s-\frac{3}{2}\right)}{\Gamma(s)} - \frac{G(\beta)b^{1/2-s}}{\sqrt{4\pi}} \frac{\Gamma\left(s-\frac{1}{2}\right)}{\Gamma(s)}$$

$$+ \frac{b^{1/2-s}}{\sqrt{4\pi}} \frac{\Gamma\left(s-\frac{1}{2}\right)}{\Gamma(s)} \left[-\frac{1}{2}\log b + \frac{\Psi\left(s-\frac{1}{2}\right)-C}{2} \right]$$

$$+ \frac{2^{-s}b^{3/4-s}}{\sqrt{2\pi}\Gamma(s)} \int_0^\infty dz z^{s-1/2} K_{3/2-s}(\sqrt{b}z)$$

$$\times \left[\log(\cosh z - 1) - 2\log\frac{z}{2} \right], \tag{3.33}$$

where the last integral is convergent.

It should be noticed that, due to the presence of parabolic elements, the meromorphic structure of this ζ function contains double poles at $s = 1/2 - k$, $k = 0,1,2,\ldots$; moreover, this ζ function is analytic in $s = 0$, and its derivative reads

$$\ln(\det L_b) = -\zeta'(0|L_b) = -\frac{V_0(R_0)b^{3/2}}{6\pi} - G(\beta)\sqrt{b} - F_b, \tag{3.34}$$

where F_b is a constant (independent from β) given by

$$F_b = \sqrt{b}\left[\frac{1}{2}\log b + C + \log 2 - 1 \right] + \frac{b^{3/4}}{\sqrt{2\pi}} \int_0^\infty dz\, z^{-1/2} K_{\frac{3}{2}}(\sqrt{b}z)$$

$$\times \left[\log(\cosh z - 1) - 2\log\frac{z}{2} \right]. \tag{3.35}$$

IV. FIRST QUANTUM CORRECTION TO THE ENTROPY OF THE BTZ BLACK HOLE

The first on-shell quantum correction to the Bekenstein-Hawking entropy may be computed within the Euclidean semiclassical approximation [16] and we shall follow this approach in this section. A pure gravitational quantum correction to the BTZ entropy has been presented in [19], making use of Chern-Simons representation of three-dimensional gravity [22]. Very recently in [23] the first quantum correction to the entropy and the back reaction of the BTZ black hole also have been studied. Here, for the sake of simplicity, we assume that the quantum degrees of freedom of the massive black hole are represented by the quantum scalar field [described as usual by the action (3.5)] propagating outside the black hole [3], and we shall make use of the results of [19] as well as the ones obtained in Sec. III.

Recall that within the Euclidean approach, the one-loop approximation gives, for the partition function in the BTZ background,

$$Z_{BTZ} = e^{-I_M}(\det L_b)_M^{-1/2}, \tag{4.1}$$

where I_M is the classical action related to the massive BTZ solution (see, for example, [19]). It reads

$$I_M = I_{BTZ} + B_{BTZ}, \tag{4.2}$$

in which I_{BTZ} is the Hilbert-Einstein action, while B_{BTZ} is the usual boundary term which depends on the extrinsic curvature at a large spatial distance. We recall that the total classical action is divergent; the geometry is noncompact and one has to introduce the reference background $\mathcal{H}_0^3$, at least at the tree level [24], and the related volume cutoffs R and R_0. Thus, one may also consider the related ground state partition function

$$Z_{BTZ_0} = e^{-I_0}(\det L_b)_0^{-1/2}, \tag{4.3}$$

where I_0 is the classical action related to the massless BTZ solution, given by

$$I_0 = I_{BTZ_0} + B_{BTZ_0}. \tag{4.4}$$

A simple but crucial observation is that, in order to recover the tree-level Bekenstein-Hawking entropy, one may introduce the ''relative'' partition function

$$Z_r = \frac{Z_{BTZ}}{Z_{BTZ_0}} = \left[\frac{(\det L_b)_0}{(\det L_b)_M} \right]^{1/2} e^{-(I_M - I_0)}. \tag{4.5}$$

With this proposal, the two boundary terms of the classical contribution cancel for large r and the difference of the on-shell Euclidean classical actions leads to [19]

$$I_M - I_0 = I_{BTZ} - I_{BTZ_0} = -\frac{2}{\pi}[V(R) - V_0(R_0)] \to -2\pi r_+. \tag{4.6}$$

Restoring the correct physical dimension, it is easy to show that the on-shell tree-level partition function $Z^{(0)}$ becomes

$$\ln Z^{(0)} = \frac{\pi^2 r_+}{4\pi G}, \tag{4.7}$$

and this leads to the semiclassical Bekenstein-Hawking entropy

$$S^{(0)} = S_H = \left(r_+ \frac{\partial}{\partial r_+} + 1 \right) \ln Z^{(0)} = \frac{1}{4} \frac{2\pi r_+}{G}. \tag{4.8}$$

Furthermore, concerning the regularization of the ratio of the two functional determinants (representing the quantum corrections), our proposal implements the correct mathematical procedure that is necessary when one is dealing with functional determinants of elliptic operators on noncompact manifolds (see [25]). In fact, in our case the manifolds are noncompact and a volume regularization (such as the one previously introduced) must be used. Thus, we have

$$\ln Z_r = 2\pi r_+ + \frac{1}{2}\ln(\det L_b)_0 - \frac{1}{2}\ln(\det L_b)_M. \tag{4.9}$$

In the case of scalar fields, one can compute the functional determinants in the BTZ background. Using the ζ-function regularization and the volume cutoffs R and R_0, as well as Eq. (3.34) with $\beta = \beta_H$, one gets

$$\ln Z_r(R) = \frac{\pi r_+}{4G} + \frac{b^{3/2} V(R)}{12\pi} - \frac{1}{2} \ln \mathcal{Z}_0(2)$$
$$- \frac{b^{3/2} V_0(R_0)}{12\pi} - \frac{F_b}{2} - \frac{\sqrt{b} G(r_+)}{2}, \quad (4.10)$$

where we have introduced the function

$$\ln \mathcal{Z}_0(2) = \sum_{n=1}^{\infty} \frac{1}{n} (e^{\pi n r_+ (\sqrt{b}+1)/2} - e^{\pi n r_+ (\sqrt{b}-1)/2})^{-2}. \quad (4.11)$$

Now we can remove the volume cutoff, taking the limit $R \to \infty$. In this way the horizon divergences cancel out and the finite result can be written as

$$\ln Z_r = \frac{\pi r_+}{4G} + h(r_+), \quad (4.12)$$

where

$$h(r_+) = -\frac{1}{2} \ln \mathcal{Z}_0(2) - \frac{b^{3/2} \pi r_+}{12} - \frac{F_b}{2} - \frac{\sqrt{b} G(r_+)}{2}. \quad (4.13)$$

Here G can be identified with an effective Newton constant, and we stress that within this approach the horizon divergences have been dealt with without an ultraviolet renormalization of it. This finite relative one-loop effective action may be thought to describe an effective classical geometry belonging to the same class of the nonrotating BTZ black hole solutions. This stems from the results contained in [17], where it has been shown that the constraints for pure gravity have a unique solution. As a consequence, one may introduce a new effective radius by means of

$$\ln Z_r = \frac{\pi R_+}{4G}, \quad (4.14)$$

where

$$R_+ = r_+ + \frac{4G}{\pi} h(r_+), \quad (4.15)$$

mimicking in this way the back reaction of the quantum gravitational fluctuations. As a consequence, the new entropy is given by an effective Bekenstein-Hawking term, namely,

$$S^{(1)} = \frac{1}{4} \frac{2\pi R_+}{G}. \quad (4.16)$$

One can evaluate the asymptotic behavior of the quantity $h(r_+)$ for $r_+ \to \infty$ and $r_+ \to 0$, and then obtain the effective radius. Notice that $H(r_+)$ and $\ln \mathcal{Z}_0(2)$ are exponentially small for large r_+. Thus, c being a numerical factor, we find

$$R_+ \simeq r_+ + c\sigma G r_+, \quad (4.17)$$

where $c\sigma G r_+$ are quantum corrections, which may be small since G is the inverse of the Planck length. On the other hand, for small r_+ one has

$$R_+ \simeq r_+ + \frac{4G}{\pi} \left[\frac{\sigma^2}{16 r_+^2} + c_1 \frac{\sigma}{r_+} + O\left(\ln\left(\frac{r_+}{\sigma \pi} \right) \right) \right], \quad (4.18)$$

where c_1 is another numerical factor.

One can see that for r_+ sufficiently small the effective radius becomes larger and positive. This means that R_+ (as a function of r_+) reaches a minimum for a suitable r_+. This result is in qualitative agreement with a very recent computation of the off-shell quantum correction to the entropy due to a scalar field in the BTZ background [13] and for the pure gravitational case [19]. In particular, it appears that the quantum gravitational corrections could become more and more important as soon as the evaporation process continues and thus they cannot be neglected. This qualitative picture does not take into account the back reaction. In order to do this, one must compute the vacuum expectation value of the stress tensor.

V. VACUUM EXPECTATION VALUE OF THE STRESS TENSOR

In this section, we shall compute the expectation value of the square of a quantum scalar field and its associated stress tensor expectation value on the black hole background. The latter will be used in the computation of the back reaction by solving the semiclassical Einstein equations

$$\mathcal{R}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \mathcal{R} + \Lambda g_{\mu\nu} = 8\pi G \langle T_{\mu\nu} \rangle. \quad (5.1)$$

With regard to this issue, it is worthwhile noticing that customarily used methods based on the correct behavior of the Green function to pick out a particular temperature for the thermal state are useless in the present contest. Indeed, such methods consider the behavior of the Green function when an argument belongs to some particular relevant point of the manifold, in particular points of the event horizons [26], and require a correct scaling limit for short distances as well as Hadamard's behavior. In the present case, no horizon appears and the singular points at $r = 0$ are not in the manifold as far as the Euclidean section of it is concerned. This is because any geodesic falling into these points spends an infinite amount of affine parameter. In the Lorentzian section, some points at $r = 0$, which are singular [9,17], belong to the manifold because, for instance, some timelike geodesics can reach such points in a finite period of proper time. Anyhow, in this case, the set of points at $r = 0$ represents a naked singularity and the use of the principles above for arguments of the Green functions fixed at $r = 0$ seems to be very problematic. On the other hand, in the Euclidean section, the request of the absence of the conical singularities does not select any temperature. For these reasons we shall deal with all possible values of the inverse temperature $\beta > 0$, so that

one has to consider the full (parabolic) isometry group of the ground state [whereas, in the case of the zero-temperature state, one should deal only with the element γ_{p_2} of Eq. (2.22)].

Let us now consider a nonminimally coupled scalar field ϕ, described by the action (3.5). We recall that within the ζ-function regularization, one has [27,28]

$$\langle \phi^2(\mathbf{x}) \rangle = \lim_{s \to 0} [\zeta(s+1,\mathbf{x}|L_b) + s\zeta'(s+1,\mathbf{x}|L_b) \ln \mu^2].$$
(5.2)

The substantial equivalence between the formula above and the result of a point splitting procedure has been analyzed in [28]. In $D=3$, the local ζ function is regular at $s=1$ and the dependence on the scale parameter μ^2 drops out; thus one has

$$\langle \phi^2(y) \rangle = \frac{\sqrt{b}}{(4\pi)^{3/2}} \Gamma\left(-\frac{1}{2}\right) + \frac{b^{1/4} 2^{3/2}}{(4\pi)^{3/2}}$$
$$\times \sum_{\mathbf{n} \neq \mathbf{0}} \frac{\sqrt{\sigma_{\mathbf{n}}(y)}}{\sinh \sigma_{\mathbf{n}}(y)} K_{1/2}[\sqrt{b}\sigma_{\mathbf{n}}(y)]$$
$$= \frac{-\sqrt{b}}{4\pi} + \frac{1}{4\pi} \sum_{\mathbf{n} \neq \mathbf{0}} \frac{e^{-\sqrt{b}\sigma_{\mathbf{n}}(y)}}{\sinh \sigma_{\mathbf{n}}(y)}.$$
(5.3)

Notice that the $\mathbb{H}^3$ case corresponds to the first term in the above equation and it turns out that the contribution is negative; namely, one has

$$\langle \phi^2(y) \rangle^{\mathbb{H}^3} = -\frac{\sqrt{b}}{4\pi}.$$
(5.4)

The second term can be referred to as the ''topological term'' and may be rewritten noticing that

$$\sigma_{\mathbf{n}}(y) = \ln(1 + C_{\mathbf{n}} + \sqrt{C_{\mathbf{n}}^2 + 2C_{\mathbf{n}}}),$$
(5.5)

where we have introduced the function

$$C_{\mathbf{n}}(\mathbf{x},\mathbf{x}') = \frac{(y-y')^2 + (x_1 - x_1' - \beta n_1)^2 + (x_2 - x_2' - 2\pi n_2)^2}{2yy'},$$
(5.6)

which on the diagonal reads

$$C_{\mathbf{n}}(y) = \frac{2b_{\mathbf{n}}^2}{y^2}, \quad b_{\mathbf{n}}^2 = \frac{\beta^2 n_1^2}{4} + \pi^2 n_2^2.$$
(5.7)

A direct computation of the field fluctuation as a function of $C_{\mathbf{n}}$ leads to

$$\langle \phi^2(y) \rangle = -\frac{\sqrt{b}}{4\pi} + \sum_{\mathbf{n} \neq \mathbf{0}} \mathcal{H}(C_{\mathbf{n}}(y)),$$
(5.8)

with

$$\mathcal{H}(C_{\mathbf{n}}) = \frac{2^{\sqrt{b}-3}}{\pi} \left(\frac{1}{\sqrt{C_{\mathbf{n}}}} - \frac{1}{\sqrt{C_{\mathbf{n}}+2}} \right) (\sqrt{C_{\mathbf{n}}} + \sqrt{C_{\mathbf{n}}+2})^{1-2\sqrt{b}}.$$
(5.9)

The series which appears on the right hand side of Eq. (5.8) is convergent as soon as $b>0$.

A similar computation in the BTZ case, namely, $M>0$, yields the same result, but with

$$C_n(\mathbf{x},\mathbf{x}') = \frac{(N^{n/2}y - N^{-n/2}y')^2 + (N^{n/2}x_1 - N^{-n/2}x_1')^2 + (N^{n/2}x_2 - N^{-n/2}x_2')^2}{2yy'},$$
(5.10)

in place of $C_{\mathbf{n}}$, where $\ln N = 2\pi r_+$, and on the diagonal

$$C_n(r) = \frac{r^2}{r_+^2} \sinh^2 2\pi n r_+.$$
(5.11)

In particular, in the massless conformally invariant case, one has

$$\langle \phi^2(r) \rangle^{\mathrm{BTZ}} = -\frac{1}{8\pi} + \frac{1}{2\sqrt{2}\pi} \sum_{n=1}^{\infty} \left(\frac{1}{\sqrt{C_n}} - \frac{1}{\sqrt{C_n+2}} \right),$$
(5.12)

in agreement with the result reported in [9].

As far as the expectation value of the stress tensor related to the field ϕ is concerned, in $D=3$ we have [29,30]

$$\langle T_{\mu\nu}(\mathbf{x}) \rangle = \zeta_{\mu\nu}(1,\mathbf{x}|L_b),$$
(5.13)

where the right hand side of the equation above is defined (in the sense of the analytical continuation) as

$$\zeta_{\mu\nu}(s,\mathbf{x}|L_b) = \sum_n \lambda_n^{-s} T_{\mu\nu}(\phi_n^*,\phi_n),$$
(5.14)

with ϕ_n representing the eigenfunctions of the Laplace-like operator L_b and $T_{\mu\nu}(\phi_n^*,\phi_n)$ being the classical stress tensor evaluated on the modes. The latter is defined as

$$T_{\mu\nu}(\phi^*,\phi) = \frac{2}{\sqrt{g}} \frac{\delta I[\phi^*,\phi]}{\delta g^{\mu\nu}},$$
(5.15)

where $I[\phi^*,\phi]_{\phi^*=\phi}$ is the associated classical action.

Furthermore, it is possible to show that [29,30]

$$\zeta_{\mu\nu}(s,\mathbf{x}|L_b)=L_{\mu\nu}\zeta(s,\mathbf{x}|L_b)-\frac{1}{2}g_{\mu\nu}\zeta(s-1,\mathbf{x}|L_b)$$

$$+\bar{\zeta}_{\mu\nu}(s,\mathbf{x}|L_b), \qquad (5.16)$$

where

$$L_{\mu\nu}=\xi\mathcal{R}_{\mu\nu}+\left(\xi-\frac{1}{4}\right)g_{\mu\nu}\Delta-\xi\nabla_\mu\nabla_\nu \qquad (5.17)$$

and, again in the sense of the analytical continuation,

$$\bar{\zeta}_{\mu\nu}(s,\mathbf{x}|L_b)=\frac{1}{2}\sum_n \lambda_n^{-s}(\partial_\mu\phi_n^*\partial_\nu\phi_n+\partial_\nu\phi_n^*\partial_\mu\phi_n). \qquad (5.18)$$

As a result, in $D=3$, since $\zeta(0,\mathbf{x}|L_b)=0$, one has

$$\langle T_{\mu\nu}(\mathbf{x})\rangle=\lim_{s\to 1}[L_{\mu\nu}\zeta(s,\mathbf{x}|L_b)+\bar{\zeta}_{\mu\nu}(s,\mathbf{x}|L_b)]. \quad (5.19)$$

Now, recalling that we are dealing with quotient manifolds $\mathbb{H}^3/\Gamma$, the image sum method can be applied. In general, our ζ functions are thus the sum of two contributions, namely,

$$\zeta(s,\mathbf{x},\mathbf{x}'|L_b)=\zeta^{\mathbb{H}^3}(s,\mathbf{x},\mathbf{x}'|L_b)+\zeta^\Gamma(s,\mathbf{x},\mathbf{x}'|L_b). \qquad (5.20)$$

Thus, in our case, the expectation value of the stress tensor splits into the sum of the related contributions:

$$\langle T_{\mu\nu}(\mathbf{x})\rangle=\langle T_{\mu\nu}(\mathbf{x})\rangle^{\mathbb{H}^3}+\langle T_{\mu\nu}(\mathbf{x})\rangle^\Gamma. \qquad (5.21)$$

Let us compute the first contribution. Now $\zeta^{\mathbb{H}^3}(s,\mathbf{x}|L_b)$ is independent from $\mathbf{x}$, and thus

$$\lim_{s\to 1} L_{\mu\nu}\zeta^{\mathbb{H}^3}(s,\mathbf{x}|L_b)=-2\xi\zeta^{\mathbb{H}^3}(1,\mathbf{x}|L_b)g_{\mu\nu}^{\mathbb{H}^3}=\frac{\xi\sqrt{b}}{2\pi}g_{\mu\nu}^{\mathbb{H}^3},$$

$$(5.22)$$

$g_{\mu\nu}^{\mathbb{H}^3}$ being the $\mathbb{H}^3$ metric. Furthermore, making use of the eigenfunctions reported in the Appendix, one easily finds the following analytical continuation:

$$\bar{\zeta}_{\mu\nu}^{\mathbb{H}^3}(s,\mathbf{x}|L_b)=\frac{1}{12\pi^2\Gamma(s)}\left[\Gamma\left(\frac{3}{2}\right)\Gamma\left(s-\frac{3}{2}\right)b^{3/2-s}\right.$$

$$\left.+\Gamma\left(\frac{5}{2}\right)\Gamma\left(s-\frac{5}{2}\right)b^{5/2-s}\right]g_{\mu\nu}^{\mathbb{H}^3}. \qquad (5.23)$$

In this way we obtain the result

$$\langle T_{\mu\nu}(\mathbf{x})\rangle^{\mathbb{H}^3}=\frac{\sqrt{b}}{4\pi}\left(\frac{b-1}{3}+2\xi\right)g_{\mu\nu}^{\mathbb{H}^3}=-\frac{m^2}{3}\langle\phi^2(y)\rangle^{\mathbb{H}^3}g_{\mu\nu}^{\mathbb{H}^3},$$

$$(5.24)$$

with the related trace

$$g_{\mathbb{H}^3}^{\mu\nu}\langle T_{\mu\nu}(\mathbf{x})\rangle^{\mathbb{H}^3}=-m^2\langle\phi^2(y)\rangle^{\mathbb{H}^3} \qquad (5.25)$$

in agreement with the general formula[1] [27]

$$g^{\mu\nu}\langle T_{\mu\nu}(\mathbf{x})\rangle=\zeta(0,\mathbf{x}|A)-\left(m^2+\frac{\xi-\xi_D}{4\xi_D-1}\Delta\right)\langle\phi^2(\mathbf{x})\rangle.$$

$$(5.26)$$

In particular, in the massless conformally coupled case one has $\langle T_{\mu\nu}(\mathbf{x})\rangle^{\mathbb{H}^3}=0$, in agreement with the fact that $\mathbb{H}^3$ is a homogeneous symmetric space and that the conformal anomaly vanishes in odd dimensions.

For the topological nontrivial part $\langle T_{\mu\nu}(\mathbf{x})\rangle^\Gamma$, it is convenient to proceed as follows.

Making use of the of the eigenvalues equation for the scalar eigenfunctions,

$$L_b\phi_n=\lambda_n\phi_n, \qquad (5.27)$$

and the background metric form, a standard calculation for the stress tensor evaluated on the modes (5.15) leads to

$$2T_{\mu\nu}(\phi_n^*,\phi_n)(\mathbf{x})$$

$$=(1-2\xi)(\nabla_\mu\phi_n^*\nabla_\nu\phi_n+\phi_n^*\nabla_\mu\nabla_\nu\phi_n)$$

$$+\left(2\xi-\frac{1}{2}\right)g_{\mu\nu}[(\nabla|\phi_n|)^2+\phi_n^*\Delta\phi_n]-\frac{m^2}{3}g_{\mu\nu}|\phi_n|^2$$

$$+\left(\frac{1}{3}g_{\mu\nu}\phi_n^*\Delta\phi_n-\phi_n^*\nabla_\mu\nabla_\nu\phi_n\right)$$

$$+\{\phi_n\to\phi_n^*,\phi_n^*\to\phi_n\}-\frac{\lambda_n g_{\mu\nu}}{3}|\phi_n|^2. \qquad (5.28)$$

Then, we can make use of Eq. (5.13), noticing that the last term in the equation above cannot product a contribution to the final stress tensor because it should be proportional to

$$g_{\mu\nu}[\zeta(0,\mathbf{x}|L_b)-\zeta^{\mathbb{H}^3}(0,\mathbf{x}|L_b)]=0, \qquad (5.29)$$

which vanishes since $D=3$ is odd so that both ζ functions above vanish for $s=0$ (remember that there is no conformal anomaly in odd-dimensional space times). Moreover, following the analysis contained in [30], it is possible to prove that the function $\zeta_{\mu\nu}^\Gamma(1,\mathbf{x}|L_b)$ of the topological nontrivial part of the stress tensor can be computed as the coincidence limit of the corresponding off-diagonal ζ function. This is because the corresponding series does not contain the identity element which gives rise to divergences. In general, the equivalence drops out for this element just because of the existence of a singularity at the coincidence limit. In practice, concern-

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[1]The coefficient $1/2\xi_D$ which appears in Eq. (13) of [27] is misprinted, and has to be corrected to $1/(4\xi_D-1)$.

ing the nontrivial topological part of the stress tensor, from Eqs. (5.28) and (5.13), one finds that it reduces to

$$\langle T_{\mu\nu}(\mathbf{x})\rangle^{\Gamma} = (1-2\xi)A_{\mu\nu} + \left(2\xi - \frac{1}{2}\right)g_{\mu\nu}A$$
$$+ \frac{1}{3}g_{\mu\nu}B - B_{\mu\nu} - \frac{m^2}{3}g_{\mu\nu}\zeta^{\Gamma}(1,\mathbf{x}|L_b),$$

(5.30)

where we have defined

$$A_{\mu\nu} = \lim_{\mathbf{x}'\to\mathbf{x}} \frac{1}{2}[(\nabla_\mu\nabla'_\nu + \nabla'_\mu\nabla_\nu)$$
$$+ (\nabla_\mu\nabla_\nu + \nabla'_\mu\nabla'_\nu)]\zeta^{\Gamma}(1,\mathbf{x},\mathbf{x}'|L_b),$$

(5.31)

with $A = g^{\mu\nu}A_{\mu\nu}$ and

$$B_{\mu\nu} = \lim_{\mathbf{x}'\to\mathbf{x}} \frac{1}{2}[(\nabla_\mu\nabla_\nu + \nabla'_\mu\nabla'_\nu)]\zeta^{\Gamma}(1,\mathbf{x},\mathbf{x}'|L_b),$$

(5.32)

with $B = g^{\mu\nu}B_{\mu\nu}$. Moreover, since

$$\zeta^{\Gamma}(1,\mathbf{x},\mathbf{x}'|L_b) = \sum_{\mathbf{n}\neq\mathbf{0}} \mathcal{H}(C_{\mathbf{n}}(\mathbf{x},\mathbf{x}')),$$

(5.33)

a direct calculation in the coordinate system (y,x_1,x_2) leads to

$$A_{\mu\nu} = \sum_{\mathbf{n}\neq\mathbf{0}} \left[\left(\frac{8b_{\mathbf{n}}^4\mathcal{H}''}{y^6} + \frac{2b_{\mathbf{n}}^2\mathcal{H}'}{y^4}\right)\delta_{\mu 0}\delta_{\nu 0} + \frac{2b_{\mathbf{n}}^2\mathcal{H}'}{y^2}g_{\mu\nu}^{\mathrm{H}^3}\right],$$

(5.34)

$$A = \sum_{\mathbf{n}\neq\mathbf{0}} \left[\frac{8b_{\mathbf{n}}^4\mathcal{H}''}{y^4} + \frac{8b_{\mathbf{n}}^2\mathcal{H}'}{y^2}\right],$$

(5.35)

$$B_{\mu\nu} = \sum_{\mathbf{n}\neq\mathbf{0}} \left[\left(\frac{4b_{\mathbf{n}}^4}{y^6}\delta_{\mu 0}\delta_{\nu 0} + \frac{\beta^2 n_1^2}{y^4}\delta_{\mu 1}\delta_{\nu 1}\right.\right.$$
$$\left.\left. + \frac{4\pi^2 n_2^2}{y^4}\delta_{\mu 2}\delta_{\nu 2}\right)\mathcal{H}'' + \left(g_{\mu\nu}^{\mathrm{H}^3} + \frac{2b_{\mathbf{n}}^2}{y^2}g_{\mu\nu}^{\mathrm{H}^3}\right)\mathcal{H}'\right],$$

$$B = \sum_{\mathbf{n}\neq\mathbf{0}} \left[\left(\frac{4b_{\mathbf{n}}^4}{y^4} + \frac{4b_{\mathbf{n}}^2}{y^2}\right)\mathcal{H}'' + \left(3 + \frac{6b_{\mathbf{n}}^2}{y^2}\right)\mathcal{H}'\right],$$

(5.36)

where the prime means derivatives with respect to $C_{\mathbf{n}}$.

Summarizing, we have found that the complete renormalized stress tensor is that written on the right hand side of Eq. (5.21) where the former term is given in Eq. (5.24) taking account of Eq. (5.4), and the latter is given in Eq. (5.30) taking account of Eqs. (5.9), (5.33) and the expressions for $A_{\mu\nu}, A, B_{\mu\nu}, B$ written above. Moreover, notice that the dependence on ξ and m^2 arises only from $\mathcal{H}$ and its derivatives, and is given by Eq. (5.9); the β dependence is instead due to $b_{\mathbf{n}}$ and $\mathcal{H}$ and is given by Eqs. (5.7) and (5.9).

In the zero-temperature case one has the same result, but replacing $b_{\mathbf{n}}^2$ with $\pi^2 n_2^2$, dropping the term proportional to β in $B_{\mu\nu}$, and considering only the sum over n_2.

With regard to the stress tensor trace one finally has

$$g^{\mu\nu}\langle T_{\mu\nu}(\mathbf{x})\rangle^{\Gamma} = 4\left(\xi - \frac{1}{8}\right)A - m^2\zeta^{\Gamma}(1,\mathbf{x},|L_b), \quad (5.37)$$

so that the total contribution reads

$$g^{\mu\nu}\langle T_{\mu\nu}(\mathbf{x})\rangle = \langle T\rangle = 4\left(\xi - \frac{1}{8}\right)A - m^2\langle\phi^2(\mathbf{x})\rangle$$
$$= \left[2\left(\xi - \frac{1}{8}\right)\Delta - m^2\right]\langle\phi^2(\mathbf{x})\rangle, \quad (5.38)$$

again in agreement with Eq. (5.26) and [27].[2] Thus, for a massless and conformally coupled scalar field, one also has a vanishing contribution.

VI. BACK REACTION ON THE METRIC

In this section, we shall discuss the back reaction on the BTZ ground state due to the quantum fluctuations. Since any temperature is admissible, we choose $\beta = \infty$, which corresponds to fixing the temperature of the ground state at the lowest possible value $T = 0$.

To begin with, we rewrite the semiclassic Einstein equations in the form ($\Lambda = -1$)

$$\mathcal{R}_{\mu\nu} = -2g_{\mu\nu} + 8\pi G(\langle T_{\mu\nu}\rangle - g_{\mu\nu}\langle T\rangle)$$
$$= -2g_{\mu\nu} + 8\pi G\langle\hat{T}_{\mu\nu}\rangle, \quad (6.1)$$

where we have used the result

$$\mathcal{R} = -6 - 16\pi G\langle T\rangle. \quad (6.2)$$

Now, we have found the general expressions of the expectation values $\langle T_{\mu\nu}\rangle$. As a consequence, the semiclassic metric shows a nonconstant scalar curvature as well as a nonconstant Ricci tensor. Furthermore, these nonconstant quantities are singular in the limit $r \to 0$. In the conformally coupled case, $\langle T\rangle$ is vanishing, but $\mathcal{R}_{\mu\nu}$ is still not constant and eventually one has to deal with a "distorted" black hole solution, whose nature comes from solving the semiclassical back reaction equations at first order in the Planck length G. To this aim, it is an usual approach starting from the general static radial symmetric solution in the coordinates (t,r,φ), the ones of our background, namely, the ground state of the BTZ solution. Now a subtle point arises: in this background the one-loop approximation may break down (fluctuations in $\langle T_{\mu\nu}\rangle$ would be of the same order of $\langle T_{\mu\nu}\rangle$). In order to cure this flaw, a possible trick consists in considering N independent scalar fields instead of one, with N very large such that $NG = \hat{G}$ is small and fixed [31,32,9]. This has two effects: from one side the ratio of the fluctuations to $\langle T_{\mu\nu}\rangle$ becomes

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[2]See previous footnote.

negligible in proximity of the horizon; on the other side, the one-loop approximation may become almost exact, because higher loop terms are of the order $O(1/N)$. Within this new scheme of approximation, a quite natural ansatz which is consistent with the gauge of the background is [33,11]

$$ds^2 = -e^{2\hat{G}\psi(r)}[r^2 + \hat{G}\varepsilon(r)]dt^2 + \frac{1}{r^2 + \hat{G}\varepsilon(r)}dr^2 + r^2 d\varphi^2. \tag{6.3}$$

Denoting

$$A(r) = [r^2 + \hat{G}\varepsilon(r)]^{-1}, \quad B(r) = e^{2\hat{G}\psi(r)}A^{-1}(r), \tag{6.4}$$

a standard calculation leads to

$$R_0^0 = -\frac{B''}{2AB} + \frac{B'}{4AB}\left(\frac{A'}{A} + \frac{B'}{B}\right) - \frac{B'}{2rAB}, \tag{6.5}$$

$$R_1^1 = -\frac{B''}{2AB} + \frac{B'}{4AB}\left(\frac{A'}{A} + \frac{B'}{B}\right) + \frac{A'}{2rA^2}, \tag{6.6}$$

$$R_2^2 = \frac{1}{2rA}\left(\frac{A'}{A} - \frac{B'}{B}\right). \tag{6.7}$$

The Einstein equation associated with the mixed (0,0) components gives

$$\varepsilon'(r) = 16\pi r\langle T_0^0(r)\rangle, \tag{6.8}$$

and a suitable combination of these components leads also to

$$-r\psi'(r) = 8\pi[\langle T_0^0(r)\rangle - \langle T_1^1(r)\rangle] + O(\hat{G}), \tag{6.9}$$

where, in the second equation, we have retained only the leading term in $\hat{G}$. As solutions of the two differential equations above, we may take

$$\varepsilon(r) = 16\pi\int dr\, r\langle T_0^0(r)\rangle, \tag{6.10}$$

$$\psi(r) = 8\pi\int \frac{dr}{r}[\langle T_1^1(r)\rangle - \langle T_0^0(r)\rangle], \tag{6.11}$$

the constants of integration chosen in order to have the ground state ($M=0$) solution when the back reaction is switched off. In the conformally coupled case, the computation is easier and, within our choice of the integration constants, one has

$$\varepsilon(r) = -\frac{\zeta_R(3)}{\pi^3 r} + \Phi(r), \tag{6.12}$$

$$\psi(r) = -\frac{1}{4l}\sum_{n=1}^{\infty}\left(1 + \frac{\pi^2 n^2 r^2}{l^2}\right)^{-3/2}, \tag{6.13}$$

where

$$\Phi(r) = \frac{1}{2\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}[\pi^2 n^2 r^2(1 + \pi^2 n^2 r^2)^{-3/2} + 2(1 + \pi^2 n^2 r^2)^{-1/2}]. \tag{6.14}$$

Notice that the two series $\psi(r)$ and $\Phi(r)$ converge as long as $r>0$.

As anticipated, a curvature singularity is present at $r=0$, but this singularity may be hidden by the quantum corrections as soon as there exist positive real solutions to the equation $g^{11}=0$, i.e,

$$\hat{G}\Phi(r) = \frac{\hat{G}\zeta_R(3)}{\pi^3 r} - r^2. \tag{6.15}$$

Let us consider this equation for $r>0$.

$\Phi(r)$ is a smooth, monotonically nonincreasing, and strictly positive function of r with a unique flex at $r=r_f$ near $r=0$; moreover, it takes the limit $\zeta_R(2)/\pi^2$ for $r\to 0^+$ and vanishes for $r\to +\infty$. On the other hand, the function which appears on the right hand side of Eq. (6.15) is smooth and monotonically nonincreasing too; furthermore, it is positive for $r^3 < \hat{G}\zeta_R(3)/\pi^3$, divergent in the limit $r\to 0^+$, and shows a unique flex in $r^3 = \hat{G}\zeta_R(3)/\pi^3$ where the function takes the only zero in the considered domain.

In this way, it remains proved that, for each values of $\hat{G}$, there exists at least one and at most three positive and real solutions to the Eq. (6.15), so that the singularity $r=0$ is always shielded by an event horizon, the radius of which coincides with the rightmost zero where g^{11} changes sign (such a zero always exists); notice that, after that zero, $g^{11} >0$. In any cases, when $\hat{G}$ is small sufficiently [$\hat{G} < \pi^3 r_f^3/\zeta_R(3)$], only one zero arises where g^{11} changes sign. Restoring the correct physical dimensions, the event horizon satisfies

$$0 < r_+ < \left[\frac{\hat{G}l^2\zeta(3)}{\pi^3}\right]^{1/3}, \tag{6.16}$$

which, anyhow, cannot be arbitrarily large. Qualitatively, we expect the nonconformally coupled case to be similar to the one discussed here. Furthermore, the singularity dressing phenomenon illustrated here for the massless BTZ black hole has a four-dimensional analogue [34] associated with the recent discovery of a class of four-dimensional AdS topological black holes [35–37].

VII. CONCLUDING REMARKS

In this paper, one-loop quantum properties of the ground state of the BTZ black hole have been discussed in detail,

considering a scalar quantum field propagating in the classical background of the massless BTZ black hole. No restriction to the gravitational coupling and the mass of the scalar field has been assumed, and the one-loop effective action and the expectation value for the energy-momentum stress tensor have been computed. As applications of these results, the leading order quantum correction to the BTZ black hole entropy and the back reaction to the classical metric due to the quantum fluctuations have been presented. With regard to the latter, we have confirmed that, in the presence of N conformally coupled scalar fields and in the large N limit, the quantum fluctuations tend to dress the original naked singularity, similarly to the effect found in the four-dimensional case [35]. This may be interpreted as a quantum implementation of the cosmic censorship hypothesis.

APPENDIX

In this appendix, we shall briefly outline the computation of the heat-kernel trace for the scalar Laplace operator on the noncompact hyberbolic space $\mathbb{H}^3$, starting from the spectral theorem. Although the final result is well known (see, for example, [38,39]), we think that it is useful to present here an elementary derivation.

To begin with, let us introduce the operator $L=-\Delta-1$, Δ being the Laplace operator on $\mathbb{H}^3$. Thus

$$K_t^{\mathbb{H}^3}(\mathbf{x},\mathbf{x}'|\Delta)=\langle\mathbf{x}|e^{t\Delta}|\mathbf{x}'\rangle=e^{-t}\langle\mathbf{x}|e^{-tL}|\mathbf{x}'\rangle, \quad (\text{A1})$$

where $\mathbf{x}=(y,\mathbf{w})\in\mathbb{H}^3$. Our aim is so to compute the heat kernel $\langle\mathbf{x}|e^{-tL}|\mathbf{x}'\rangle$. The eigenvalues equation for L is

$$L\psi=[-y^2(\Delta_2+\partial_y^2)+y\partial_y-1]\psi=\lambda^2\psi, \quad (\text{A2})$$

where Δ_2 is the Laplace operator on $\mathbb{R}^2$ (the transverse manifold), which satisfies the eigenvalues equation

$$-\Delta_2 f_{\mathbf{k}}(\mathbf{w})=k^2 f_{\mathbf{k}}(\mathbf{w}), \quad (\text{A3})$$

where

$$f_{\mathbf{k}}(\mathbf{w})=\frac{e^{i\mathbf{k}\cdot\mathbf{w}}}{2\pi}, \quad k^2=\mathbf{k}\cdot\mathbf{k}. \quad (\text{A4})$$

With the ansatz

$$\psi=\phi(y)f_{\mathbf{k}}(\mathbf{w}), \quad (\text{A5})$$

one gets the equation

$$y^2\phi''-y\phi'+(\lambda^2+1-k^2y^2)\phi=0, \quad (\text{A6})$$

whose solutions are MacDonald's functions

$$\phi(y)=yK_{i\lambda}(ky), \quad (\text{A7})$$

with λ non-negative. As a result, the spectrum is continuous and the generalized eigenfunctions are

$$\psi_{\lambda,\mathbf{k}}(\mathbf{x})=yK_{i\lambda}(ky)f_{\mathbf{k}}(\mathbf{w}). \quad (\text{A8})$$

The nontrivial spectral measure, which plays an important role, is given by

$$\mu(\lambda)=\frac{2}{\pi^2}\lambda\sinh\pi\lambda. \quad (\text{A9})$$

The spectral theorem yields

$$\langle\mathbf{x}|e^{-tL}|\mathbf{x}'\rangle=\int_0^\infty d\lambda\,\mu(\lambda)e^{-t\lambda^2}\int\frac{d^2k}{2\pi}e^{i\mathbf{k}\cdot\mathbf{u}}yy'$$
$$\times K_{i\lambda}(ky)K_{i\lambda}(ky'), \quad (\text{A10})$$

where $\mathbf{u}=\mathbf{w}-\mathbf{w}'$. The integration over $\mathbf{k}$ can be performed, making use of polar coordinates in the plane, and gives

$$\int_0^\infty dk\,k\int_0^{2\pi}d\theta\,e^{iku\cos\theta}yy'K_{i\lambda}(ky)K_{i\lambda}(ky')$$
$$=\frac{\lambda^2}{\mu(\lambda)}\frac{P_{i\lambda-1/2}^{-1/2}(\cosh\sigma(\mathbf{x},\mathbf{x}'))}{\sqrt{2\pi\sinh\sigma(\mathbf{x},\mathbf{x}')}}. \quad (\text{A11})$$

Since

$$P_{i\lambda-1/2}^{-1/2}[\cosh\sigma(\mathbf{x},\mathbf{x}')]=\sqrt{\frac{2}{\pi}}\frac{\sin\lambda\sigma(\mathbf{x},\mathbf{x}')}{\sqrt{\sinh\sigma(\mathbf{x},\mathbf{x}')}}, \quad (\text{A12})$$

an elementary integration over λ gives

$$K_t^{\mathbb{H}^3}(\mathbf{x},\mathbf{x}'|L)=\frac{1}{(4\pi t)^{3/2}}\frac{\sigma(\mathbf{x},\mathbf{x}')}{\sinh\sigma(\mathbf{x},\mathbf{x}')}\exp\left[-\frac{\sigma^2(\mathbf{x},\mathbf{x}')}{4t}\right], \quad (\text{A13})$$

from which Eq. (3.8) easily follows.

Along the same lines, we determine the generalized eigenfunctions of the Laplace operator on the ground state solution $\mathcal{H}_0^3$. It is convenient again to deal with the operator L. One has a continuous and discrete spectrum, because now

the transverse manifold is a compact two-dimensional torus. One has

$$\psi_{\lambda,\mathbf{0}}(x) = y^{1+i\lambda}, \quad \psi_{\lambda,\mathbf{k}}(x) = y K_{i\lambda}(ny) \frac{e^{i\mathbf{w}\cdot\mathbf{n}}}{\sqrt{2\pi\beta}}, \tag{A14}$$

where $\mathbf{x} = (2\pi n_1/\beta, n_2)$. As a result, the kernel of the opera-

tor $F(L)$, where $F(\cdot)$ is a smooth function, reads

$$\langle\mathbf{x}|F(L)|\mathbf{x}'\rangle = \int_0^\infty d\lambda F(\lambda) y'^{\,1-i\lambda} y^{1+i\lambda}$$

$$+ \sum_{\mathbf{k}\neq\mathbf{0}} \frac{yy'}{2\pi\beta} \int_0^\infty d\lambda\,\mu(\lambda) F(\lambda)$$

$$\times e^{i\mathbf{n}\cdot\mathbf{u}} K_{i\lambda}(ny) K_{i\lambda}(ny'). \tag{A15}$$

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