Quantum chaos in compact lattice QED

B. A. Berg
Department of Physics, The Florida State University, Tallahassee, Florida 32306
and Supercomputer Computations Research Institute, The Florida State University, Tallahassee, Florida 32306

H. Markum
Institut für Kernphysik, Technische Universität Wien, A-1040 Vienna, Austria

R. Pullirsch
Department of Physics, The Florida State University, Tallahassee, Florida 32306
and Institut für Kernphysik, Technische Universität Wien, A-1040 Vienna, Austria
(Received 10 December 1998; published 12 April 1999)

Complete eigenvalue spectra of the staggered Dirac operator in quenched 4D compact QED are studied on $8^3\times4$ and $8^3\times6$ lattices. We investigate the behavior of the nearest-neighbor spacing distribution $P(s)$ as a measure of the fluctuation properties of the eigenvalues in the strong coupling and the Coulomb phase. In both phases we find agreement with the Wigner surmise of the unitary ensemble of random-matrix theory indicating quantum chaos. Combining this with previous results on QCD, we conjecture that quite generally the non-linear couplings of quantum field theories lead to a chaotic behavior of the eigenvalues of the Dirac operator. [S0556-2821(99)05409-0]

PACS number(s): 11.15.Ha, 05.45.Mt, 12.38.Gc

I. MOTIVATION

The fluctuation properties of the eigenvalues of the Euclidean lattice QCD Dirac operator have attracted much attention in the past few years. In Ref. [1] it was first shown for SU(2) lattice gauge theory that certain features of the spectrum of the Dirac operator are described by random-matrix theory (RMT). In particular the so-called nearest-neighbor spacing distribution $P(s)$, i.e. the distribution of the spacings s of adjacent eigenvalues on the ''unfolded'' scale (see below), agrees with the Wigner surmise of RMT. According to the Bohigas-Giannoni-Schmit conjecture [2], quantum systems whose classical counterparts are chaotic have a $P(s)$ given by RMT whereas systems whose classical counterparts are integrable obey a Poisson distribution $P(s)=e^{-s}$. Therefore, the specific form of $P(s)$ is often taken as a criterion for ''quantum chaos.'' However, there is no accepted proof of the Bohigas-Giannoni-Schmit conjecture yet. The field of quantum chaos is still developing and there are many open conceptual problems [3]. Applying this conjecture it was recently demonstrated that QCD is chaotic, both in the confinement and the quark gluon plasma phase [4].

A number of interesting results have been established for chaotic dynamics in classical gauge theories. Lattice gauge theories are chaotic as classical Hamiltonian dynamical systems [5]. Furthermore, it was found that the leading Lyapunov exponent of SU(2) Yang-Mills field configurations indicates that configurations corresponding to the deconfinement phase are chaotic although they are less chaotic than in the strong coupling phase at finite temperature [6]. The scaling of the maximal Lyapunov exponent in the classical continuum limit was studied in Ref. [7]: It was suggested that Abelian gauge theories behave regularly in the continuum limit whereas non-Abelian gauge theories are chaotic in the continuum, although the exact scaling relation is still an open problem. Chaos to order transitions were observed in a spatially homogeneous SU(2) Yang-Mills-Higgs system and in a spatially homogeneous SU(2) Yang-Mills Chern-Simons Higgs system [8,9]. In Ref. [8] a chaos to order transition was also seen on the quantum level, i.e. a smooth transition from a Wigner to a Poisson distribution was found. A transition in $P(s)$ from Wigner to Poisson behavior was further observed at the metal-insulator transition of the Anderson model [10]. Recently, the suppression of the characteristic manifestations of dynamical chaos by quantum fluctuations was analyzed in the context of spatially homogeneous scalar electrodynamics [11] and for a $0+1$-dimensional space-time N-component ϕ^4 theory in the presence of an external field [12]. These chaos to order transitions were seen in spatially homogeneous models and not for the full classical field theory. The relationship to properties of the quantum field theory is an interesting issue.

Here we focus on the Dirac operator for quenched 4D compact QED to search for the possible existence of a transition from chaotic to regular behavior in Abelian lattice gauge theories. In particular, we are interested in the nearest-neighbor spacing distribution of the eigenvalues of the Dirac operator across the phase transition from the strong coupling to the Coulomb phase. In the strong coupling region Abelian as well as non-Abelian lattice gauge theories are in a confined phase [13]. For compact QED this means that for couplings $\beta<\beta_c\approx1.01$ the electron is confined. However, when crossing the phase transition the conventional Coulomb phase is observed. There are some interesting properties of the two phases which can be studied in lattice QED. In the confinement phase the photons form massive bound states similar as the gluons bind to glue-balls in lattice QCD. When crossing the phase transition a massless photon is found [14] whereas in lattice QCD the gluon is a massive particle in the deconfinement region. U(1) lattice gauge theory contains Dirac magnetic monopoles in addition to photons [15] and it was demonstrated via numerical simulations that the vacuum in the confined phase is populated by monopole currents which become rare in the Coulomb phase [16]. It is an interesting question if the difference between the Coulomb phase in QED and the quark-gluon plasma phase in QCD has an influence on the level repulsion of the corresponding Dirac spectra.

II. ANALYSIS

We generated gauge field configurations using the standard Wilson plaquette action for U(1) gauge theory,

$$S_G(U_l)=\beta\sum_P (1-\cos\Theta_P), \qquad (2.1)$$

where $U_l \equiv U_{x\mu}=\exp(i\Theta_{x\mu})$, with $\Theta_{x\mu}\in[-\pi,\pi)$, are the field variables defined on the links $l\equiv(x,\mu)$. The plaquette angles are $\Theta_P=\Theta_{x,\mu}+\Theta_{x+\hat{\mu},\nu}-\Theta_{x+\hat{\nu},\mu}-\Theta_{x,\nu}$. We simulated $8^3\times4$ and $8^3\times6$ lattices at various values of the inverse gauge coupling $\beta=1/e^2$ both in the strong coupling and the Coulomb phase. Typically we discarded the first 10000 sweeps for reaching equilibrium and produced 20 independent configurations separated by 1000 sweeps for each β. Because of the spectral ergodicity property of RMT one can replace ensemble averages by spectral averages [17] if one is only interested in the bulk properties. Thus a few independent configurations are sufficient to compute $P(s)$.

On the lattice the Dirac operator $\not{D}=\not{\partial}+ie\not{A}$ for staggered fermions

$$M_{x,x'}=\frac{1}{2}\sum_{\mu=1}^{4}\eta_{x\mu}(\delta_{x+\hat{\mu},x'}U_{x,\mu}-\delta_{x-\hat{\mu},x'}U^{\dagger}_{x,\mu}) \qquad (2.2)$$

is anti-Hermitian so that all eigenvalues are imaginary. For convenience we denote them by $i\lambda_n$ and refer to the λ_n as the eigenvalues in the following. Because of $\{\not{D},\gamma_5\}=0$ the λ_n occur in pairs of opposite sign. All spectra were checked against the analytical sum rules

$$\sum_n \lambda_n=0 \quad \text{and} \quad \sum_{\lambda_n>0}\lambda_n^2=V, \qquad (2.3)$$

where V is the lattice volume [18]. We further checked our spectra by calculating the chiral condensate

$$\langle\bar{\chi}\chi\rangle=V^{-1}\left\langle\sum_n (i\lambda_n+m)^{-1}\right\rangle \qquad (2.4)$$

for $m=0.04$ and found agreement with results in the literature [19].

To construct the nearest-neighbor spacing distribution $P(s)$ from the eigenvalues, one has to ''unfold'' the spectra.

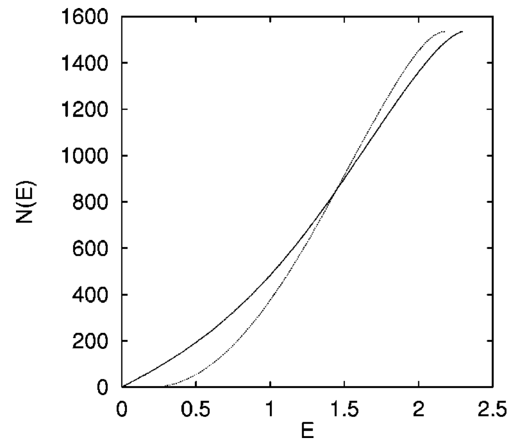

FIG. 1. Staircase function $N(E)$ representing the number of positive eigenvalues $\leqslant E$ for a typical configuration of compact U(1) theory on an $8^3\times6$ lattice in the strong coupling phase $\beta=0.90$ (solid line) and in the Coulomb phase $\beta=1.10$ (dotted line).

This procedure is a local rescaling of the energy scale so that the mean level spacing $\bar{s}$ is equal to unity on the unfolded scale [20]: One first defines the staircase function $N(E)$ to be the number of eigenvalues with $\lambda\leqslant E$. This staircase function is decomposed into an average part and a fluctuating part, $N(E)=N_{\rm av}(E)+N_{\rm fl}(E)$. The smooth average part is extracted by fitting $N(E)$ to a smooth curve, e.g. to a low-order Chebyshev polynomial. One then defines the unfolded energies to be $x_n=N_{\rm av}(E_n)$. As a consequence the sequence $\{x_n\}$ has mean level spacing equal to unity. Ensemble and spectral averages are only meaningful after unfolding. Figure 1 shows a typical staircase function for $\beta=0.90$ (strong coupling phase) and $\beta=1.10$ (Coulomb phase) on an $8^3\times6$ lattice. It exhibits a decrease of small eigenvalues due to the restoration of chiral symmetry across the transition.

III. RESULTS AND DISCUSSION

In RMT one has to distinguish between different universality classes which are determined by the symmetries of the system. So far the classification for the QED Dirac operator has not been done. Our calculations show that in the case of the staggered 4D compact QED Dirac matrix the appropriate ensemble is the unitary ensemble. Although from a mathematical point of view this is the simplest one, the RMT result for the nearest-neighbor spacing distribution is still rather complicated. It can be expressed in terms of so-called prolate spheroidal functions, see Ref. [21] where $P(s)$ has also been tabulated. A good approximation to $P(s)$ is provided by the Wigner surmise for the unitary ensemble

$$P(s)=\frac{32}{\pi^2}s^2e^{-(4/\pi)s^2}. \qquad (3.1)$$

We have simulated $8^3\times4$ lattices at $\beta=0,0.90,0.95,1.00,1.05,1.10,1.50$ and $8^3\times6$ lattices at $\beta=0.90,1.10,1.50$. All results are similar to those selected for the plots. Figure 2 shows the nearest-neighbor spacing distribution $P(s)$ for $\beta=0.90$ in the confined phase averaged

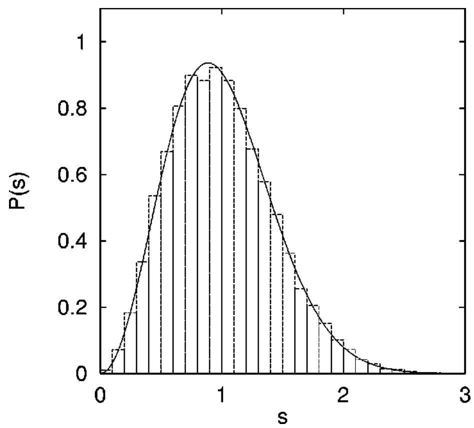

FIG. 2. Nearest-neighbor spacing distribution $P(s)$ of the Dirac operator for compact U(1) theory in the strong coupling phase for $\beta=0.90$. The histogram represents the lattice data on an $8^3\times 6$ lattice averaged over 20 independent configurations. The full curve is the Wigner distribution of Eq. (3.1) for the unitary ensemble of RMT.

over 20 independent configurations on the $8^3\times 6$ lattice compared with the Wigner surmise for the unitary ensemble of RMT of Eq. (3.1). Good agreement is found. According to the Bohigas-Giannoni-Schmit conjecture this means the system can be regarded as chaotic in the strong coupling region. Figure 3 shows the nearest-neighbor spacing distribution $P(s)$ for $\beta=1.10$ in the Coulomb phase again averaged over 20 independent configurations and compared with the Wigner surmise (3.1). The agreement of the lattice data with the RMT predictions is interpreted as a signal that quantum chaos survives the phase transition. We find no deviation up to the maximum coupling considered, $\beta=1.50$.

In the strong coupling phase the result holds down to $\beta=0$. Therefore, we tend to interpret our, as well as previous [4,1], results in the sense that the disorder of the gauge field

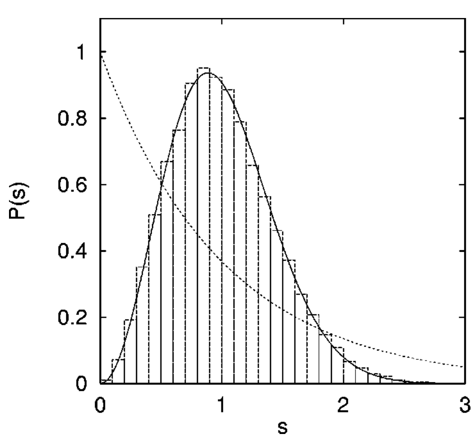

FIG. 3. Nearest-neighbor spacing distribution $P(s)$ of the Dirac operator for compact U(1) theory in the Coulomb phase for $\beta=1.10$. The histogram represents the lattice data on an $8^3\times 6$ lattice averaged over 20 independent configurations. The full curve is the Wigner distribution of Eq. (3.1) for the unitary ensemble of RMT. For comparison the Poisson distribution $P(s)=e^{-s}$ is also indicated by the dashed line.

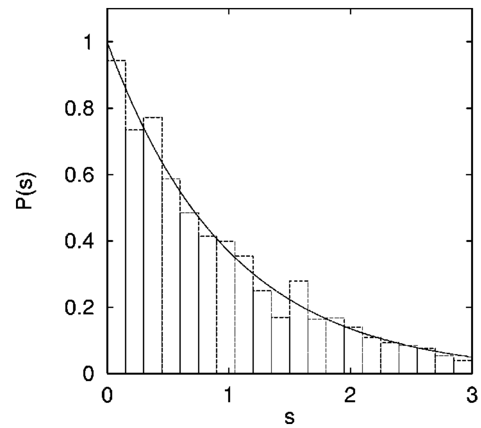

FIG. 4. Nearest-neighbor spacing distribution $P(s)$ of the analytically calculated eigenvalues of Eq. (3.2) for a free Dirac operator on a $53\times 47\times 43\times 41$ lattice (histogram) compared with the Poisson distribution $P(s)=e^{-s}$ (solid line).

configurations [5,6] is responsible for the chaotic characteristics of the spectrum of the Dirac operator. In contrast to that: The free fermion theory is non-chaotic and the corresponding nearest-neighbor spacing distribution obeys a Poisson distribution. This is illustrated in Fig. 4 where $P(s)$ is obtained from the analytical eigenvalues of the free Dirac operator on a $53\times 47\times 43\times 41$ lattice:

$$a^2\lambda^2=\sum_{\mu=1}^{4}\sin^2\left(\frac{2\pi n_\mu}{L_\mu}\right). \tag{3.2}$$

Here a is the lattice constant, L_μ is the number of lattice sites in μ-direction, and $n_\mu=0,\ldots,L_\mu-1$. We used an asymmetric lattice with L_μ being primes and restricted the range to $(L_\mu-1)/2$ instead of $L_\mu-1$ in each direction to avoid degeneracies of the free spectrum [22].

IV. CONCLUSION

We have analyzed the nearest-neighbor spacing distribution $P(s)$ of the eigenvalues of the Dirac operator in quenched 4D QED on $8^3\times 4$ and $8^3\times 6$ lattices both in the strong coupling region and in the Coulomb phase. In both phases we found excellent agreement of the lattice data with the Wigner surmise of the unitary ensemble of RMT. Our results evidence that the fermions in U(1) gauge theory show quantum chaos in the confined as well as in the Coulomb phase. Dynamical fermions are not expected to affect the Wigner distribution as has been demonstrated for SU(3) [4]. In accordance with previous findings [4,1] we conjecture that the eigenvalues of the Dirac operator of interacting quantum field theories quite generally reveal chaos due to the disorder of the gauge field configurations. The free Dirac operator, in absence of a covariant derivative and minimal gauge coupling, exhibits regular behavior.

It would be interesting to study the relationship between chaos to order transitions in classical [5–9,11,12] and quantum field theories. However, this faces several difficulties: The available investigations of classical field theories focus mainly on the gauge sector, whereas the numerical methods employed here are only efficient for the fermion sector of quantum field theory. A similar accurate determination of the eigenvalue spectrum of the gauge sector necessitates to construct the corresponding Fock space and to diagonalize high-dimensional matrices which seems to be out of reach for 4D QED or QCD. On the other hand, for the classical limit fermion sector studies of chaos have not yet been attempted.

ACKNOWLEDGMENTS

This research was partially funded by the Department of Energy under Contracts No. DE-FG02-97ER41022 and DE-FG05-85ER2500 and by the ''Fonds zur Förderung der Wissenschaftlichen Forschung'' under Contract P11456-PHY. We also acknowledge useful discussions with T.S. Biró, K. Rabitsch, and T. Wettig.

[1] M. A. Halasz and J. J. M. Verbaarschot, Phys. Rev. Lett. **74**, 3920 (1995); M. A. Halasz, T. Kalkreuter, and J. J. M. Verbaarschot, Nucl. Phys. B (Proc. Suppl.) **53**, 266 (1997).
[2] O. Bohigas, M.-J. Giannoni, and C. Schmit, Phys. Rev. Lett. **52**, 1 (1984).
[3] Reviews on the subject of quantum chaos are M. C. Gutzwiller, *Chaos in Classical and Quantum Mechanics* (Springer, New York, 1990); *Chaos and Quantum Physics*, Proceedings of the LII Session of the Les Houches School, edited by M.-J. Giannoni and A. Voros (North-Holland, Amsterdam, 1991); *Quantum Chaos: Between Order and Disorder*, edited by G. Casati and B. V. Chirikov (Cambridge University Press, Cambridge, England, 1995).
[4] R. Pullirsch, K. Rabitsch, T. Wettig, and H. Markum, Phys. Lett. B **427**, 119 (1998).
[5] T. S. Biró, C. Gong, B. Müller, and A. Trayanov, Int. J. Mod. Phys. C **5**, 113 (1994).
[6] T. S. Biró, M. Feurstein, and H. Markum, Acta Phys. Hung. New Ser.: Heavy Ion Phys. **7**, 235 (1998).
[7] L. Casetti, R. Gatto, and M. Pettini, chao-dyn/9810033; H. B. Nielsen, H. H. Rugh, and S. E. Rugh, in *ICHEP 96*, Proceedings of the 28th International Conference, Warsaw, Poland, edited by Z. Ajduk and A. Wroblewski (World Scientific, Singapore, 1997), hep-th/9611128; B. Müller, chao-dyn/9607001; H. B. Nielsen, H. H. Rugh, and S. E. Rugh, chao-dyn/9605013.
[8] L. Salasnich, Mod. Phys. Lett. A **12**, 1473 (1997).
[9] C. Mukku, M. S. Sriram, J. Segar, B. A. Bambah, and S. Lakshmibala, J. Phys. A **30**, 3003 (1997).
[10] B. L. Al'tshuler, I. Kh. Zharekeshev, S. A. Kotochigova, and B. I. Shklovskiĭ, Zh. Éksp. Teor. Fiz. **94**, 343 (1988) [Sov. Phys. JETP **67**, 625 (1988)]; B. I. Shklovskiĭ, B. Shapiro, B. R. Sears, P. Lambrianides, and H. B. Shore, Phys. Rev. B **47**, 11487 (1993).
[11] S. G. Matinyan and B. Müller, Phys. Rev. Lett. **78**, 2515 (1997).
[12] L. Casetti, R. Gatto, and M. Modugno, Phys. Rev. E **57**, R1223 (1998).
[13] K. G. Wilson, Phys. Rev. D **10**, 2445 (1974).
[14] B. Berg and C. Panagiotakopoulos, Phys. Rev. Lett. **52**, 94 (1984).
[15] A. M. Polyakov, Phys. Lett. **59B**, 82 (1975); T. Banks, R. J. Myerson, and J. Kogut, Nucl. Phys. **B129**, 493 (1977); S. Mandelstam, Phys. Rep., Phys. Lett. **23C**, 245 (1976); G. 't Hooft, in *High Energy Physics*, edited by A. Zichichi (Editrice Compositori, Bologna, 1976).
[16] T. A. DeGrand and T. Toussaint, Phys. Rev. D **22**, 2478 (1980); A. Hoferichter, V. K. Mitrjushkin, M. Müller-Preussker, T. Neuhaus, and H. Stüben, Nucl. Phys. **B434**, 358 (1995).
[17] T. Guhr, A. Müller-Groeling, and H. A. Weidenmüller, Phys. Rep. **299**, 189 (1998); J. B. French, P. A. Mello, and A. Pandey, Phys. Lett. **80B**, 17 (1978).
[18] T. Wettig (private communication).
[19] A. Hoferichter, V. K. Mitrjushkin, and M. Müller-Preussker, Phys. Lett. B **338**, 325 (1994).
[20] O. Bohigas and M. J. Giannoni, in *Mathematical and Computational Methods in Nuclear Physics*, Proceedings of the Sixth Granada Workshop, Granada, Spain, edited by F. S. Dehesa, F. M. G. Gomet, and A. Polls, Lecture Notes in Physics Vol. 209 (Springer, New York, 1984), p. 1.
[21] M. L. Mehta, *Random Matrices*, 2nd ed. (Academic, San Diego, 1991).
[22] J. J. M. Verbaarschot (unpublished).