Critical behavior in gravitational collapse of a perfect fluid

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To confirm the scenario of critical behavior in gravitational collapse in a previous paper we carry out a rather complete analysis for perfect fluids with pressure proportional to density, in a wide range of the adiabatic index γ. In particular, the uniqueness of the relevant mode around a fixed point is established by a Lyapunov analysis. This shows that critical phenomena occur not only for the radiation fluid but also for perfect fluids with $1<\gamma\lesssim 1.889$. We also analyze the stability of other self-similar solutions and also discuss the universality class. [S0556-2821(98)04324-0]

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I. INTRODUCTION

Choptuik [2] found a critical behavior in gravitational collapse by a numerical study of a self-gravitating scalar field. Evans and Coleman [3] found similar phenomena in spherically symmetric collapse of a radiation fluid, with continuous self-similarity. The present authors [1] showed that both qualitative and quantitative understanding is possible by introducing renormalization group ideas: The *uniqueness* of the relevant (unstable) eigenmode of the linear perturbations around the critical self-similar space-time is essential for the scaling and the universality to be observed, and that the critical exponent $\beta_{\rm BH}$ is given by $\beta_{\rm BH}=1/\kappa$ where κ is the first Lyapunov exponent. The value obtained was $\beta_{\rm BH}=0.35580192$, whose difference from those obtained for a scalar field (0.37) [3] and a gravitational wave (0.38) [4] was beyond the possible numerical errors in their simulations. This showed that there is no universality between radiation fluid collapse and other systems. Maison [5] applied the method to systems of perfect fluids with pressure proportional to density and showed how $\beta_{\rm BH}$ depends on the equation of state, under the assumption that the same phenomena occur and the same scenario holds as in the case of a radiation fluid.

In this paper we present a thorough analysis of perfect fluid collapse, with various adiabatic indices γ. In particular, we go beyond [1] to confirm the uniqueness of the relevant mode around the critical solution by a new method called *Lyapunov analysis*. A Lyapunov analysis and the shooting method for an ordinary differential equation adopted in [1] are complementary to each other in the following sense: (1) The former extracts eigenmodes in descending order of the real parts of their eigenvalues κ, whereas the latter provides information on κ only in a finite region. (2) The Lyapunov analysis can provide information on the continuous spectrum. In Sec. VI we show that the real part of the continuous spectrum is negative even if the eigenmodes do not form a complete set. (3) It is easier in the latter than in the former to numerically obtain accurate values of the eigenvalues κ and hence the critical exponent $\beta_{\rm BH}$. The uniqueness of the relevant mode in particular implies the existence of critical behavior like that observed in [3] also for $\gamma\neq 4/3$ and the self-similar solutions in [5] are in fact responsible for the critical behavior.

We also find various self-similar solutions in Sec. III and discuss their stability (numbers of relevant modes) in Sec. V. Self-similar solutions with more than one relevant mode are responsible for multicritical behavior. We also discuss universality class in Sec. VII (the universality class is independently discussed by Gundlach and Martín-García [6]). By this we can predict that a class of equations of motion which are not necessarily scale invariant exhibits the same critical behavior.

For detailed calculations and explanations see Ref. [7].

II. EQUATIONS OF MOTION

The line element of any spherically symmetric space-time is written as

$$ds^2=-\alpha^2(t,r)dt^2+a^2(t,r)dr^2+r^2(d\theta^2+\sin^2\theta d\phi^2). \tag{2.1}$$

We assume that the matter content is a perfect fluid with energy-momentum tensor $T_{ab}=\rho u_a u_b+p(g_{ab}+u_a u_b)$, where ρ is the density, p is the pressure, and u_a is a unit time like (co)vector whose components are given by $u_t=-\alpha/\sqrt{1-V^2}$, $u_r=aV/\sqrt{1-V^2}$, with V being the three-velocity of fluid particles. We consider the case $p=(\gamma-1)\rho$, where $\gamma\in(1,2)$ is a constant (adiabatic index). In terms of the variables $s\equiv-\ln(-t)$, $x\equiv\ln(-r/t)$, and introducing $N\equiv\alpha/ae^x$, $A\equiv a^2$, $\omega\equiv 4\pi r^2a^2\rho$, we can write the equations of motion (EOM) in an autonomous form, which makes the scale invariance of the system transparent:

$$\frac{A_{,x}}{A}=1-A+\frac{2\omega[1+(\gamma-1)V^2]}{1-V^2}, \tag{2.2a}$$

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$$\frac{N_{,x}}{N} = -2 + A - (2-\gamma)\omega, \tag{2.2b}$$

$$\frac{A_{,s}}{A} + \frac{A_{,x}}{A} = -\frac{2\gamma NV\omega}{1-V^2}, \tag{2.2c}$$

$$\frac{\omega_{,s}}{\omega} + \frac{\gamma V V_{,s}}{1-V^2} + (1+NV)\frac{\omega_{,x}}{\omega} + \frac{\gamma(N+V)V_{,x}}{1-V^2} = \frac{3(2-\gamma)}{2}NV - \frac{2+\gamma}{2}ANV + (2-\gamma)NV\omega, \tag{2.2d}$$

$$(\gamma-1)V\frac{\omega_{,s}}{\omega} + \frac{\gamma V_{,s}}{1-V^2} + (\gamma-1)(N+V)\frac{\omega_{,x}}{\omega} + \frac{\gamma(1+NV)V_{,x}}{1-V^2} = -(\gamma-2)(\gamma-1)N\omega + \frac{7\gamma-6}{2}N + \frac{2-3\gamma}{2}AN. \tag{2.2e}$$

Only four out of the above five equations are independent: Equation (2.2c) is automatically satisfied by solutions of the set (2.2a), (2.2b), (2.2d), and (2.2e), as long as they satisfy a boundary condition $A(s,-\infty)=1$, $V(s,-\infty)=\omega(s,-\infty)=0$. We pick up the above set of four equations as our basic equations of motion, and use Eq. (2.2c) as an auxiliary equation.

The only coordinate transformation which preserves the form of Eq. (2.1) is $t \mapsto F^{-1}(t)$, which corresponds to

$$(s,x) \mapsto (f^{-1}(s), x-s+f^{-1}(s)), \tag{2.3}$$

where $F^{-1}(t) = -e^{-f^{-1}(s)}$ with $\dot{f} \equiv df/ds \neq 0$. Under this transformation, the variables h $(h=A,N,\omega,V)$ transform to $\tilde{h}$ given by

$$\tilde{h}(s,x) = \begin{cases} h(s,x) & (h=A,\omega,V), \\ \dot{f}(s)h(s,x) & (h=N). \end{cases} \tag{2.4}$$

Equations (2.2) are of course invariant under this transformation. We can determine the coordinate system completely by fixing the value of N at a point, for example, at $x=0$, on each constant s line. We shall retain the degree of freedom here and fix it later.

III. CRITICAL (SELF-SIMILAR) SOLUTION

To carry out the first step of our scenario in [1,7], we first have to find out self-similar solutions, i.e., fixed points of renormalization group transformations. The behavior of self-similar solutions has been extensively discussed by Bogoyavlenskii [8], followed by others [9,10].

Let us first require that the space-time be self-similar, i.e., that N and A depend only on x: $N=N_{ss}(x)$, $A=A_{ss}(x)$. [In fact (see [11]), any spherically symmetric self-similar space-times can be expressed in that form if a freedom of coordinate transformation (2.3) is used.] Then it follows from Eqs. (2.2) that ω_{ss} and V_{ss} are also functions of x only: $\omega=\omega_{ss}(x)$, $V=V_{ss}(x)$. In this sense, the space-time we are interested in corresponds to a *fixed point* of the renormalization group transformation. The equations for a self-similar solution are then given by omitting terms containing s derivatives in Eqs. (2.2), which we will call Eqs. (2.2)′. From Eqs. (2.2a)′ and (2.2c)′, one obtains an algebraic identity

$$1 - A + \frac{2\omega[1+(\gamma-1)V^2]}{1-V^2} = -\frac{2\gamma NV\omega}{1-V^2}. \tag{3.1}$$

Instead of eliminating one variable, we will keep using four variables (A,N,ω,V) and use Eq. (3.1) as a check at appropriate stages of our numerical calculation.

Under a self-similar ansatz, the coordinate freedom $t \mapsto F^{-1}(t)$ in Sec. II reduces to $t \mapsto \tilde{t} = kt$ with constant k, which corresponds to a translation in x [the other transformations alter the constant x lines in (t,r) space]. This freedom of the coordinate transformation allows one to adjust the value of N arbitrarily at a given point. We fix the coordinate system by requiring that the sonic point [9] be at $x=0$.

Since we are interested in self-similar solutions which could represent *intermediate asymptocis* of the (near-)critical space-times, they should have regular spatial profiles. Namely, we require the following.

(i) The self-similar solution be analytic (or at least smooth, in the sense of having an asymptotic expansion to all orders) for all $x \in \mathbb{R}$,

(ii) The space-time and the matter be regular, $A=1$ and $V=0$, at the center $(x=-\infty)$.

Now, (2.2)′ is a set of ordinary differential equations (ODE's) for four variables (A,N,ω,V), which satisfies the Lipschitz condition except at the so-called *sonic point* (see below).

Equations (2.2d)′ and (2.2e)′ can be written in the form

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \omega_{,x} \\ V_{,x} \end{pmatrix} = \begin{pmatrix} e \\ f \end{pmatrix}, \tag{3.2}$$

where a, b, c, d, e, and f are functions, none of which depends on derivatives of ω and V. Solving the above in favor of $\omega_{,x}$ and $V_{,x}$, the resulting equation violates the Lipschitz condition at the *sonic point* where the determinant of the coefficient matrix of $\omega_{,x}$ and $V_{,x}$ vanishes: $ad-bc=0$. Physically the sonic point is where the velocity of fluid particles seen from the observer on the constant x line is equal to the speed of sound, $\sqrt{\gamma-1}$. To have finite derivatives $\omega_{,x}$ and $V_{,x}$ at the sonic point the rows of Eq. (3.2) must be proportional to each other: $af-ec=0$. These two conditions together with the requirement that the variables allow power series expansions imply that the self-similar solution which is regular at the sonic point is characterized by a single parameter $V_0 \equiv V_{ss}(0)$ [12].

Asymptotic behavior of solutions of ODE's (2.2)′ as $x \to \pm\infty$ can be most easily understood by viewing Eqs. (2.2)′ as a dynamical system and considering its fixed points [12]. Condition (ii) implies that the asymptotic behavior as $x \to -\infty$ is

$$A_{s.s.}(x) \sim 1 + A_{-\infty}e^{2x}, \quad N_{ss}(x) \sim N_{-\infty}e^{-x},$$

$$\omega_{\rm ss}\sim\omega_{-\infty}e^{2x},\quad V_{\rm ss}\sim V_{-\infty}e^{x},\tag{3.3}$$

where coefficients $A_{-\infty}$, $N_{-\infty}$, $\omega_{-\infty}$, $V_{-\infty}$ satisfy $A_{-\infty}=2\omega_{-\infty}/3$, $N_{-\infty}V_{-\infty}=-2/3\gamma$.

In the numerical calculation, we first fix the value V_0 at the sonic point $x=0$, and use the power series expansion there to derive the derivatives. Then starting from the sonic point $x=0$, we solve ODE's (2.2)′ towards $x=-\infty$, using a Runge-Kutta fourth order integrator, with reliable error estimates. If at some $x<0$ either (1) $A<1$ or (2) $\det=ad-bc=0$, we stop solving and conclude that this V_0 does not give rise to a desired self-similar solution. The first case ($A<1$) is excluded because it is easily shown that solutions regular at the center (i.e., $A\to1$ as $x\to-\infty$) must satisfy $A\geqslant1$ for all $x\in\mathbb{R}$. The second case implies the existence of another sonic point between $x=0$ and $x=-\infty$. We can always find such a solution with more than one sonic point by searching for its part between its leftmost sonic point and $x=-\infty$, and then solving Eqs. (2.2)′ from the leftmost sonic point to the right (towards $x=\infty$).[1]

Our numerical results show that each allowed V_0 leads to either of (1) or (2) above. Moreover, it is seen, both numerically and mathematically rigorously, that a desired self-similar solution exists at those values of V_0, where V_0+0 leads to case (1) and V_0-0 leads to case (2).

We have searched for self-similar solutions which satisfy the above conditions (i) and (ii) numerically, for all the allowed values of V_0. As has been extensively studied in [8], there exists a sequence of self-similar solutions, each of which is characterized by the number of zeros of V. The number of zeros is an integer starting from 1, and the one with exactly one zero is the solution cited in [3], which we call the Evans-Coleman solution. Because of a numerical difficulty, we could find only self-similar solutions with an odd number of zeros of V.

IV. PERTURBATION

We studied perturbations around self-similar solutions by two different methods. The first one is to directly find out eigenmodes by solving the system of ODE's below. The second one is to apply the so-called Lyapunov analysis. The former has the advantage that it reduces the problem to that of solving ODE's, and gives us fairly accurate estimates of eigenvalues, but has the disadvantage that it is almost impossible to search for *every possible* eigenvalue. The latter has the advantage that it can easily find out eigenvalues in descending order of their real part, but has disadvantages that (1) eigenvalue estimates are not so accurate and (2) we have to solve a partial differential equation (PDE). However, we here emphasize that the PDE we have to solve for the Lyapunov analysis is a very regular one, and can be handled by standard techniques for solving PDE's.

[1]However, it is quite unlikely that such solutions with more than one sonic point exist, because the existence of only one sonic point already reduces the freedom of parameters to discrete ones, and other possible sonic points will further reduce it.

To simplify the notation, we consider perturbations of $\overline{A}=\ln A$, $\overline{N}=\ln N$, $\overline{\omega}=\ln\omega$, and V. For example, the $\overline{A}$ version of Eq. (4.1) in fact means $\overline{A}(s,x)=\overline{A}_{\rm ss}(x)+\epsilon\overline{A}_{\rm var}(s,x)$ where $\overline{A}(s,x)\equiv\ln A(s,x)$, $\overline{A}_{\rm ss}(x)\equiv\ln A_{\rm ss}(x)$.

Perturbation equations are obtained by taking the first order variation in Eqs. (2.2) from the self-similar solution $H_{\rm ss}$:

$$h(s,x)=H_{\rm ss}(x)+\epsilon h_{\rm var}(s,x),\tag{4.1}$$

where h represents each of $(\overline{A},\overline{N},\overline{\omega},V)$. In the following, we write V for $V_{\rm ss}$ and write $V_{\rm var}$ for perturbations. From Eqs. (2.2a), (2.2b), (2.2d), and (2.2e) we have equations of the form

$$\left[\begin{pmatrix}O&O\\O&P_s\end{pmatrix}\partial_s+\begin{pmatrix}I&O\\O&P_x\end{pmatrix}\partial_x\right]\mathbf{h}_{\rm var}=Q\mathbf{h}_{\rm var},\tag{4.2}$$

where P_s and P_x are 2×2 matrices and Q is a 4×4 matrix, all of which determined by the self-similar solution, and $\mathbf{h}_{\rm var}=(\overline{A}_{\rm var},\overline{N}_{\rm var},\overline{\omega}_{\rm var},V)^T$. See [7] for explicit expressions. From Eq. (2.2c) we also have

$$\partial_s\overline{A}_{\rm var}+\partial_x\overline{A}_{\rm var}=-\frac{2\gamma NV\omega}{1-V^2}(\overline{N}_{\rm var}+\overline{\omega}_{\rm var})-\frac{2\gamma N\omega(1+V^2)}{(1-V^2)^2}V_{\rm var}.\tag{4.3}$$

We note that there is freedom of the coordinate transformation of order ϵ, namely, Eq. (2.3) with $f(s)=s+\epsilon f_1(s)$. The transformation (2.4) now becomes [taking $O(\epsilon)$ terms, $'=d/dx$, $\dot{}=d/ds$]

$$\tilde{h}_{\rm var}(s,x)=\begin{cases}h_{\rm var}(s,x)+f_1(s)h'_{\rm ss}(x) & (h=\overline{A},\overline{\omega},V),\\ h_{\rm var}(s,x)+f_1(s)h'_{\rm ss}(x)+\dot{f}_1(s) & (h=\overline{N}).\end{cases}\tag{4.4}$$

This in particular means that one can always require $\overline{N}_{\rm var}(s,x_0)\equiv0$ for a fixed x_0.

V. EIGENMODE ANALYSIS

A. Equations and boundary conditions

We consider eigenmodes of the form $h_{\rm var}(s,x)=h_{\rm p}(x)e^{\kappa s}$, with $\kappa\in\mathbb{C}$ being a constant. Substituting this into Eqs. (4.2) and (4.3) yields a set of linear, homogeneous first order ODE's for $(\overline{N}_{\rm p},\overline{A}_{\rm p},\overline{\omega}_{\rm p},V_{\rm p})$, which we call Eqs. (4.2)′ and (4.3)′, respectively. Equations (4.3)′ and the first row of Eq. (4.2)′ provide an algebraic identity:

$$(\kappa-A)\overline{A}_{\rm p}+\left(\frac{2\gamma NV\omega}{1-V^2}\right)\overline{N}_{\rm p}+\left(\frac{2\omega\{1+(\gamma-1)V^2+\gamma NV\}}{1-V^2}\right)\overline{\omega}_{\rm p}+\left(\frac{2\gamma\omega\{N(1+V^2)+2V\}}{(1-V^2)^2}\right)V_{\rm p}=0,\tag{5.1}$$

just as in the case of self-similar solutions [cf. Eq. (3.1)]. As in our treatment of self-similar solutions, we here treat

$\overline{A}_{\rm p}, \overline{N}_{\rm p}, \overline{\omega}_{\rm p}, V_{\rm p}$ as four unknown functions, and use Eq. (5.1) as a check at appropriate steps of computation. We require $\overline{N}_{\rm p}(s,0)=0$ to fix the coordinate freedom (4.4).

As in the case of self-similar solutions, we require (i) that the perturbations be analytic for all $x \in \mathbb{R}$ and (ii) that the perturbed space-times be regular at the center ($\overline{A}_{\rm p}=0$ at $x=-\infty$).

The perturbation solutions satisfying the analyticity condition (i) at the sonic point are specified by one free parameter κ, apart from the overall multiplicative factor. This can be seen as follows. First, we note that the sonic point is a *regular singular point* for the perturbations. That is, the system of ODE's for perturbations $(4.2)'$ is singular where the determinant of the matrix on the left-hand side (LHS) of Eq. $(4.2)'$ vanishes: $\det P_x=0$. This is identical to the sonic point condition for self-similar solutions. Second, in order to have a smooth solution at the sonic point, the third and fourth rows of Eq. $(4.2)'$ must be proportional to each other at the sonic point, like in the case of the self-similar solution. This yields an algebraic relation between $\overline{A}_{\rm p}(0)$, $\overline{N}_{\rm p}(0)$, $\overline{\omega}_{\rm p}(0)$, and $V_{\rm p}(0)$. Third, we have another algebraic relation between $\overline{A}_{\rm p}(0)$, $\overline{N}_{\rm p}(0)$, $\overline{\omega}_{\rm p}(0)$, and $V_{\rm p}(0)$ by Eq. (5.1). Fourth, the above two algebraic relations, together with our choice of gauge $\overline{N}_{\rm p}(0)=0$, enable us to express $\overline{\omega}_{\rm p}(0)$ and $V_{\rm p}(0)$ in terms of $\overline{A}_{\rm p}(0)$ and κ. Then higher order expansion coefficients of perturbations are given in terms of $\overline{A}_{\rm p}(0)$ and κ. Because the system is linear and homogeneous, overall normalization $\overline{A}_{\rm p}(0)$ is irrelevant. We thus see that the solution which satisfies (i) is characterized by a single parameter κ.

This, together with the regularity condition (ii) at the center, in general allows only discrete values for κ. Indeed, in view of Eq. (3.3), the system of ODE's for perturbations $(4.2)'$ takes on a simple form as $x \to -\infty$, given by

$$\partial_x \begin{pmatrix} \overline{A}_p \\ \overline{N}_p \\ \overline{\omega}_p \\ V_p \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \dfrac{2-3\gamma}{2(\gamma-1)} & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix} \begin{pmatrix} \overline{A}_p \\ \overline{N}_p \\ \overline{\omega}_p \\ V_p \end{pmatrix}. \tag{5.2}$$

This linear homogeneous ODE with constant coefficients has four independent solutions, given by

$$\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} e^{-2x}, \quad \begin{pmatrix} 1 \\ -1 \\ \dfrac{2-3\gamma}{2(\gamma-1)} \\ 0 \end{pmatrix} e^{-x}. \tag{5.3}$$

The last one should be excluded in view of the identity (5.1), and thus the asymptotic behavior of arbitrary solutions of ODE's is described by linear combinations of the first three. Now, one of them (the third one) blows up as $x \to -\infty$. Therefore we have to choose κ so as to eliminate the unwanted expanding mode in order to get a solution which satisfies our boundary condition.

B. Gauge mode

In searching for possible eigenmodes, one has to pay attention to a ''gauge mode'' which emerges from a coordinate transformation applied to the self-similar solution.

Suppose one has a self-similar solution $h(s,x)=H_{\rm ss}(x)$. Seen from another coordinate [related by the coordinate transformation (2.3)], its perturbation is given by setting $h_{\rm var} \equiv 0$ in $\tilde{h}$ of Eq. (4.4):

$$\tilde{h}_{\rm var}(s,x) = \begin{cases} f_1(s) h'_{\rm ss}(x) & (h=\overline{A}, \overline{\omega}, V), \\ f_1(s) h'_{\rm ss}(x) + \dot{f}_1(s) & (h=\overline{N}). \end{cases} \tag{5.4}$$

For general f_1, this does not behave like an eigenmode. However, with the choice of $f_1(s) \equiv e^{\overline{\kappa} s}$ ($\overline{\kappa}$ arbitrary), $\tilde{h}$ does behave as an eigenmode with eigenvalue $\overline{\kappa}$:

$$\tilde{h}_{\rm gauge}(s,x;\overline{\kappa}) = e^{\overline{\kappa} s} \begin{pmatrix} h'_{\rm ss}(x) \\ h'_{\rm ss}(x) + \overline{\kappa} \end{pmatrix} \begin{matrix} (h=\overline{A}, \overline{\omega}, V) \\ (h=\overline{N}). \end{matrix} \tag{5.5}$$

The mode emerges via the coordinate transformation of the self-similar solution $h_{\rm ss}$, and thus can be considered as a result of a pure gauge transformation, like a ''zero mode'' in translation-invariant systems. Because $\overline{\kappa} \in \mathbb{C}$ is *arbitrary*, this pure gauge mode forms a one-parameter family.

Similarly, for any (nongauge) eigenmode $h_{\rm p}$, Eq. (4.4) for $h_{\rm var}(s,x)=h_{\rm p}(x)e^{\kappa s}$ becomes

$$\tilde{h}_{\rm var}(s,x) = \begin{cases} h_{\rm p}(x) e^{\kappa s} + f_1(s) h'_{\rm ss}(x) & (h=\overline{A}, \overline{\omega}, V), \\ h_{\rm p}(x) e^{\kappa s} + f_1(s) h'_{\rm ss}(x) + \dot{f}_1(s) & (h=\overline{N}). \end{cases} \tag{5.6}$$

As long as $h_{\rm p} \not\equiv 0$, by taking $f_1(s)=e^{\kappa s}$, this means transformed $\tilde{h}_{\rm var}$ is also an eigenmode, with the same κ. Thus we have a one-parameter family of eigenmodes (with the same eigenvalue κ), which are mutually related by the gauge mode (5.5).

C. Numerical results

Based on the above observations, we searched for the desired eigenmodes as follows. We first fix a value of κ, and then starting from $x=0$ (the sonic point), integrate Eq. $(4.2)'$ to $x=-\infty$. When there appear nonzero components of the expanding modes (for sufficiently large $|x|$), we judge that this κ does not give a desired eigenmode and stopped.

Here we show the results for $\gamma=4/3$. Those for other γ are summarized in Sec. VIII. The eigenvalue with the largest Re κ is $\kappa \simeq 2.81055255$, which corresponds to the exponent value $\beta_{\rm BH} \simeq 0.35580192$, and which we believe to be exact to the last digit. This is in good agreement with the value of [3]. The profile of the eigenmode is found in Fig. 2 of [1]. In our gauge, where $\overline{N}_{\rm p}(s,0) \equiv 0$, we observe the gauge mode

with $\bar{\kappa}=-(d\bar{N}_{\rm ss}/dx)(0)\simeq 0.35699$, as explained in Sec. V B.[2]

To confirm our scenario, we performed a thorough search of other eigenmodes in the region $0\leqslant \mathrm{Re}\,\kappa\leqslant 15$, $|\mathrm{Im}\,\kappa|\leqslant 14$, and found *none*, except for the above-mentioned relevant mode and the gauge mode. We also performed a less complete search in the region $-1.5\leqslant \mathrm{Re}\,\kappa<0$, $|\mathrm{Im}\,\kappa|<2$. There is an eigenmode with $\mathrm{Re}\,\kappa\lesssim -1.4$, which is consistent with the results of the Lyapunov analysis in Sec. VI.

Our search of eigenmodes in this section has a drawback in that it is theoretically impossible to cover the whole values of $\kappa\in\mathbb{C}$ (unless, of course, one employs more sophisticated mathematical techniques). Moreover, it is not *a priori* obvious whether the eigenmodes form a complete set of basis functions. To fill these gaps, and to further confirm our scenario, we perform a Lyapunov analysis in Sec. VI.

D. Modes for other self-similar solutions

As has been stated in Sec. III, there exists a series of self-similar solutions (specified by the number of zeros of V), in which the Evans-Coleman solution can be considered as the first one (exactly one zero for V). We have searched for relevant eigenmodes for the first several self-similar solutions and found *more than one* relevant modes (see Table I). This implies that the other self-similar solutions are irrelevant for the generic critical behavior. However, our analysis is less complete for these higher self-similar solutions, because (1) we have not done the analysis for (possible) self-similar solutions with an even number of zeros of V and (2) there might be more relevant modes than are reported in Table I.

VI. LYAPUNOV ANALYSIS

To further confirm the uniqueness of the relevant mode we performed a Lyapunov analysis around the critical solution.

A. Numerical methods

The Lyapunov analysis is a method of extracting eigenvalues in descending order. It involves time integration of the linearized EOM (4.2) around the self-similar solution. It takes advantage of the fact that the eigenmodes with large $\mathrm{Re}\,\kappa$ dominate at late times and that the volume, defined by an (arbitrarily chosen) inner product on Γ, of the parallelepiped spanned by the eigenmodes corresponding $\kappa_1,\kappa_2,\ldots,\kappa_n$ approaches $e^{\kappa_1\kappa_2\cdots\kappa_n}$. During the integration, vectors in Γ are orthonormalized by the Gram-Schmidt procedure. This is essential for avoiding numerical overflows and underflows. For details see the Appendix. We can also have information on the imaginary part of eigenvalues by the period of oscillation in time s.

We have performed the above calculation by numerically solving the PDE's (4.2) for the linear perturbation. We emphasize that solving the PDE's (4.2) is much easier than solving the full EOM close to the critical point, because linear perturbations are well behaved in contrast with near-critical solutions whose derivatives are diverging in (t,r) coordinates.

We used the first order Lax scheme, and the second order Lax-Wendroff scheme for time evolution. We show the results from the Lax-Wendroff scheme. To simplify the coding, we employed a new gauge $\bar{N}_{\rm var}(s,-\infty)=0$. In this gauge we observe the gauge mode at $\bar{\kappa}=1$. We have made appropriate coordinate transformations to stabilize the integration schemes.

We imposed the free boundary condition at the sonic point $\xi\equiv e^x=1$. This can be done because no information can come in from outside the sonic point. The latter fact is easily seen mathematically by studying the characteristic curves or physically by recalling that spherically symmetric metric perturbations contain no gravitational waves and are determined by the matter degrees of freedom.

B. Numerical results

Figure 1 shows the dependence of $\mathrm{Re}\,\kappa$ on γ. The largest eigenvalue is real and agrees well with the first eigenvalue (which is real) found by solving the two-point boundary problem of an ODE in Sec. V. The second largest eigenvalue is that of the gauge mode, which should be exactly equal to 1. For $\gamma\leqslant 1.2$ the third and fourth largest eigenvalues are real, and the fifth is imaginary. For $\gamma=1.3,1.4$ the third is real and the fourth and fifth are imaginary, which are complex conjugate. Those for $\gamma\geqslant 1.4$ are complex. We can consider that there is a crossover of Lyapunov exponents corre-

TABLE I. Relevant modes found for other self-similar solutions, which are labeled by the number of zeros of V.

Self-similar solution	κ
1 (Evans-Coleman)	2.8105525488
3	8.456
	3.464
	1.665
	0.497
5	15.80
	7.13
	3.22
	1.51
	0.500
7	15.97
	6.92
	3.20
	1.50

[2]Maison [5] states that the gauge mode reported in [1] (same as reported here) ''seems to be erroneous.'' The fact is that our report in [1] (and of course here) is correct. The confusion seems to be due to the different gauges used in the analysis. Here and in [1] we use the gauge $\bar{N}_p(s,0)=0$ at the sonic point, while Maison [5] and we in Sec. VI use the gauge $\bar{N}_p(s,-\infty)=0$. The former gauge gives the gauge mode at $\bar{\kappa}\simeq 0.35699$, while the latter gives it at $\bar{\kappa}=1$; the difference in the values of $\bar{\kappa}$ is well understood in view of Eq. (5.5).

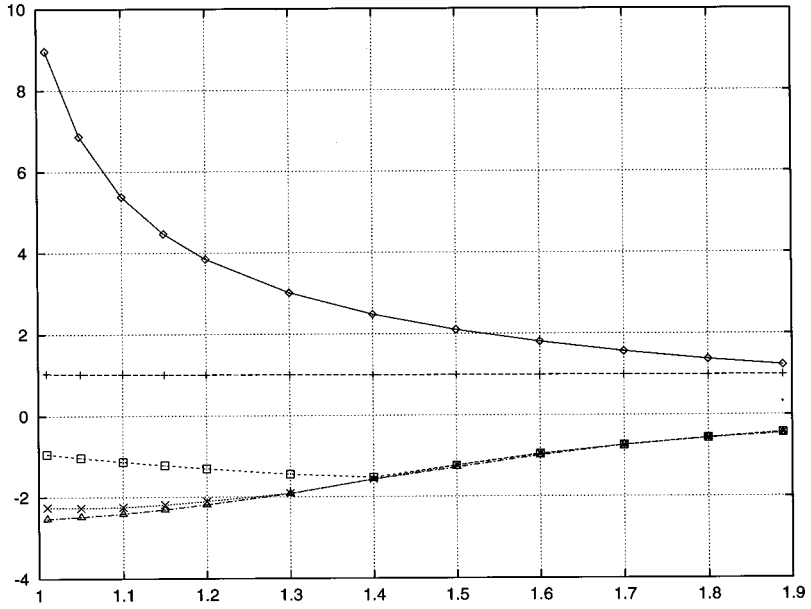

FIG. 1. Dependence of Lyapunov exponents Re κ on γ. The lines show the Lyapunov exponents. The top line, labeled ''relevant,'' represents the relevant mode, and the second line, labeled ''gauge,'' represents the gauge mode, whose Lyapunov exponent equals unity theoretically. For $\gamma<1.3$, the third and the fourth largest modes are real (labeled ''real''). For $\gamma=1.3$ and $\gamma=1.4$ the third is real and the fourth and fifth are complex conjugate. For $\gamma>1.4$ the third and fourth are complex conjugate (labeled ''cc''). We see that the complex conjugate pair takes over the real eigenvalue twice in $1<\gamma\lesssim 1.889$. The graph shows there is a *unique* relevant mode for all values of γ analyzed.

sponding to the real eigenvalue and the complex pair between $\gamma=1.2$ and $\gamma=1.3$. For $\gamma\geqslant 1.5$ the third and fourth are a complex conjugate pair, so that there is another crossover of Lyapunov exponents corresponding to the real eigenvalue and the complex pair between $\gamma=1.4$ and $\gamma=1.5$. For large γ it seems that many eigenmodes with close values of Re κ are present. Our preliminary computation of ten modes for $\gamma=1.889$ shows that there are at least nine modes in the range $-0.6\leqslant \mathrm{Re}\ \kappa\leqslant -0.4$, though it seems that not all of them are degenerate in Re κ.

Figure 1 establishes the uniqueness of the relevant mode in the collapse of perfect fluids with $p=(\gamma-1)\rho, 1<\gamma\lesssim 1.889$. From this we can conclude that critical behavior must be observed for all of these models with $1<\gamma\lesssim 1.889$. It should be noted that the results from our Lyapunov analysis in particular show that the continuous spectrum is confined to the region Re $\kappa<-0.4$ because otherwise we would find a solution which grows or decays more slowly than $e^{-0.4s}$ in the cospace of the unique relevant mode. The existence of a nonzero gap below zero for irrelevant modes and the continuous spectrum is essential for sharp critical behavior to be observed.

VII. UNIVERSALITY CLASS

Though we have concentrated our analysis on perfect fluids with equation of state $p=(\gamma-1)\rho$, we can predict by a renormalization group analysis that the same critical behavior (with the same critical exponent) can be observed for a class of equation of states, a *universality class*. The main idea is as follows. Under the scaling transformation which induces the renormalization group transformation *on phase space*, the EOM is transformed into a new EOM for scaled variables. We call this transformation a renormalization group transformation on the *space of EOM's*, whose fixed points correspond to scale-invariant systems. If a renormalization group drives the EOM to a fixed point, and if the fixed point shows a critical behavior, the original system exhibits the same critical behavior; i.e., it is in the same universality class as the fixed point. See [13] or [7] for details. The universality class is discussed independently by Gundlach [6].

As an example, let us consider an equation of state

$$L_6 := p-(\gamma-1)\rho-f(\rho)=0. \tag{7.1}$$

The EOM for this system is given by Eqs. (2.2) and (7.1). [We write L_6 in Eq. (7.1) because it is the sixth of the EOM.] The system with $f\equiv 0$ is a perfect fluid that we have studied in detail, and it exhibits critical behavior. Let us consider a scaling transformation which acts on ρ and p as

$$\rho^{(s)}\equiv e^{-2s}\rho(e^{-s}t,e^{-s}r), \quad p^{(s)}\equiv e^{-2s}p(e^{-s}t,e^{-s}r). \tag{7.2}$$

Under the renormalization group transformation induced from the scaling transformation, the EOM (2.2) are invariant, while L_6 is transformed into

$$L_6^{(s)}=p^{(s)}-(\gamma-1)\rho^{(s)}-f^{(s)}(\rho^{(s)}),$$

$$f^{(s)}(x)\equiv e^{-2s}f(e^{2s}x). \tag{7.3}$$

The system with $f\equiv 0$, where $L_6^*\equiv p-(\gamma-1)\rho$, is a fixed point.

Our problem is to determine the models (forms of f) which belong to the same universality class as the $f\equiv 0$ model. The model belongs to the same universality class as that of $f\equiv 0$, as long as (for fixed x)

$$f^{(s)}(x)\to 0 \quad (s\to\infty). \tag{7.4}$$

This is the sufficient condition we are after. An example of f which shows the same critical behavior as $f\equiv 0$ is given by $f(x)=x^\delta$ with $0\leqslant\delta<1$.

VIII. SUMMARY OF RESULTS

We have performed the analysis presented in Secs. III–VI for perfect fluid, with $1<\gamma\leqslant 2$. As has been noted in [8–10], a self-similar solution of Evans-Coleman type (i.e., with one zero of V) ceases to exist for $\gamma\gtrsim 1.889$. For Evans-Coleman-type self-similar solutions, the Lyapunov analysis of Sec. VI establishes that there is a unique relevant mode, and thus we can observe the critical behavior. Then, the precise values of the relevant eigenvalue κ are obtained by the shooting method of Sec. V, and the critical exponent is given by $\beta_{\rm BH}=1/\kappa$.

The result is shown in Table II. In particular, it is confirmed that the relevant eigenmodes found by Maison [5] are actually unique, and thus are responsible for the critical behavior. The value of the critical exponent $\beta_{\rm BH}$ depends strongly on γ. Moreover, the limit $\gamma\to 1$ seems to be discontinuous. (For the dust, $\gamma=1$, we expect a trivial critical behavior with $\beta_{\rm BH}=1$.) This may be because the domain of attraction of the Evans-Coleman-type self-similar solution vanishes as $\gamma\to 1^+$.

TABLE II. Values of κ and $\beta_{\rm BH}=1/\kappa$ for $1<\gamma\leqslant 1.889$. The last digit is rounded.

γ	κ	$\beta_{\rm BH}$
1.00001	9.4629170	0.10567566
1.0001	9.45592488	0.10575380
1.001	9.38660322	0.10653481
1.01	8.74868715	0.11430286
1.03	7.61774326	0.13127247
1.04	7.16334221	0.13959964
1.05	6.76491004	0.14782163
1.06	6.41269915	0.15594058
1.08	5.81789124	0.17188358
1.1	5.33435815	0.18746398
1.12	4.93282886	0.20272343
1.15	4.44235059	0.22510605
1.18	4.0484584	0.2470076
1.2	3.82545008	0.26140715
1.22	3.62729455	0.27568756
1.25	3.36750228	0.29695600
1.28	3.14337431	0.31812947
1.3	3.00990875	0.33223599
1.32	2.88714829	0.34636253
4/3	2.81055255	0.35580192
1.36	2.66838221	0.37475891
1.38	2.57025726	0.38906611
1.4	2.47850858	0.40346844
1.42	2.39245265	0.41798110
1.44	2.31150728	0.43261815
1.46	2.23517329	0.44739260
1.48	2.16301995	0.46231659
1.5	2.09467339	0.47740140
1.52	2.02980720	0.49265763
1.55	1.93841621	0.51588508
1.58	1.85338883	0.53955219
1.6	1.79989076	0.5555893
1.62	1.74873002	0.5718436
1.64	1.69974510	0.5883235
1.66	1.65278973	0.6050376
1.68	1.60773076	0.6219947
1.7	1.56444628	0.6392038
1.72	1.52282404	0.6566747
1.74	1.48276003	0.6744180
1.76	1.44415717	0.6924454
1.78	1.40692422	0.7107703
1.8	1.37097467	0.7294081
1.88	1.2383842	0.8075039
1.888	1.2259859	0.8156700
1.889	1.2244458	0.8166960

IX. DISCUSSION

There remain several open questions to be answered.

First, it should be emphasized that our analysis in Secs. II–VIII confirms only the *local* behavior of the renormalization group flow, around specific self-similar solutions. Although the self-similar solutions with more than one zero of V will be irrelevant for generic critical behavior (because all such solutions analyzed have more than one relevant mode), it would be interesting to know the *global* behavior of the flow, in particular, how these self-similar solutions are related.

Second, now that various kinds of critical behavior for different models have been observed, it would be interesting to ask what happens in a mixed system (e.g., perfect fluid+scalar fields). We performed a preliminary analysis in this direction. That is, we considered a ''mixed'' system of radiation fluid with a real scalar field, whose energy-momentum tensor is given by $T_{ab}=\rho u_a u_b+p(u_a u_b+g_{ab})+\nabla_a\phi\nabla_b\phi-(1/2)g_{ab}\nabla^c\phi\nabla_c\phi$, and studied the linear stability of the Evans-Coleman self-similar solution in this mixed system. (It is easily seen that eigenmodes decouple.) Lyapunov analysis in Sec. VI, applied to this mixed system, shows that the Evans-Coleman self-similar solution has a unique relevant mode (given by the relevant mode of the radiation fluid), and all scalar eigenmodes are irrelevant. This means that the Evans-Coleman self-similar solution is stable under the scalar perturbation, and we will observe the same radiation fluid critical behavior for the mixed system (at least when the scalar field is sufficiently weak). It would

be interesting to investigate the behavior of the system without the restriction that the scalar field is weak; in particular, where does the crossover between two types (fluid, scalar) of critical behavior occur?

A ''critical behavior'' such as observed in gravitational collapse is not limited to self-gravitating systems. In fact, similar phenomena can be observed in much simpler systems, such as the nonlinear heat equation, where we can rigorously carry out the analysis. However, if a solution of the EOM blows up in such simple systems, it usually means that the equation is not physically applicable in the blowup region, and the blowup is an artifact of a bad approximation. General relativity provides rare examples where the blowup of solutions does have a physical meaning—the formation of a black hole.

It is extremely desirable to develop a mathematically rigorous analysis of the critical behavior for a physically interesting model such as a radiation fluid or a scalar field. We are planning to come back to this problem in the near future.

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APPENDIX: THE LYAPUNOV ANALYSIS

The Lyapunov method has been extensively used in nonlinear analysis [14]. To make our presentation self-contained and transparent, we present a concise mathematical description of the method, restricted to the situation we are interested in.

Let us consider the linearization of a flow on a real Hilbert space (or a real Hilbert manifold) around an orbit. It is determined by an equation

$$\frac{df}{ds}(s)=A(s)f(s), \tag{A1}$$

where each $f(s)$ is an element of real Hilbert space V, a complete vector space with inner product, and $A(s)$ is a real function determined by the (background) orbit which we are considering. Let $(\cdot,\cdot)$ denote the inner product and $\|\cdot\| \equiv\sqrt{(\cdot,\cdot)}$ denote the norm defined by the inner product. We have $A(s)=A$ in Eq. (A1) when the orbit is a fixed point, and $A(s+\Delta)=A(s)$ when it is a periodic orbit with periodicity Δ. We concentrate on these cases below.

Let us define time evolution operator $T_s:V\to V$ by

$$f(0)=F, \quad f(s)=T_sF, \tag{A2}$$

where f is a solution of Eq. (A1). We wish to find eigenvalues κ and eigenvectors E^{c} of T_s satisfying

$$T_sE^{\mathrm{c}}=e^{\kappa s}E^{\mathrm{c}}, \tag{A3}$$

in particular those with large Re κ. Here κ is a complex number and E^{c} is a complex vector with Re E^{c},Im $E^{\mathrm{c}}\in V$. In Eq. (A3) and below, s (and σ) can be any number in the case of a fixed point, whereas it is restricted to be an integer multiple of Δ in the case of a periodic orbit.

We assume that the eigenvalue problem (A3) has a complete set of discrete eigenvectors, denoted by $\mathcal{E}^{\mathrm{c}} \equiv\{E_i^{\mathrm{c}}\}_{i=1,2,\ldots}$, with the corresponding eigenvalues $\{\kappa_i\}_{i=1,2,\ldots}$,Re $\kappa_1\geqslant$Re $\kappa_2\geqslant$Re $\kappa_3\geqslant\ldots$. Each Re κ_i is called the ith Lyapunov exponent. (Here we implicitly assume that we can always find κ_i with the largest real part after we have defined $\kappa_1,\ldots,\kappa_{i-1}$.) We observe that if κ_i is not real, there is an integer $i^*\neq i$ with $\kappa_{i*}=\kappa_i'$ and $E_{i*}^{\mathrm{c}}=E_i^{\mathrm{c}*}$, where asterisks (except for those on i or j) denote a complex conjugate in this appendix. We define $\mathcal{E}=\{E_i\}_{i=1,2,\ldots}$ by $E_i =E_i^{\mathrm{c}}$ if κ_i is real, and $E_i=\mathrm{Re}\,E_i^{\mathrm{c}}$, $E_{i*}=\mathrm{Im}\,E_i^{\mathrm{c}}$ if κ_i is not real, $i<i^*$ and $E_{i*}^{\mathrm{c}}=E_i^{\mathrm{c}*}$. Any real vector F can be expanded by $\mathcal{E}$ or $\mathcal{E}^{\mathrm{c}}$ as

$$F=\sum_{i=1}^{\infty} f^iE_i=\mathrm{Re}\sum_{i=1}^{\infty} f_{\mathrm{c}}^iE_i^{\mathrm{c}}, \tag{A4}$$

where $f_{\mathrm{c}}^i=f_{\mathrm{c}}^{i^*}{}^*=(1/2)(f^i+\sqrt{-1}f^{i^*})$. Without loss of generality, we assume all eigenvectors are normalized so that $\|\mathrm{Re}\,E_i^{\mathrm{c}}\|^2+\|\mathrm{Im}\,E_i^{\mathrm{c}}\|^2=1$, although E_i's are not necessarily orthogonal with respect to the inner product $(\cdot,\cdot)$.

Let $\mathcal{F}=\{F_i\}_{1\leqslant i\leqslant n}$ denote an n-frame of (linearly independent, real) vectors and let span $\mathcal{F}$ be the subspace of V spanned by $\mathcal{F}$. Below we often omit $1\leqslant i\leqslant n$ in $\mathcal{F} =\{F_i\}_{1\leqslant i\leqslant n}$ if no confusion occurs. We define several operations on frames. First given an operator $X:V\to V$ we define $\mathcal{X}$, an operator on frames induced from X, by $\mathcal{X}\mathcal{F} \equiv\{X(F_i)\}_{1\leqslant i\leqslant n}$. We in the following consider the cases where X is either a linear operator (such as T_σ) or the normalization operator N: $NF\equiv F/\|F\|$. Second, we define the orthogonalization operator $\mathcal{O}\mathcal{F}\mapsto\mathcal{F}'=\{F_i'\}_{1\leqslant i\leqslant n}$, where $\mathcal{F}'$ is defined by

$$F_1'\equiv F_1, \quad F_i'\equiv F_i-\sum_{j=1}^{i-1}\frac{(F_j',F_i)}{\|F_j'\|^2}F_j', \quad 2\leqslant i\leqslant n. \tag{A5}$$

We can prove the following.[3]

Proposition. For almost every $\mathcal{F}$, i.e., except for measure-zero cases, we have

$$\mathrm{Re}\;\kappa_i=\lim_{m\to\infty}\frac{1}{m\sigma}\ln\frac{\|(\mathcal{O}\mathcal{T}_{m\sigma}\mathcal{F})_i\|}{\|F_i\|} \tag{A6}$$

[3]We think that the proposition is well known, but since we did not find a proof in the literature, we refer the reader to Appendix D of [7] where we gave a simple proof for the case of periodic A considered here.

$$= \lim_{m\to\infty} \frac{1}{m\sigma} \sum_{l=1}^{m} \lambda_i(l,\sigma), \tag{A7}$$

for $1 \leqslant i \leqslant n$, where $\lambda_i(l,\sigma) \equiv \ln\|[\mathcal{O}\mathcal{T}_\sigma(\mathcal{N}\mathcal{O}\mathcal{T}_\sigma)^{l-1}\mathcal{F}]_i\|$.

Remark. The first expression (A6) of the proposition is mathematically straightforward, but is not suitable for numerical calculations, due to serious overflow and underflow problems in large time integration. To overcome this difficulty, we employ the second expression (A7) of the proposition, and we performed the calculation in the following manner in practice.

(1) Prepare $\mathcal{F}$ and let $\Lambda_i = 0$.

(2) Evolve $\mathcal{F}$ in time by T_σ, and define $\mathcal{F}' = \mathcal{T}_\sigma \mathcal{F}$.

(3) Find $\mathcal{F}'' = \mathcal{O}\mathcal{T}_\sigma \mathcal{F}$ and $\mathcal{F}''' = \mathcal{N}\mathcal{O}\mathcal{T}_\sigma \mathcal{F}$ by the Gram-Schmidt procedure:

$$F_1'' = F_1', \quad F_1''' = F_1''/\|F_1''\|,$$
$$F_i'' = F_i' - \sum_{j=1}^{i-1} (F_j''', F_i') F_j''', \quad F_i''' = F_i''/\|F_i''\|, \quad i \geqslant 2. \tag{A8}$$

(4) Add $\lambda_i = \ln\|F_i''\|$ to Λ_i.

(5) Define new $\mathcal{F} = \mathcal{F}'''$ and go back to (2).

$\Lambda_i/m\sigma$, where m is the number of iteration, gives the Lyapunov exponent Re κ_i according to the proposition, because $\mathcal{N}\mathcal{O}$ is equivalent to the Gram-Schmidt procedure.

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