Spherically symmetric closed universe as an example of a 2D dilatonic model

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We study a two-dimensional dilatonic model describing a massless scalar field minimally coupled to the spherically reduced Einstein-Hilbert gravity. The general solution of this model is given in the case when a Killing vector is present. When interpreted in four dimensions, the solution describes either a static or a homogeneous collision of incoming and outgoing null dust streams with spherical symmetry. The homogeneous universe is closed. [S0556-2821(99)01710-5]

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I. INTRODUCTION

In a recent paper [1] we gave the general solution of the Einstein field equations describing the collision of spherically symmetric null dust streams when the resulting spacetime is static. For special parameter values the solution was found independently by Kramer [2], who obtained also static and stationary solutions describing the collision of cylindrical null dust beams [3]. The spherically symmetric static solution has the two dimensional (2D) interpretation [1] of a scalar field in minimal coupling with dilatonic gravity, which arises from the dimensional reduction of the Einstein-Hilbert action.

In this paper we present a general solution in the case when an additional Killing vector (not necessarily timelike) is present. This corresponds to the $so(3)\oplus R$ algebra in a four dimensional (4D) picture. The system is characterized by the action

$$S=\int d^2x\sqrt{-g}\rho\left[\mathcal{R}[g]+\frac{1}{2}g^{\alpha\beta}\nabla_\alpha \ln\rho\nabla_\beta \ln\rho+\frac{2}{\rho}\right]-\frac{1}{2}\int d^2x\sqrt{-g}g^{\alpha\beta}\nabla_\alpha\varphi\nabla_\beta\varphi. \tag{1}$$

Here ρ is the dilaton, $g_{\alpha\beta}$ the two-metric, $\mathcal{R}$ the Ricci scalar of the $g_{\alpha\beta}$-compatible two-connection ∇ and φ the scalar field. We obtain the dynamical equations in the following way. First we pass to a conformal metric $\eta_{\alpha\beta}=h^{-1}g_{\alpha\beta}$ and vary the action (1) with respect to ρ, $\eta^{\alpha\beta}$ and φ. (The variation with respect to h gives nothing new but the trace of the equation obtained by varying with respect to $\eta^{\alpha\beta}$). Then we impose the flatness of the metric $\eta_{\alpha\beta}$ and choose null coordinates $x^\pm$. In this way the 4D line element takes the form $ds^2=-h(x^+,x^-)dx^+dx^-+\rho(x^+,x^-)d\Omega^2$ and the equations are

$$\delta\varphi:\quad \varphi_{,+-}=0 \tag{2}$$

$$\delta\eta^{++}:\quad \rho_{,++}-\rho_{,+}(\ln\sigma)_{,+}=-\frac{1}{2}(\varphi_{,+})^2 \tag{3}$$

$$\delta\eta^{--}:\quad \rho_{,--}-\rho_{,-}(\ln\sigma)_{,-}=-\frac{1}{2}(\varphi_{,-})^2 \tag{4}$$

$$\delta\eta^{+-}:\quad \rho_{,+-}+\frac{1}{2}\rho^{-1/2}\sigma=0 \tag{5}$$

$$\delta\rho:\quad (\ln\sigma)_{,+-}-\frac{1}{4}\rho^{-3/2}\sigma=0. \tag{6}$$

Here commas denote derivatives and we have introduced the notation $\sigma=h\rho^{1/2}$. In deriving Eq. (6) the trace of the $\delta\eta^{\alpha\beta}$ equation was employed. Inserting $\rho_{,+-}$ from Eq. (5) in the ∂_- derivative of Eq. (3), we find Eq. (6). This interdependence of the equations is not surprising, as there are Bianchi identities to be satisfied. The wave equation (2) leaves us with the D'Alembert solution $\varphi=\varphi^+(x^+)+\varphi^-(x^-)$, a sum of left- and right-mover fields. Before proceeding to solve the problem in the case when a symmetry is present, we review how the solution emerges in the cases when no or only one component of the scalar field is present.

II. VACUUM SOLUTION

It is easy to solve the remaining three equations (3)–(5) in the vacuum case $\varphi=0$. After dividing by $\rho_{,+}$ and $\rho_{,-}$ respectively, Eqs. (3) and (4) can immediately be integrated to obtain

$$\rho_{,\pm}=H^\mp\sigma, \tag{7}$$

where $H^\mp(x^\mp)$ are arbitrary integration functions depending only on one coordinate. Next we eliminate σ from the above two equations and Eq. (5). The resulting system can be integrated once more, finding

$$H^\pm\rho_{,\pm}=2m^\pm-\rho^{1/2}, \tag{8}$$

where $m^\pm(x^\pm)$ form a second set of integration functions. A comparison with Eq. (7) however leaves us with $m^\pm=M$ $=$const and the algebraic relation

$$\sigma=\frac{2M-\rho^{1/2}}{H^+H^-}. \tag{9}$$

Inserting this expression of σ in Eq. (7) an integration yields $2M\ln|2M-\rho^{1/2}|+\rho^{1/2}=K^\pm-F^\mp$, where $F^\mp(x^\mp)$ $=\int^{x^\mp}d\tilde{x}^\mp/2H^\mp(\tilde{x}^\mp)$ and $K^\pm(x^\pm)$ are integration functions, which can be again eliminated by comparing the right hand sides: $K^{\pm}+F^{\pm}=2A=$const. Thus a second algebraic relation between ρ and σ has emerged:

$$|2M-\rho^{1/2}|\exp\left(\frac{\rho^{1/2}}{2M}\right)=\exp\left(\frac{2A-F^{+}-F^{-}}{2M}\right). \qquad (10)$$

Our choice of the null coordinates $x^{\pm}$ is not unique. They can be changed by a coordinate transformation belonging to the conformal subgroup of diffeomorphisms to $\hat{x}^{\pm}=\chi^{\pm}(x^{\pm})$. Under such a transformation the variable ρ remains unchanged while σ transforms as $\hat{\sigma}=\sigma/(\chi^{+})_{,+}(\chi^{-})_{,-}$. By choosing the new null coordinates defined by the differential equation $4MH^{\pm}(\chi^{\pm})_{,\pm}+\chi^{\pm}=0$ and fixing the integration constants in a convenient way (to annihilate the constant A), Eqs. (9), (10) become

$$\sigma=\frac{16M^{2}(2M-\rho^{1/2})}{\hat{x}^{+}\hat{x}^{-}}$$

$$(2M-\rho^{1/2})\exp\left(\frac{\rho^{1/2}}{2M}\right)=2|M|\hat{x}^{+}\hat{x}^{-}. \qquad (11)$$

The remaining freedom in the choice of the null coordinates is to scale one of them by a constant and the other one by the reciprocal of this constant. In the second equation of Eqs. (11) we have used that the expressions $\hat{x}^{+}\hat{x}^{-}$ and $2M-\rho^{1/2}$ have the same sign, as can be seen from the positiveness of σ. Inserting $\rho^{1/2}=R$ and $\sigma=hR$ we obtain the Schwarzschild solution with mass M (positive or negative) with curvature coordinate R and conformal factor h written in terms of the Kruskal coordinates. A similar derivation was given by Synge [4].

For the value $M=0$ of the first integration constant the derivation has to be slightly modified. Then in place of Eq. (10) we have $\rho^{1/2}=2A-F^{+}-F^{-}$ and the new null coordinates $\hat{x}^{\pm}=\pm 2(A-F^{\pm})$ are chosen. The solution is the flat space-time

$$h=1, \quad \hat{x}^{+}-\hat{x}^{-}=2R. \qquad (12)$$

III. CHIRAL SOLUTION

The chiral case, when only one component of the scalar field is present, has been solved for a quite general class of dilatonic Lagrangians, obtaining generalized Vaidya solutions [5]. We illustrate how the solution emerges for the system (1) in the case of the left-mover field $\varphi=\varphi^{+}$ (thus $\varphi^{-}=0$). Then Eq. (4) can be integrated, obtaining Eq. (7) containing $\rho_{,-}$. By inserting σ from Eq. (5) and integrating as in the vacuum case we find the $(+)$ equation of Eqs. (8). Inserting this into Eq. (3) and employing again Eq. (5) a first order differential equation for the function m^{+} emerges, with the solution

$$m^{+}=M-\frac{1}{4}\int dx^{+}H^{+}(\varphi_{,+})^{2}. \qquad (13)$$

We define a convenient null coordinate V by $dV=-dx^{+}/H^{+}$. (This is related to the Kruskal coordinate introduced in the vacuum case by $dV=4Md\ln\hat{x}^{+}$.) Expressed as a function of V, the integration function m^{+} becomes

$$m^{+}=M+\frac{1}{4}\int dV(\varphi_{,V})^{2}. \qquad (14)$$

In terms of the curvature coordinate R and conformal factor $h'=-H^{+}h$, the remaining field equations [the $\rho_{,-}$ equation from Eqs. (7) and the $\rho_{,+}$ equation from Eqs. (8)] give

$$2R_{,-}=-h', \quad 2R_{,V}=1-\frac{2m^{+}(V)}{R}. \qquad (15)$$

Writing the line element (with h' and V in place of h and x^{+}) in terms of the coordinates (R,V) we obtain the incoming Vaidya solution [6] $ds^{2}=-[1-2m^{+}(V)/R]dV^{2}+2dRdV+R^{2}d\Omega^{2}$, with m^{+} as the mass function. Waugh and Lake [7] have shown that a closed form of this solution can be given in double null coordinates only for linear and exponential mass functions. This implies by Eq. (14) that the scalar field has to be also a linear or exponential function in order to be able to integrate the second equation (15).

IV. SOLUTION WITH SYMMETRY

Until now we have discussed the cases where at least one of the two components of the scalar field vanishes. If both components are present, we can use them in the construction of new null coordinates $\tilde{x}^{\pm}=\varphi^{\pm}/\sqrt{2}$. In terms of these null coordinates, dropping the tildes, we obtain the system

$$\rho_{,++}-\rho_{,+}(\ln\sigma)_{,+}=-1 \qquad (16)$$

$$\rho_{,--}-\rho_{,-}(\ln\sigma)_{,-}=-1 \qquad (17)$$

$$2\rho_{,+-}+\rho^{-1/2}\sigma=0. \qquad (18)$$

The general solution for the system (16)–(18) is not yet known in closed form. (Miković [8] has given the solution in the form of a perturbative series in powers of the outgoing energy-momentum component).

Our purpose is to generalize the solution given in [1] for any Killing vector tangent to the surface defined by $\theta=$const and $\varphi=$const. From the Killing equations we find that the Killing vector $\mathcal{K}=(\mathcal{K}^{+}(x^{+}),\mathcal{K}^{-}(x^{-}),0,0)$ satisfies $\mathcal{K}^{+}\rho_{,+}+\mathcal{K}^{-}\rho_{,-}=0$ and $(\mathcal{K}^{+}h)_{,+}+(\mathcal{K}^{-}h)_{,-}=0$. The last relation implies the existence of a potential N defined by

$h\mathcal{K}^{\pm}=\pm N_{,\mp}$. The remaining Killing equations in terms of the potential N are

$$N_{,-}\rho_{,+}=N_{,+}\rho_{,-} \tag{19}$$

$$N_{,\pm\pm}=N_{,\pm}\left(\frac{\rho_{,\pm}}{2\rho}+(\ln\sigma)_{,\pm}\right). \tag{20}$$

The potential N can be eliminated from the system (19), (20) in the following way. We express $N_{,+-}$ from the ∂_- derivative of Eq. (19) and we substitute it into the ∂_+ derivative of Eq. (19) in which also $N_{,\pm\pm}$ are replaced by their expressions (20). By inserting N_+ from Eq. (19), we obtain a differential equation in ρ and σ. Eliminating the second derivatives $\rho_{,\pm\pm}$ from Eqs. (16) and (17), we find $\rho_{,+}=c\rho_{,-}$ where $c^2=1$. This implies that ρ depends on a single variable $x^+ + cx^-$. It is immediate to show that the same property holds for σ and N. The Killing vector is $\mathcal{K}=\alpha(c,-1,0,0)$, where α is a constant. Thus Eqs. (16)–(18) lead to a system of ordinary differential equations

$$\frac{d^2\rho}{d(x^+ + cx^-)^2}+\frac{c\sigma}{2\rho^{1/2}}=0$$

$$\frac{d\rho}{d(x^+ + cx^-)}\,\frac{d\ln\sigma}{d(x^+ + cx^-)}=1-\frac{c\sigma}{2\rho^{1/2}}. \tag{21}$$

We introduce timelike and spacelike coordinates t and r respectively by $x^{\pm}=t\pm r$ in terms of which the metric is $ds^2=h(-dt^2+dr^2)+\rho d\Omega^2$. There are two cases to discuss: (a) when all metric functions depend only on r (static case) or (b) when they depend solely on t (homogeneous case).

(a) We have discussed this case in detail in [1]. The fact that the dilaton $\rho=R^2$ has only radial dependence suggests to introduce R as a new radial coordinate. The metric becomes $ds^2=-hdt^2+f^{-1}dR^2+R^2d\Omega^2$, where the metric function f is defined by the differential equation

$$\left(\frac{dR}{dr}\right)^2=fh. \tag{22}$$

By introducing the notation $\beta=2/h$ we recover, from Eqs. (21), Eqs. (2.7), (2.8) of [1].

(b) Now the dilaton depends only on time. We introduce R as a new time variable and obtain the metric $ds^2=-f^{-1}dR^2+hdr^2+R^2d\Omega^2$, where the metric function f is defined similarly to Eq. (22):

$$\left(\frac{dR}{dt}\right)^2=fh. \tag{23}$$

From Eqs. (21) we find equations very much similar to Eqs. (2.7), (2.8) of [1]. Both are written concisely as

$$f\frac{d\ln f}{d\ln R}=-\beta-f-c$$

$$f\frac{d\ln\beta}{d\ln R}=-\beta+f+c, \tag{24}$$

where $c=-1$ refers to the static case and $c=1$ to the homogeneous case.

The solving procedure of this system follows closely [1], the only difference being in the definition of the new variables $P=\pm(2f\beta)^{1/2}$ and $L=\pm(1-c\beta)/(2f\beta)^{1/2}$, allowed to take either positive or negative values. We find

$$cCR=e^{L^2}-2L\Phi_B(L),\quad \Phi_B(L)=B+\int^L e^{x^2}dx \tag{25}$$

$$\beta=\frac{CR}{e^{L^2}},\quad f=\frac{2\Phi_B^2(L)}{e^{L^2}CR} \tag{26}$$

$$ds^2=-\frac{2ce^{L^2}R}{C}dL^2+\frac{e^{L^2}}{cCR}dr^2+R^2d\Omega^2 \tag{27}$$

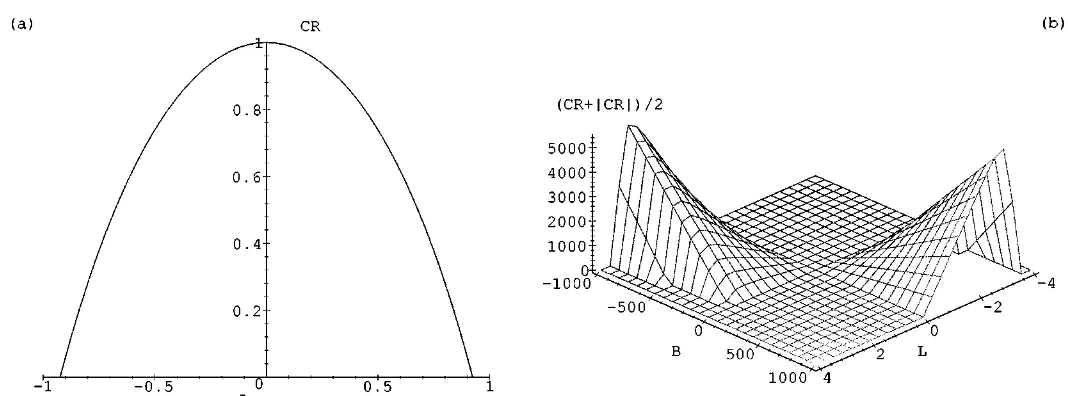

FIG. 1. (a) The function CR is positive in a domain of L centered at the origin, when the parameter $B=0$ is chosen. (b) The plot of $(CR+|CR|)/2$ for a wide range of the constant B shows a shift of the admissible domain of L in the positive or negative direction depending on the sign of B.

where B and $C>0$ are constants and Eq. (25) determines R as function of L.

V. CONCLUDING REMARKS

The metric (27), either static (then t replaces r and L is a radial coordinate) or homogeneous, has a true singularity at $R=0$. Although the Ricci scalar vanishes $\mathcal{R}=0$, other scalars, like $\mathcal{R}_{ab}\mathcal{R}^{ab}=2C^2/e^{2L^2}R^2$ and the Kretschmann scalar $\mathcal{R}_{abcd}\mathcal{R}^{abcd}$ which has the denominator $(CR)^6$, are ill-behaved at $R=0$.

The 4D interpretation of the source is again twofold. First we have the general picture of colliding null dust streams, valid even without the assumption of a Killing vector. Second we can interpret the source as an anisotropic fluid with no tangential pressures and both the energy density and radial pressure equal to $\beta/8\pi R^2=C/e^{L^2}R$ (these also became infinite at $R=0$).

Note that in contrast with [1], the transcendental function $\Phi_B(L)$ is *not* constrained to be positive. In the static case, because $\Phi_B(L)=-\Phi_{-B}(-L)$, the negative values of L lead just to another copy of the solution written in [1] for positive L, as was explained in [9]. The consequences in the homogeneous case are deeper, as will be seen in what follows. A natural requirement the new time variable L should satisfy is to be a monotonous function of t. From Eqs. (25) the relation $d(CR)/dL=-2\Phi_B(L)$ emerges. By fixing the sign in the square root of Eq. (23) appropriately ($-$ for $\Phi>0$ and $+$ for $\Phi<0$), we get $d(CR)/dt=-2\Phi/R$. In conclusion

$$\frac{dL}{dt}=R>0. \tag{28}$$

Another remark is that at $d(CR)/dL=0$ the function f vanishes; thus the metric written in the coordinates (R,r,θ,ϕ) has a coordinate singularity, which however does not appear in the form (27) of the metric, when the coordinates (L,r,θ,ϕ) are employed. This feature is closely related to the fact that both the transformations $t\rightarrow R$ and $L\rightarrow R$ are ill-behaved at $\Phi_B(L)=0$, whereas the direct transformation $t\rightarrow L$, given by Eq. (28), is regular.

A delicate issue is the signature of the metric (27). To have a homogeneous metric $(c=1)$ with L as time and r as radial coordinate, as claimed, the condition $R>0$ should be satisfied. This translates to have L confined to a finite range $L\in(L_{0-},L_{0+})$, as can be seen in Fig. 1 from the numerical plot of Eqs. (25). The constant C provides a scale as in the static case. The other constant B shifts the admissible domain of L to negative values when $B>0$ and conversely. The singularity is on the boundaries $L_{0\pm}$ of the admissible range of L.

The function R is the time-dependent radius of the Kantowski-Sachs type homogeneous universe described by Eq. (27). It is first increasing to a maximum value after which it decreases to zero. This universe, filled by a two-component radiation, is born from a singularity and collapses into another singularity.

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