Bounds on negative energy densities in static space-times

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(Received 14 December 1998; published 23 April 1999)

Certain exotic phenomena in general relativity, such as backward time travel, appear to require the presence of matter with negative energy. While quantum fields are a possible source of negative energy densities, there are lower bounds—known as quantum inequalities—that constrain their duration and magnitude. In this paper, we derive new quantum inequalities for scalar fields in static space-times, as measured by static observers with a choice of sampling function. Unlike those previously derived by Pfenning and Ford, our results do not assume any specific sampling function. We then calculate these bounds in static three- and four-dimensional Robertson-Walker universes, the de Sitter universe, and the Schwarzschild black hole. In each case, the new inequality is stronger than that of Pfenning and Ford for their particular choice of sampling function. [S0556-2821(99)06610-2]

PACS number(s): 04.62.+v

I. INTRODUCTION

In recent years, there has been much interest in various exotic solutions of general relativity—such as traversable wormholes [1,2], the Alcubierre ''warp drive'' [3], and the Krasnikov ''tube'' [4]—that permit hyperfast or backward time travel. However, these space-times without exception require the presence of matter which possess *negative* energy densities [5–8], and hence violate the standard energy conditions.

Now, it is well known that quantum field theory, unlike classical physics, allows energy density to be unboundedly negative at a point in space-time [9]. Should the theory place no restrictions on this negative energy, quantum fields could be used to produce gross macroscopic effects such as those mentioned above, or even a violation of cosmic censorship or the second law of thermodynamics. It is therefore important to have a quantitative handle on the permitted amount of negative energy in a neighborhood of a space-time point.

Ford and Roman [10,11] have found inequalities which constrain the duration and magnitude of negative energy densities for quantized free, real scalar fields in Minkowski space. They show that a static observer, who samples the energy density by time-averaging it against the Lorentzian function

$$f(t)=\frac{t_0}{\pi}\frac{1}{t^2+t_0^2}, \tag{1}$$

obtains a result which is bounded from below by a negative quantity depending inversely on the characteristic timescale t_0. For example, in the case of a massless scalar field in four dimensions, the averaged renormalized energy density ρ in any quantum state satisfies

$$\rho \geqslant -\frac{3}{32\pi^2 t_0^4}. \tag{2}$$

This means the more negative the energy density that is present in an interval, the shorter the duration of this interval must be. Thus, this ''quantum inequality''—in a way reminiscent of the uncertainty principle of quantum mechanics[1]—serves to limit any large-scale, long-time occurrence of negative energy. In the infinite sampling time limit $t_0 \to \infty$, it reduces to the usual averaged weak energy condition (for quantum fields [12,13]).

Eveson and one of the present authors [14] have recently presented a different derivation of the quantum inequalities for a massive scalar field in n-dimensional Minkowski space (with $n \geqslant 2$). The method used is straightforward—involving only the canonical commutation relations and the convolution theorem of Fourier analysis—and has the virtue of being valid for any smooth, non-negative and even sampling function decaying sufficiently quickly at infinity. Furthermore, the resulting bounds turn out to be stronger than those obtained by Ford and Roman [10,11] when the Lorentzian sampling function is applied.

In the present paper, we extend this method to derive quantum inequalities for scalar fields in generally curved but static space-times using arbitrary smooth, non-negative (although not necessarily even, as assumed in [14]) sampling functions of sufficiently rapid decay. We obtain a lower bound on the averaged normal-ordered energy density in the

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[1]However, the derivation of the quantum inequalities does not depend on any putative time-energy uncertainty principle.

Fock space built on the static vacuum in terms of the appropriate mode functions. Since the normal-ordered energy density in a given state is the difference between the renormalized energy density in this state and the (generally nonzero and potentially negative) renormalized energy density of the static vacuum, our bound also constrains the renormalized energy density (cf. [12]).

We apply our bound to several examples where it can be explicitly evaluated, namely the three- and four-dimensional Robertson-Walker universes, the de Sitter universe, and the Schwarzschild black hole. In all these cases, we obtain bounds which are up to an order of magnitude stronger than those previously derived by Pfenning and Ford [12,13,15] for the specific sampling function they used.

II. DERIVATION OF THE QUANTUM INEQUALITY

We shall consider $(n+1)$-dimensional space-times that are globally static, with time-like Killing vector ∂_t. The metric of such a space-time takes the general form

$$ds^2 = -|g_{tt}(\boldsymbol{x})|dt^2 + g_{ij}(\boldsymbol{x})dx^i dx^j, \tag{3}$$

where $\boldsymbol{x}=(x^1,x^2\ldots,x^n)$ and $i,j=1,2,\ldots,n$. The equation of a free, real scalar field ϕ of mass $\mu \geqslant 0$ in this space-time is

$$-\frac{1}{|g_{tt}|}\partial_t^2\phi + \nabla^i\nabla_i\phi - \mu^2\phi = 0. \tag{4}$$

Suppose it admits a complete, orthonormal set of positive frequency solutions. We write these mode functions as

$$f_\lambda(t,\boldsymbol{x}) = U_\lambda(\boldsymbol{x})e^{-i\omega_\lambda t}, \tag{5}$$

where λ denotes the set of quantum numbers needed to specify the mode (which may be continuous or discrete). A general quantum scalar field can then be expanded as

$$\phi = \sum_\lambda (a_\lambda f_\lambda + a_\lambda^\dagger f_\lambda^*) \tag{6}$$

in terms of creation and annihilation operators $a_\lambda^\dagger$, a_λ obeying the canonical commutation relations

$$[a_\lambda, a_{\lambda'}^\dagger] = \delta_{\lambda\lambda'}\mathbb{1}, \quad [a_\lambda, a_{\lambda'}] = [a_\lambda^\dagger, a_{\lambda'}^\dagger] = 0, \tag{7}$$

and which generate the Fock space built on the static vacuum state $|0\rangle$. We shall be interested in the energy density of ϕ along the world-line $x^\mu(t)=(t,\boldsymbol{x}_0)$ of a *static* observer, with $\boldsymbol{x}_0$ kept fixed. If the field is in a normalized quantum state $|\psi\rangle$, the normal-ordered energy density as measured by such an observer at time t is [12,13]

$$\begin{aligned}\langle :T_{\mu\nu}u^\mu u^\nu: \rangle = \mathrm{Re}\sum_{\lambda,\lambda'}\Bigg\{ &\frac{\omega_\lambda\omega_{\lambda'}}{|g_{tt}|}[U_\lambda^* U_{\lambda'}\langle a_\lambda^\dagger a_{\lambda'}\rangle e^{i(\omega_\lambda-\omega_{\lambda'})t} - U_\lambda U_{\lambda'}\langle a_\lambda a_{\lambda'}\rangle e^{-i(\omega_\lambda+\omega_{\lambda'})t}] \\ &+ [\nabla^i U_\lambda^* \nabla_i U_{\lambda'}\langle a_\lambda^\dagger a_{\lambda'}\rangle e^{i(\omega_\lambda-\omega_{\lambda'})t} + \nabla^i U_\lambda \nabla_i U_{\lambda'}\langle a_\lambda a_{\lambda'}\rangle e^{-i(\omega_\lambda+\omega_{\lambda'})t}] \\ &+ \mu^2[U_\lambda^* U_{\lambda'}\langle a_\lambda^\dagger a_{\lambda'}\rangle e^{i(\omega_\lambda-\omega_{\lambda'})t} + U_\lambda U_{\lambda'}\langle a_\lambda a_{\lambda'}\rangle e^{-i(\omega_\lambda+\omega_{\lambda'})t}]\Bigg\},\end{aligned} \tag{8}$$

where $u^\mu = (|g_{tt}|^{-1/2},\boldsymbol{0})$ is the observer's four-velocity, and U_λ and its derivatives are evaluated at $\boldsymbol{x}_0$. We have also written $\langle\cdot\rangle \equiv \langle\psi|\cdot|\psi\rangle$ for brevity. Recall that the normal-ordered energy density is the difference between the renormalized energy density in the two states $|\psi\rangle$ and $|0\rangle$.

We now define a weighted energy density

$$\rho = \int_{-\infty}^{\infty} dt \langle :T_{\mu\nu}u^\mu u^\nu: \rangle f(t), \tag{9}$$

where f is any smooth, non-negative function decaying at least as fast as $\mathrm{O}(t^{-2})$ at infinity, and normalized to have unit integral. Ford and coworkers [10–13,15] employ the Lorentzian function (1), whose specific properties play a key role in their arguments [in particular, the Fourier transform of Eq. (1) is simply the function $\exp(-|\omega|t_0)$]; we emphasize that our arguments apply to general f. Substituting from Eq. (8), the weighted energy density measured by the observer is

$$\begin{aligned}\rho = \mathrm{Re}\sum_{\lambda,\lambda'}\Bigg\{ &\frac{\omega_\lambda\omega_{\lambda'}}{|g_{tt}|}[U_\lambda^* U_{\lambda'}\langle a_\lambda^\dagger a_{\lambda'}\rangle \hat{f}(\omega_{\lambda'}-\omega_\lambda) - U_\lambda U_{\lambda'}\langle a_\lambda a_{\lambda'}\rangle \hat{f}(\omega_\lambda+\omega_{\lambda'})] + [\nabla^i U_\lambda^* \nabla_i U_{\lambda'}\langle a_\lambda^\dagger a_{\lambda'}\rangle \hat{f}(\omega_{\lambda'}-\omega_\lambda) \\ &+ \nabla^i U_\lambda \nabla_i U_{\lambda'}\langle a_\lambda a_{\lambda'}\rangle \hat{f}(\omega_\lambda+\omega_{\lambda'})] + \mu^2[U_\lambda^* U_{\lambda'}\langle a_\lambda^\dagger a_{\lambda'}\rangle \hat{f}(\omega_{\lambda'}-\omega_\lambda) + U_\lambda U_{\lambda'}\langle a_\lambda a_{\lambda'}\rangle \hat{f}(\omega_\lambda+\omega_{\lambda'})]\Bigg\},\end{aligned} \tag{10}$$

where we define the Fourier transform of f by

$$\hat{f}(\omega)=\int_{-\infty}^{\infty} dt\, f(t)e^{-i\omega t}. \tag{11}$$

By applying the inequality (A2), proved in the Appendix, to each of the cases $q_\lambda=(\omega_\lambda/|g_{tt}|^{1/2})U_\lambda$, $\nabla_i U_\lambda$, and μU_λ, we obtain the following manifestly negative lower bound for ρ:

$$\rho \geqslant -\frac{1}{2\pi}\int_0^\infty d\omega \sum_\lambda \left(\frac{\omega_\lambda^2}{|g_{tt}|} U_\lambda^* U_\lambda + \nabla^i U_\lambda^* \nabla_i U_\lambda + \mu^2 U_\lambda^* U_\lambda \right) |\widehat{f^{1/2}}(\omega+\omega_\lambda)|^2. \tag{12}$$

Using the field equation satisfied by the spatial mode function [12,13]:

$$\nabla^i \nabla_i U_\lambda + \left(\frac{\omega_\lambda^2}{|g_{tt}|} - \mu^2 \right) U_\lambda = 0, \tag{13}$$

this inequality can be rewritten as

$$\rho \geqslant -\frac{1}{\pi}\int_0^\infty d\omega \sum_\lambda \left(\frac{\omega_\lambda^2}{|g_{tt}|} + \frac{1}{4}\nabla^i \nabla_i \right) |U_\lambda|^2 |\widehat{f^{1/2}}(\omega+\omega_\lambda)|^2. \tag{14}$$

This is the desired quantum inequality, which is valid for general sampling functions $f(t)$, subject to the above-stated conditions. Another useful form of it can be obtained by introducing the new variable $u=\omega+\omega_\lambda$:

$$\rho \geqslant -\frac{1}{\pi}\int_{\omega_{\min}}^\infty du |\widehat{f^{1/2}}(u)|^2 \sum_{\lambda \text{ s.t. } \omega_\lambda \leqslant u} \left(\frac{\omega_\lambda^2}{|g_{tt}|} + \frac{1}{4}\nabla^i \nabla_i \right) |U_\lambda|^2, \tag{15}$$

with $\omega_{\min} \equiv \min_\lambda \omega_\lambda$.

To simplify it any further would require a specific choice of $f(t)$. For example, with the even sampling function

$$f(t)=\frac{2}{\pi}\frac{t_0^3}{(t^2+t_0^2)^2} \tag{16}$$

that is peaked at $t=0$, we have

$$|\widehat{f^{1/2}}(\omega)|^2 = 2\pi t_0 e^{-2|\omega| t_0}. \tag{17}$$

In this case, the quantum inequality can be expressed in terms of the Euclidean Green's function

$$G_E(t,\boldsymbol{x};t',\boldsymbol{x}')=\sum_\lambda U_\lambda^*(\boldsymbol{x})U_\lambda(\boldsymbol{x}')e^{\omega_\lambda(t-t')} \tag{18}$$

quite compactly as

$$\rho \geqslant -\tfrac{1}{4}\Box_E G_E(-t_0,\boldsymbol{x};t_0,\boldsymbol{x}), \tag{19}$$

where $\Box_E \equiv (1/|g_{tt}|)\partial_{t_0}^2 + \nabla^i \nabla_i$ is the Euclidean wave operator. This bound is, in fact, identical to one that was derived in [12,13] assuming the Lorentzian sampling function (1). But because Eq. (16) is a more sharply peaked function [half the area under the Lorentzian function lies within $|t|<t_0$, while this figure is $\frac{1}{2}+1/\pi \simeq 0.82$ for Eq. (16)], this is a first indication that the inequality derived here is a stronger result.

Finally, we record the fact that for the Lorentzian function,

$$|\widehat{f^{1/2}}(\omega)|^2 = \frac{4t_0}{\pi} K_0(t_0|\omega|)^2, \tag{20}$$

where $K_0(x)$ is the modified Bessel function of zeroth order. In the rest of this paper, we shall consider the quantum inequality in specific examples of globally static space-times where the left-hand side of Eqs. (14) or (15) can be explicitly evaluated. As these examples have been considered previously by Pfenning and Ford [12,13,15], we shall at times be brief and refer to their papers for more details. For the most part we will closely follow their notation and conventions.

III. MINKOWSKI SPACE

We begin with a review of the quantum inequality in $(n+1)$-dimensional Minkowski space, the case that was treated in [14]. The mode functions for a free scalar field of mass μ are

$$U_{\boldsymbol{k}}(\boldsymbol{x}) = \frac{1}{[(2\pi)^n 2\omega_{\boldsymbol{k}}]^{1/2}} e^{i\boldsymbol{k}\cdot\boldsymbol{x}}, \quad \omega_{\boldsymbol{k}} = \sqrt{|\boldsymbol{k}|^2+\mu^2}, \tag{21}$$

with each component of the n-dimensional (spatial) momentum covector $\boldsymbol{k}$ satisfying $-\infty<k_i<\infty$. The quantum inequality (14) becomes

$$\begin{aligned}\rho &\geqslant -\frac{1}{2\pi}\int_0^\infty d\omega \int \frac{d^n\boldsymbol{k}}{(2\pi)^n}\omega_{\boldsymbol{k}} |\widehat{f^{1/2}}(\omega+\omega_{\boldsymbol{k}})|^2 \\ &= -\frac{C_n}{2\pi}\int_0^\infty d\omega \int_\mu^\infty d\omega'\, \omega'^2 (\omega'^2-\mu^2)^{n/2-1} \\ &\quad \times |\widehat{f^{1/2}}(\omega+\omega')|^2,\end{aligned} \tag{22}$$

where C_n is equal to the area of the unit $(n-1)$-sphere divided by $(2\pi)^n$, i.e.,

$$C_n \equiv \frac{1}{2^{n-1}\pi^{n/2}\Gamma(\frac{1}{2}n)}. \tag{23}$$

If we make the change of variables $u=\omega+\omega'$ and $v=\omega'$, the quantum inequality (22) can be rewritten as

$$\rho \geqslant -\frac{C_n}{2\pi(n+1)}\int_\mu^\infty du\, |\widehat{f^{1/2}}(u)|^2 u^{n+1} Q_n\left(\frac{u}{\mu}\right), \tag{24}$$

where the functions $Q_n(x)$ are defined by

$$Q_n(x) = (n+1)x^{-(n+1)}\int_1^x dy\, y^2 (y^2-1)^{n/2-1}. \tag{25}$$

There are several special cases in which this bound can be evaluated analytically [14], notably massless fields in two and four dimensions with the sampling function (1). In the former, the bound is four times stronger than that derived by Ford and Roman [11], but $1\frac{1}{2}$ times weaker than the optimal one of Flanagan [16]. In the latter case, the present bound is $\frac{9}{64}$ of Ford and Roman's result.

IV. THREE-DIMENSIONAL CLOSED UNIVERSE

The line element for the static, three-dimensional closed universe is

$$ds^2 = -dt^2 + a^2(d\theta^2 + \sin^2\theta\, d\varphi^2), \tag{26}$$

where a is the radius of the two-sphere at each constant time-slice, and the angular variables take values $0 \leqslant \theta \leqslant \pi$, $0 \leqslant \varphi < 2\pi$ (and will do so for all the space-times considered in this paper). We consider the massive scalar field equation on this background with a coupling of strength ξ to the scalar curvature $R = 2/a^2$:

$$\Box\phi - (\mu^2 + \xi R)\phi = 0, \tag{27}$$

whose mode-function solutions are given in terms of the usual spherical harmonics $Y_{lm}(\theta,\varphi)$ by [13,15]

$$U_{lm}(\boldsymbol{x}) = \frac{1}{(2\omega_l a^2)^{1/2}} Y_{lm}(\theta,\varphi), \tag{28}$$

for $l = 0,1,2,\ldots$ and $m = -l, -l+1, \ldots, l$, with

$$\omega_l = a^{-1}\sqrt{l(l+1) + 2\xi + (a\mu)^2}. \tag{29}$$

The $Y_{lm}(\theta,\varphi)$ obey the sum rule

$$\sum_{m=-l}^{l} |Y_{lm}(\theta,\varphi)|^2 = \frac{2l+1}{4\pi}, \tag{30}$$

which can be used in Eqs. (14) and (15) to show that

$$\begin{aligned}\rho \geqslant & -\frac{1}{8\pi^2 a^2}\int_0^\infty d\omega \sum_{l=0}^{\infty} (2l+1)\omega_l |\widehat{f^{1/2}}(\omega+\omega_l)|^2 \\ & = -\frac{1}{8\pi^2 a^2}\int_{\omega_0}^\infty du\, |\widehat{f^{1/2}}(u)|^2 \sum_{l=0}^{N(u)} (2l+1)\omega_l .\end{aligned} \tag{31}$$

Here, $N(u) \equiv \max\{n \in \mathbb{Z} : \omega_n \leqslant u\}$, i.e.,

$$N(u) = \left\lfloor \frac{\sqrt{1 - 4[2\xi + (a\mu)^2 - (au)^2]} - 1}{2} \right\rfloor, \tag{32}$$

where $\lfloor x \rfloor$ denotes the integer part of x.

While the bound in Eq. (31) can be readily evaluated using numerical techniques, it may be worthwhile to first simplify it analytically as much as possible. This may be useful if one should want to draw conclusions about its general properties. In particular, we shall present a general strategy for approximating finite summations like that in Eq. (31).

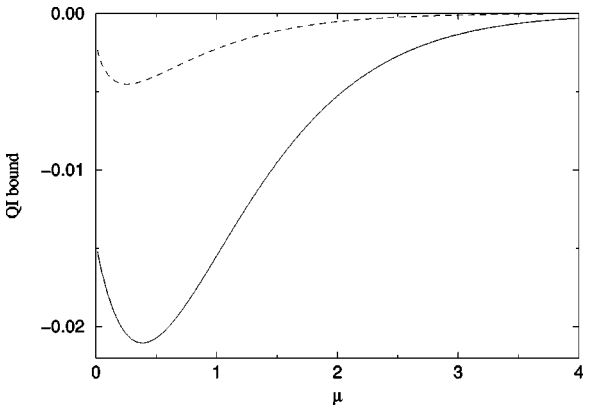

FIG. 1. Graph of the QI bound for the 3D closed universe (dashed line), and that obtained by Pfenning and Ford (solid line), against μ.

The summation in Eq. (31) can be evaluated using the trapezoidal rule of numerical integration [e.g., see Eq. (3.6.1) of [17]]:

$$\sum_{n=0}^{N} g(n) = \int_0^N dx\, g(x) + \frac{1}{2}[g(0) + g(N)] + \frac{N}{12} g''(\zeta) \tag{33}$$

for some $\zeta \in (0,N)$. In the present case, $g(x) = (2x+1)\sqrt{x(x+1) + 2\xi + (a\mu)^2}$, and the integral in Eq. (33) can be evaluated analytically. Furthermore, $g''(\zeta)$ is non-decreasing in the interval in question, so its occurrence in Eq. (31) can be replaced by $g''(N)$, at the expense of weakening the bound slightly. We obtain the final inequality

$$\begin{aligned}\rho \geqslant & -\frac{1}{8\pi^2 a^3}\int_{\omega_0}^\infty du\, |\widehat{f^{1/2}}(u)|^2 \Bigg\{ \int_0^{N(u)} dx\, g(x) \\ & + \frac{1}{2}[g(0) + g(N(u))] + \frac{N(u)}{12} g''(N(u)) \Bigg\},\end{aligned} \tag{34}$$

with

$$\int dx\, g(x) = \frac{2}{3}[x(x+1) + 2\xi + (a\mu)^2]^{3/2},$$

$$\begin{aligned}g''(x) = & \frac{3(2x+1)}{\sqrt{x(x+1) + 2\xi + (a\mu)^2}} \\ & - \frac{1}{4}\frac{(2x+1)^3}{[x(x+1) + 2\xi + (a\mu)^2]^{3/2}}.\end{aligned} \tag{35}$$

The graph of the bound in Eq. (31) is plotted against mass in Fig. 1, for $a = 1$ and $\xi = 0$. As usual, the sampling function $f(t)$ is taken to be the Lorentzian function (1), with $t_0 = 1$. When plotted on the same scale, that of Eq. (34) is almost indistinguishable from the former graph. For comparison, the

corresponding bound obtained by Pfenning and Ford [13,15] is also plotted in Fig. 1. It is clear that our bound is stronger for all values of mass.

V. FOUR-DIMENSIONAL ROBERTSON-WALKER UNIVERSE

We shall first consider the case of the open universe, before proceeding to the closed universe. The line element is

$$ds^2 = -dt^2 + a^2[d\chi^2 + \sinh^2\chi(d\theta^2 + \sin^2\theta d\varphi^2)], \quad (36)$$

where a characterizes the scale of the spatial section, and $0 \leqslant \chi < \infty$. The mode functions for a scalar field of mass μ are [18]

$$U_{qlm}(\boldsymbol{x}) = \frac{1}{(2\omega_q a^3)^{1/2}} \Pi_{ql}^{(-)}(\chi) Y_{lm}(\theta,\varphi),$$

$$\omega_q = \sqrt{\frac{q^2+1}{a^2} + \mu^2}, \quad (37)$$

with $0 < q < \infty$ and l,m as usual. The functions $\Pi_{ql}^{-}(\chi)$ satisfy

$$\Pi_{ql}^{(-)}(\chi) \propto \sinh^l\chi \left(\frac{d}{d\cosh\chi}\right)^{l+1} \cos q\chi, \quad (38)$$

and obey the sum rule

$$\sum_{l,m} |\Pi_{ql}^{(-)}(\chi) Y_{lm}(\theta,\varphi)|^2 = \frac{q^2}{2\pi^2}. \quad (39)$$

The right-hand side does not depend on the angular variables, as is expected of a system with isotropic symmetry. Hence, the quantum inequality (14) becomes

$$\rho \geqslant -\frac{1}{4\pi^3 a^3} \int_0^\infty d\omega \int_0^\infty dq\, \omega_q q^2 |\widehat{f^{1/2}}(\omega + \omega_q)|^2. \quad (40)$$

Note that this bound is identical in form to that in (four-dimensional) Minkowski space. Both Eqs. (22) and (40) can, in fact, be written as

$$\rho \geqslant -\frac{1}{4\pi^3} \int_0^\infty d\omega \int_C d\omega'\, \omega' \sqrt{\omega'^2 - C^2} |\widehat{f^{1/2}}(\omega + \omega')|^2$$

$$= -\frac{1}{16\pi^3} \int_C^\infty du\, |\widehat{f^{1/2}}(u)|^2 u^4 Q_3\left(\frac{u}{C}\right), \quad (41)$$

where

$$C \equiv \sqrt{\frac{\epsilon}{a^2} + \mu^2}, \quad \epsilon = \begin{cases} 0 & \text{Minkowski space;} \\ 1 & \text{open universe,} \end{cases} \quad (42)$$

and an explicit expression (and graph) for $Q_3(x)$ can be found in [14]. The Minkowski space result is obviously recovered in the limit of infinite a. Furthermore, since $Q_3(x)$ is an increasing function on $[1,\infty)$, the bound for general a is tighter than that in Minkowski space for all sampling functions $f(t)$. Pfenning and Ford [13,15] also noted this for their particular choice of $f(t)$.

We now turn to the closed or Einstein universe, with line element

$$ds^2 = -dt^2 + a^2[d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2\theta d\varphi^2)], \quad (43)$$

where $0 \leqslant \chi \leqslant \pi$. The mode functions are [18]

$$U_{nlm}(\boldsymbol{x}) = \frac{1}{(2\omega_n a^3)^{1/2}} \Pi_{nl}^{(+)}(\chi) Y_{lm}(\theta,\varphi),$$

$$\omega_n = \sqrt{\frac{n(n+2)}{a^2} + \mu^2}, \quad (44)$$

with $n = 0,1,2,\ldots$, $l = 0,1,\ldots,n$, $m = -l, -l+1 \ldots, l$, and

$$\Pi_{nl}^{(+)}(\chi) \propto \sin^l\chi \left(\frac{d}{d\cos\chi}\right)^{l+1} \cosh(n+1)\chi. \quad (45)$$

Using the sum rule

$$\sum_{l,m} |\Pi_{nl}^{(+)}(\chi) Y_{lm}(\theta,\varphi)|^2 = \frac{(n+1)^2}{2\pi^2}, \quad (46)$$

we obtain the quantum inequality

$$\rho \geqslant -\frac{1}{4\pi^3 a^3} \int_0^\infty d\omega \sum_{n=0}^\infty \omega_n (n+1)^2 |\widehat{f^{1/2}}(\omega + \omega_n)|^2$$

$$= -\frac{1}{4\pi^3 a^3} \int_\mu^\infty du |\widehat{f^{1/2}}(u)|^2 \sum_{n=0}^{N(u)} \omega_n (n+1)^2, \quad (47)$$

with

$$N(u) \equiv \lfloor \sqrt{(au)^2 - (a\mu)^2 + 1} - 1 \rfloor. \quad (48)$$

An obvious special case to investigate is $a\mu = 1$, in which $\omega_n = \mu(n+1)$. The sum in Eq. (47) may then be evaluated exactly, to give

$$\rho \geqslant -\frac{1}{16\pi^3 a^4} \int_\mu^\infty du |\widehat{f^{1/2}}(u)|^2 (N(u)+1)^2 (N(u)+2)^2. \quad (49)$$

This bound may be weakened slightly, by replacing $N(u) = \lfloor au - 1 \rfloor$ with the larger quantity $au - 1$, to give

$$\rho \geqslant -\frac{1}{16\pi^3} \int_\mu^\infty du |\widehat{f^{1/2}}(u)|^2 \left(u^4 + \frac{2u^3}{a} + \frac{u^2}{a^2}\right). \quad (50)$$

It clearly differs from the massless Minkowski bound by $O(1/a)$ terms. The bounds in Eqs. (49) and (50) are plotted in Fig. 2 against mass. The difference between these two graphs can be further minimized using the approximations below, but at the expense of having a more complicated ex-

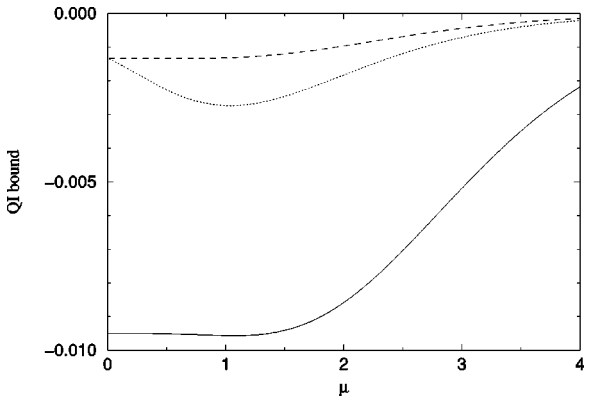

FIG. 2. Graphs of the QI bounds for the 4D closed universe with $a\mu=1$: Eq. (49) using a dashed line, Eq. (50) using a dotted line, and that obtained by Pfenning and Ford (solid line).

pression for the bound. The corresponding bound derived in [13,15] is also plotted on the same graph.

Returning to the general case, we note that Eq. (47) can be written as

$$\rho \geqslant -\frac{1}{4\pi^3 a^4}\int_\mu^\infty du\,|\widehat{f^{1/2}}(u)|^2 \sum_{n=1}^{N'} n^2\sqrt{n^2+(a\mu)^2-1}, \tag{51}$$

where $N' \equiv \lfloor a\sqrt{u^2-\mu^2}\rfloor$. The finite sum can again be approximated analytically using the trapezoidal rule, now in the form

$$\sum_{n=1}^{N} g(n) = \int_1^N dx\, g(x) + \frac{1}{2}[g(1)+g(N)] + \frac{N-1}{12}g''(\zeta) \tag{52}$$

for some $\zeta \in (1,N)$. From the fact that the second derivative of $g(x)=x^2\sqrt{x^2+(a\mu)^2-1}$ is non-decreasing in this interval, we obtain the inequality

$$\rho \geqslant -\frac{1}{4\pi^3 a^4}\int_\mu^\infty du\,|\widehat{f^{1/2}}(u)|^2 \left\{ \int_1^{N'} dx\, g(x) + \frac{1}{2}[g(1)+g(N')] + \frac{N'-1}{12}g''(N') \right\}, \tag{53}$$

with

$$\int dx\, g(x) = \frac{1}{4}x[x^2+(a\mu)^2-1]^{3/2} - \frac{1}{8}[(a\mu)^2-1]x\sqrt{x^2+(a\mu)^2-1} - \frac{1}{8}[(a\mu)^2-1]^2 \ln\left(x+\sqrt{x^2+(a\mu)^2-1}\right),$$

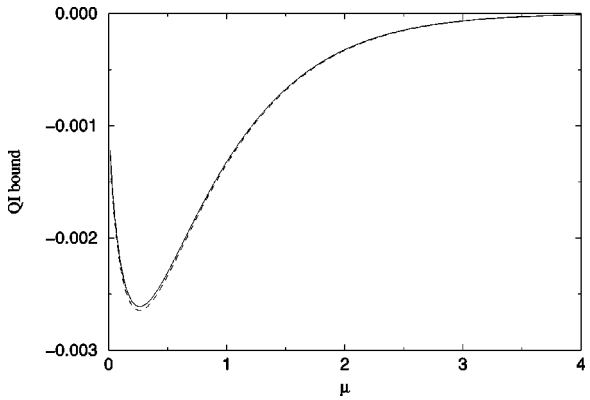

FIG. 3. Graphs of the QI bounds for the 4D closed universe: Eq. (51) using a solid line, and its approximation (53) using a dashed line.

$$g''(x) = 2\sqrt{x^2+(a\mu)^2-1} + \frac{5x^2}{\sqrt{x^2+(a\mu)^2-1}} - \frac{x^4}{[x^2+(a\mu)^2-1]^{3/2}}. \tag{54}$$

The bound in Eq. (51) and its approximation in Eq. (53) are plotted in Fig. 3 against μ, for $a=1$. As can be seen, the approximation is only very slightly weaker than the exact bound. Also plotted in Fig. 4 is the bound obtained in [13,15], for comparison with Eq. (51).

VI. de SITTER SPACE-TIME

A convenient static parametrization of the de Sitter universe is

$$ds^2 = -\left(1-\frac{r^2}{\alpha^2}\right)dt^2 + \left(1-\frac{r^2}{\alpha^2}\right)^{-1}dr^2 + r^2(d\theta^2+\sin^2\theta\, d\varphi^2), \tag{55}$$

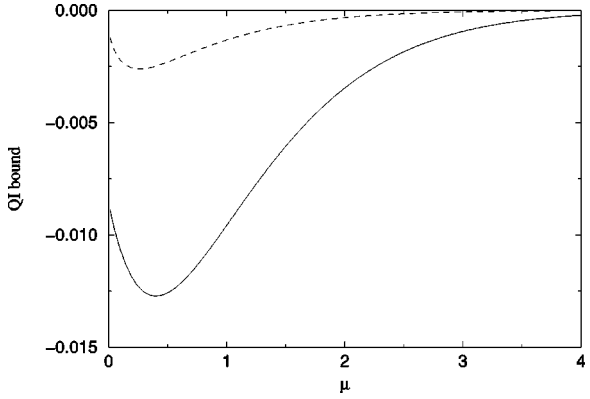

FIG. 4. Graphs of the QI bounds for the 4D closed universe: Eq. (51) using a dashed line, and that of Pfenning and Ford (solid line).

with $0 \leqslant r \leqslant \alpha$. The surface $r=\alpha$ is the particle horizon for an observer located at the origin. In this representation, the mode functions for a scalar field with mass μ and energy ω are

$$U_{klm}(\boldsymbol{x}) = \frac{1}{(4\pi\alpha^2 k)^{1/2}} f_{kl}(z) Y_{lm}(\theta,\varphi), \tag{56}$$

where we denote $z \equiv r/\alpha$ and $k \equiv \alpha\omega$. The latter continuously parametrizes the mode function from zero to infinity, while l and m are as in Sec. IV. The radial function can then be solved in terms of the hypergeometric function $F(a,b;c;z)$ as [19]

$$f_{kl}(z) = \frac{\Gamma(b_l^+)\Gamma(b_l^-)}{\Gamma(l+\frac{3}{2})\Gamma(ik)} z^l (1-z^2)^{ik/2} F\left(b_l^+, b_l^-; l+\frac{3}{2}; z^2\right), \tag{57}$$

with

$$b_l^{\pm} \equiv \frac{1}{2}\left(l+\frac{3}{2}+ik \pm \sqrt{\frac{9}{4}-\alpha^2\mu^2}\right). \tag{58}$$

Using the sum rule (30) in the quantum inequality (14), we have for an observer at the origin,

$$\begin{aligned}\rho \geqslant & -\frac{1}{64\pi^3\alpha^4}\int_0^\infty d\omega \int_0^\infty dk \sum_{l=0}^\infty \frac{2l+1}{k}\left|\frac{\Gamma(b_l^+)\Gamma(b_l^-)}{\Gamma(l+\frac{3}{2})\Gamma(ik)}\right|^2 \\ & \times \lim_{z\to 0}\left\{\frac{4k^2}{1-z^2}+\frac{1}{z^2}\partial_z[z^2(1-z^2)\partial_z]\right\} z^{2l} \\ & \times \left|F\left(b_l^+, b_l^-; l+\frac{3}{2}; z^2\right)\right|^2 |\widehat{f^{1/2}}(\omega + k/\alpha)|^2. \end{aligned} \tag{59}$$

In fact, only the $l=0$ and $l=1$ terms contribute [cf. Eqs. (4.126) and (4.127) of [13]], and the expression may be simplified to give

$$\begin{aligned}\rho \geqslant & -\frac{1}{8\pi^5\alpha^4}\int_0^\infty d\omega \int_0^\infty dk \sinh(\pi k) \\ & \times \{(k^2+\alpha^2\mu^2)|\Gamma(b_0^+)\Gamma(b_0^-)|^2 + 4|\Gamma(b_1^+)\Gamma(b_1^-)|^2\} \\ & \times |\widehat{f^{1/2}}(\omega + k/\alpha)|^2. \end{aligned} \tag{60}$$

As was noted in [12,13], there are two cases for which the gamma functions in Eq. (60) can be evaluated analytically, namely when $\mu=0$ and $\sqrt{2}/\alpha$. Assuming the Lorentzian sampling function (1) and using Eq. (20), we obtain, for the massless case,

$$\begin{aligned}\rho \geqslant & -\frac{t_0}{2\pi^4\alpha^2}\int_0^\infty d\omega \int_0^\infty d\omega' (5\omega' + 2\alpha^2\omega'^3) \\ & \times K_0(t_0(\omega+\omega'))^2. \end{aligned} \tag{61}$$

Defining the new variables $u=\omega+\omega'$ and $v=\omega'$, the bound becomes

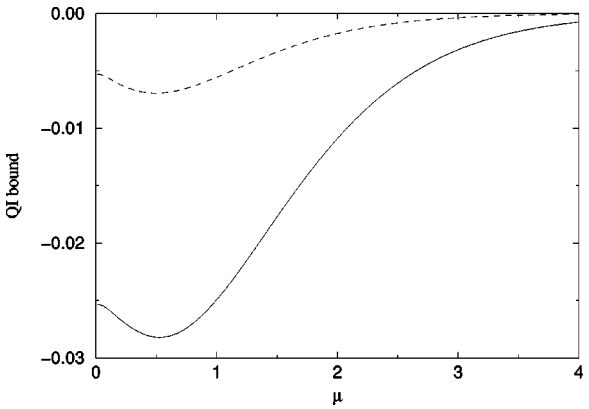

FIG. 5. Graphs of the QI bounds for de Sitter space-time: Eq. (60) using a dashed line, and that of Pfenning and Ford (solid line).

$$-\frac{t_0}{2\pi^4\alpha^2}\int_0^\infty du\, K_0(t_0 u)^2 \int_0^u dv (5v + 2\alpha^2 v^3). \tag{62}$$

This can be explicitly evaluated using the integral

$$\int_0^\infty du\, u^{\alpha-1} K_0(t_0 u)^2 = \frac{2^{\alpha-3}}{t_0^\alpha \Gamma(\alpha)}\Gamma\left(\frac{\alpha}{2}\right)^4 \tag{63}$$

to obtain

$$\rho \geqslant -\frac{3}{32\pi^2 t_0^4}\frac{9}{64}\left[1+\frac{16}{9}\frac{5}{3}\left(\frac{t_0}{\alpha}\right)^2\right]. \tag{64}$$

This bound is at least four times stronger than that obtained in [12,13]:

$$\rho \geqslant -\frac{3}{32\pi^2 t_0^4}\left[1+\frac{5}{3}\left(\frac{t_0}{\alpha}\right)^2\right]. \tag{65}$$

In the limit $\alpha\to\infty$ or $t_0\to 0$, we expect to recover the results for Minkowski space. Indeed, the bound in Eq. (64) is then $\frac{9}{64}$ that in Eq. (65), as was observed in [14].

When $\mu=\sqrt{2}/\alpha$, we similarly obtain the quantum inequality

$$\rho \geqslant -\frac{3}{32\pi^2 t_0^4}\frac{9}{64}\left[1+\frac{16}{9}\left(\frac{t_0}{\alpha}\right)^2\right], \tag{66}$$

in contrast to that derived in [12,13]:

$$\rho \geqslant -\frac{3}{32\pi^2 t_0^4}\left[1+\left(\frac{t_0}{\alpha}\right)^2\right]. \tag{67}$$

The bound in Eq. (60) and that derived in [12,13] are plotted for general μ, and $\alpha=1$, in Fig. 5.

We have, in fact, proved that for general μ, the de Sitter bound (60) differs from the Minkowski space bound (24) by terms no greater than order $\alpha^{-1/2}$ as $\alpha\to\infty$, and so our results for these cases agree in this limit. This estimate involves bounds on the integrand in Eq. (60) which are uniform in k and ω. The proof, which we omit, is accordingly somewhat technical. It is unclear whether the argument can

be strengthened to show that the deviation is in fact $O(\alpha^{-2})$ in general, as it is for the specific cases considered in Eqs. (64) and (66).

VII. SCHWARZSCHILD SPACE-TIME

As the final example, we shall examine the quantum inequalities in a black hole space-time. The line element for the Schwarzschild black hole of mass M is

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \left(1-\frac{2M}{r}\right)^{-1}dr^2 + r^2(d\theta^2+\sin^2\theta\, d\varphi^2). \tag{68}$$

For simplicity, we shall only consider a massless scalar field in this space-time. The mode functions, in the region exterior to the horizon $r>2M$, take the form [20]

$$\vec{U}_{\omega lm}(\boldsymbol{x}) = \frac{1}{(4\pi\omega)^{1/2}}\vec{R}_l(\omega|r)Y_{lm}(\theta,\varphi),$$

$$\overleftarrow{U}_{\omega lm}(\boldsymbol{x}) = \frac{1}{(4\pi\omega)^{1/2}}\overleftarrow{R}_l(\omega|r)Y_{lm}(\theta,\varphi), \tag{69}$$

where, as usual, ω is the energy of the field and $Y_{lm}(\theta,\varphi)$ are the spherical harmonics. $\vec{R}_l(\omega|r)$ and $\overleftarrow{R}_l(\omega|r)$ are the outgoing and ingoing solutions to the radial part of the wave equation, respectively. Although this equation cannot be solved analytically, the asymptotic forms of the solutions are known near the horizon and at infinity.

Again, using the sum rule (30), we see that the quantum inequality (14) becomes

$$\rho \geqslant -\frac{1}{16\pi^3}\int_0^\infty d\omega \int_0^\infty \frac{d\omega'}{\omega'}\sum_{l=0}^\infty (2l+1) \times \left\{ \frac{\omega'^2}{1-\frac{2M}{r}} + \frac{1}{4r^2}\partial_r[r^2(1-2M/r)\partial_r] \right\} \times [|\vec{R}_l(\omega'|r)|^2 + |\overleftarrow{R}_l(\omega'|r)|^2]|\widehat{f^{1/2}}(\omega+\omega')|^2. \tag{70}$$

In writing this, we are assuming that the mode functions are defined to have positive frequency with respect to the time-like Killing vector ∂_t. This is the Boulware vacuum. Now, in the two regions where $\vec{R}_l(\omega|r)$ and $\overleftarrow{R}_l(\omega|r)$ are known explicitly, we have [21]

$$\sum_{l=0}^\infty (2l+1)|\vec{R}_l(\omega|r)|^2 \simeq \begin{cases} 4\omega^2(1-2M/r)^{-1}, & r\to 2M, \\ \dfrac{1}{r^2}\displaystyle\sum_{l=0}^\infty (2l+1)|B_l(\omega)|^2, & r\to\infty, \end{cases} \tag{71}$$

and

$$\sum_{l=0}^\infty (2l+1)|\overleftarrow{R}_l(\omega|r)|^2 \simeq \begin{cases} \dfrac{1}{4M^2}\displaystyle\sum_{l=0}^\infty (2l+1)|B_l(\omega)|^2, & r\to 2M, \\ 4\omega^2, & r\to\infty. \end{cases} \tag{72}$$

If we further assume the low-energy condition $2M\omega \ll 1$, then [22]

$$B_l(\omega) \simeq \frac{(l!)^3}{(2l+1)!(2l)!}(-4iM\omega)^{l+1}. \tag{73}$$

These results can be substituted into Eq. (70), and the bound explicitly evaluated using the integral (63). However, the maximum value of l for which the expansion in Eq. (70) is valid depends on the order of the leading terms which have been dropped in $B_l(\omega)$. If Eq. (73) is exact to $O[(M\omega)^{l+2}]$, then only the $l=0$ terms should be retained, as the B_1 contribution would be smaller than the corrections to B_0 [12,13].

Near the horizon, the quantum inequality can be expressed in terms of the observer's proper time:

$$\tau_0 = \left(1-\frac{2M}{r}\right)^{1/2} t_0 \tag{74}$$

as

$$\rho \geqslant -\frac{3}{32\pi^2\tau_0^4}\left\{\frac{1}{24}\left(\frac{2M\tau_0}{r^2}\right)^2\left(1-\frac{2M}{r}\right)^{-1} + \frac{9}{64}\left[1+\left(1-\frac{2M}{r}\right)\right]+\cdots\right\}, \tag{75}$$

where the ellipsis denotes higher-order terms that have been dropped. This is to be compared with the result derived in [12,13]:

$$\rho \geqslant -\frac{3}{32\pi^2\tau_0^4}\left\{\frac{1}{6}\left(\frac{2M\tau_0}{r^2}\right)^2\left(1-\frac{2M}{r}\right)^{-1} + 1 + \left(1-\frac{2M}{r}\right)+\cdots\right\}. \tag{76}$$

The bound in Eq. (75) is between $\frac{9}{64}$ and $\frac{1}{4}$ that in Eq. (76), at least in the present approximation. Note that in either case, the bound becomes arbitrarily negative near the horizon of the black hole.

On the other hand, the quantum inequality for an observer near infinity becomes

$$\rho \geqslant -\frac{3}{32\pi^2\tau_0^4}\frac{9}{64}\left\{1-\frac{2M}{r}+\left(\frac{2M}{r}\right)^2\left[1+\frac{16}{9}\frac{1}{3}\left(\frac{\tau_0}{r}\right)^2\right]-\left(\frac{2M}{r}\right)^3\left[1+\frac{16}{9}\left(\frac{\tau_0}{r}\right)^2\right]+\cdots\right\}, \tag{77}$$

while the corresponding inequality obtained in [12,13] is

$$\rho \geqslant -\frac{3}{32\pi^2\tau_0^4}\left\{1-\frac{2M}{r}+\left(\frac{2M}{r}\right)^2\left[1+\frac{1}{3}\left(\frac{\tau_0}{r}\right)^2\right]-\left(\frac{2M}{r}\right)^3\left[1+\left(\frac{\tau_0}{r}\right)^2\right]+\cdots\right\}. \tag{78}$$

Again, the former bound is between $\frac{9}{64}$ and $\frac{1}{4}$ the latter. It gives the correct Minkowski space result in the limit $r\to\infty$ or $M\to 0$.

VIII. CONCLUDING REMARKS

In summary, we have derived new quantum inequalities (14) or (15) on the normal-ordered averaged energy density in static space-times, that are valid for quite general sampling functions. They were then applied to several standard examples using the Lorentzian sampling function. (Of course, other space-times could readily be considered, such as Rindler space, flat space with perfectly reflecting mirrors, and other black holes [12,13].) The resulting bounds are stronger than previous results, and would lead to even tighter constraints on the various exotic space-times mentioned at the beginning of the paper. Before we conclude, a few comments are in order.

An important question is whether our quantum inequalities are optimal. This could, for example, be proved by finding a quantum state that saturates the bound, which would necessarily belong to the kernel of all the operators $\mathcal{O}^{\pm}(\omega)$ in Eq. (A6). However, it is known that our bound, when applied to a massless scalar field in two-dimensional Minkowski space, is $1\frac{1}{2}$ times weaker than the optimal value obtained by Flanagan [16]. Unfortunately, his derivation relies on some special features of two-dimensional massless field theory, and does not appear to generalize to other more realistic cases.

An interesting application of our quantum inequality would be to the static Morris-Thorne–type wormholes [1]. Ford and Roman have applied the flat-space version of their quantum inequalities to this case, and have found that they constrain the size of such wormholes [5]. They justified this procedure by making the sampling time scale much shorter than the minimum characteristic curvature scale, so that space-time appears locally flat. However, it would be desirable to verify this calculation using the full curved space results; this should not be too difficult once the form of the scalar field mode functions in the wormhole space-time have been determined.

ACKNOWLEDGMENTS

C.J.F. thanks Simon Eveson for useful discussions concerning the use of the trapezoidal rule in Secs. IV and V.

APPENDIX

The inequality needed to prove the results in Sec. II is a generalization of one which was first derived in [14] using the convolution theorem. Suppose f is a smooth, nonnegative function of t, decaying at least as fast as $O(t^{-2})$ for $t\to\pm\infty$. Let operators $S^{\pm}$ be defined by

$$S^{\pm}=\mathrm{Herm}\sum_{\lambda,\lambda'}\{\hat{f}(\omega_{\lambda'}-\omega_\lambda)\overline{q_\lambda}q_{\lambda'}a_\lambda^\dagger a_{\lambda'} \pm \hat{f}(\omega_\lambda+\omega_{\lambda'})q_{\lambda'}q_\lambda a_{\lambda'}a_\lambda\}, \tag{A1}$$

where the q_λ are complex coefficients,[2] and $\mathrm{Herm}\ X\equiv\frac{1}{2}(X+X^\dagger)$ is the Hermitian part of an operator X. We will show that the expectation values $\langle S^{\pm}\rangle$ obey

$$\langle S^{\pm}\rangle \geqslant -\frac{1}{2\pi}\int_0^\infty d\omega\sum_\lambda |\widehat{f^{1/2}}(\omega+\omega_\lambda)|^2|q_\lambda|^2 \tag{A2}$$

in any normalized quantum state, where $f^{1/2}(t)\equiv\sqrt{f(t)}$ is the pointwise square-root of $f(t)$.

To obtain this result, first define a function g by

$$g(\omega)=\frac{1}{\sqrt{2\pi}}\widehat{f^{1/2}}(\omega). \tag{A3}$$

We have $\overline{g(\omega)}=g(-\omega)$ because $f^{1/2}$ is real, and

$$(g\star g)(\omega)=\hat{f}(\omega) \tag{A4}$$

by the convolution theorem, where $\star$ is given by

$$(g_1\star g_2)(\omega)=\int_{-\infty}^{\infty}d\omega' g_1(\omega-\omega')g_2(\omega'). \tag{A5}$$

Next, define the following operators on the space of quantum states:

$$\mathcal{O}^{\pm}(\omega)=\sum_\lambda\{\overline{g(\omega-\omega_\lambda)}q_\lambda a_\lambda \pm \overline{g(\omega+\omega_\lambda)q_\lambda}a_\lambda^\dagger\}. \tag{A6}$$

Using the canonical commutation relations (7) and symmetrizing in λ, λ', we obtain

[2]For clarity, we shall denote complex conjugation by an overline in this Appendix.

$$\int_0^\infty d\omega \mathcal{O}^\pm(\omega)^\dagger \mathcal{O}^\pm(\omega)$$

$$= \int_0^\infty d\omega \sum_{\lambda,\lambda'} \{ g(\omega-\omega_\lambda)\overline{g(\omega-\omega_{\lambda'})}\ \overline{q_\lambda} q_{\lambda'} a_\lambda^\dagger a_{\lambda'}$$
$$+ g(\omega+\omega_\lambda)\overline{g(\omega+\omega_{\lambda'})} q_\lambda \overline{q_{\lambda'}} a_\lambda a_{\lambda'}^\dagger$$
$$\pm g(\omega-\omega_\lambda)\overline{g(\omega+\omega_{\lambda'})}\ \overline{q_\lambda q_{\lambda'}} a_\lambda^\dagger a_{\lambda'}^\dagger$$
$$\pm g(\omega+\omega_\lambda)\overline{g(\omega-\omega_{\lambda'})} q_\lambda q_{\lambda'} a_\lambda a_{\lambda'} \}$$

$$= S^\pm + 1 \int_0^\infty d\omega \sum_\lambda |g(\omega+\omega_\lambda)|^2 |q_\lambda|^2, \qquad \text{(A7)}$$

where $S^\pm$ are given by

$$S^\pm = \text{Herm} \sum_{\lambda,\lambda'} \{ F(\omega_\lambda,\omega_{\lambda'})\overline{q_\lambda} q_{\lambda'} a_\lambda^\dagger a_{\lambda'}$$
$$\pm G(\omega_\lambda,\omega_{\lambda'}) q_{\lambda'} q_\lambda a_{\lambda'} a_\lambda \}, \qquad \text{(A8)}$$

and F, G are

$$F(\omega_\lambda,\omega_{\lambda'}) = \int_0^\infty d\omega \{ g(\omega-\omega_\lambda)\overline{g(\omega-\omega_{\lambda'})}$$
$$+ \overline{g(\omega+\omega_\lambda)} g(\omega+\omega_{\lambda'}) \}$$
$$= \int_0^\infty d\omega \{ g(\omega-\omega_\lambda) g(-\omega+\omega_{\lambda'})$$
$$+ g(-\omega-\omega_\lambda) g(\omega+\omega_{\lambda'}) \}$$
$$= \int_{-\infty}^\infty d\omega g(\omega_{\lambda'}-\omega) g(\omega-\omega_\lambda)$$
$$= (g \star g)(\omega_{\lambda'}-\omega_\lambda) = \hat{f}(\omega_{\lambda'}-\omega_\lambda), \qquad \text{(A9)}$$

$$G(\omega_\lambda,\omega_{\lambda'}) = \int_0^\infty d\omega \{ g(\omega+\omega_\lambda)\overline{g(\omega-\omega_{\lambda'})}$$
$$+ \overline{g(\omega-\omega_\lambda)} g(\omega+\omega_{\lambda'}) \}$$
$$= \int_0^\infty d\omega \{ g(\omega+\omega_\lambda) g(\omega_{\lambda'}-\omega)$$
$$+ g(\omega_\lambda-\omega) g(\omega+\omega_{\lambda'}) \}$$
$$= \int_{-\infty}^\infty d\omega\, g(\omega+\omega_\lambda) g(\omega_{\lambda'}-\omega)$$
$$= (g \star g)(\omega_\lambda+\omega_{\lambda'}) = \hat{f}(\omega_\lambda+\omega_{\lambda'}). \qquad \text{(A10)}$$

The final equalities show that Eq. (A8) agrees with the definition (A1).

Since the left-hand side of Eq. (A7) is manifestly nonnegative, we conclude that the expectation value of $S^\pm$ in any normalized quantum state must satisfy the inequality

$$\langle S^\pm \rangle \geqslant -\int_0^\infty d\omega \sum_\lambda |g(\omega+\omega_\lambda)|^2 |q_\lambda|^2, \qquad \text{(A11)}$$

which is the desired result (A2). Note that the inequality proved in [14] corresponds to the special case where the q_λ are real and f is an even function of t.

[1] M.S. Morris and K.S. Thorne, Am. J. Phys. **56**, 395 (1988).
[2] M.S. Morris, K.S. Thorne, and U. Yurtsever, Phys. Rev. Lett. **61**, 1446 (1988).
[3] M. Alcubierre, Class. Quantum Grav. **11**, L73 (1994).
[4] S.V. Krasnikov, Phys. Rev. D **57**, 4760 (1998).
[5] L.H. Ford and T.A. Roman, Phys. Rev. D **53**, 5496 (1996).
[6] M.J. Pfenning and L.H. Ford, Class. Quantum Grav. **14**, 1743 (1997).
[7] A.E. Everett and T.A. Roman, Phys. Rev. D **56**, 2100 (1997).
[8] K.D. Olum, Phys. Rev. Lett. **81**, 3567 (1998).
[9] H. Epstein, V. Glaser, and A. Jaffe, Nuovo Cimento **36**, 1016 (1965).
[10] L.H. Ford and T.A. Roman, Phys. Rev. D **51**, 4277 (1995).
[11] L.H. Ford and T.A. Roman, Phys. Rev. D **55**, 2082 (1997).
[12] M.J. Pfenning and L.H. Ford, Phys. Rev. D **57**, 3489 (1998).
[13] M.J. Pfenning, ''Quantum inequality restrictions on negative energy densities in curved spacetimes,'' Ph.D. thesis, gr-qc/9805037.
[14] C.J. Fewster and S.P. Eveson, Phys. Rev. D **58**, 084010 (1998).
[15] M.J. Pfenning and L.H. Ford, Phys. Rev. D **55**, 4813 (1997).
[16] E.E. Flanagan, Phys. Rev. D **56**, 4922 (1997).
[17] F.B. Hildebrand, *Introduction to Numerical Analysis*, 2nd ed. (Tata McGraw-Hill, New Delhi, 1974).
[18] L. Parker and S.A. Fulling, Phys. Rev. D **9**, 341 (1974).
[19] A. Higuchi, Class. Quantum Grav. **4**, 721 (1987).
[20] B.S. DeWitt, Phys. Rep. **19**, 295 (1975).
[21] P. Candelas, Phys. Rev. D **21**, 2185 (1980).
[22] B.P. Jensen, J.G. McLaughlin, and A.C. Ottewill, Phys. Rev. D **45**, 3002 (1992).