Isospin-breaking vector meson decay constants from continuous families of finite energy sum rules

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The isospin-breaking vector meson decay constants are determined from a QCD sum rule analysis of the vector current correlator $\langle O|T(V_\mu^3 V_\nu^8)|O\rangle$, using a recently proposed implementation of the finite energy sum rule approach. The analysis employs the three-loop version of the OPE and two different families of weight functions. It is shown that the requirement of consistency between results obtained using these two different weight families leads to a rather good determination of the parameter describing the deviation of the $D=6$ condensate term in the OPE from its vacuum saturation value, and that the ability to determine this value has non-trivial numerical consequences on the analysis. The phenomenological relevance of the results to experimental extractions of the isoscalar and isovector spectral functions in $e^+e^-\to$hadrons, the extraction of the strange quark mass and the determination of the 6^{th} order chiral low energy constant, Q, is also briefly discussed. [S0556-2821(99)06009-9]

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I. INTRODUCTION

Because the neutral ($a=3,8$) members of the $SU(3)_F$ octet of vector currents, $J_\mu^a=\bar{q}\gamma_\mu(\lambda^a/2)q$ (with λ^a the usual Gell-Mann matrices), couple to fermions in the standard model, it is possible to use experimental data on the spectral functions associated with correlators involving these currents to determine certain quantities of phenomenological interest. For example, defining the scalar correlators, $\Pi^{ab}(q^2)$, by means of

$$i\int d^4x \exp(iqx)\langle 0|T[J_\mu^a(x)J_\nu^b(0)]|0\rangle \equiv (q_\mu q_\nu - q^2 g_{\mu\nu})\Pi^{ab}(q^2), \tag{1}$$

and the corresponding spectral functions, $\rho^{ab}(q^2)$, as usual, by $\rho^{ab}(q^2)=(1/\pi)\ \mathrm{Im}\ \Pi^{ab}(q^2)$, one finds that (1) integrating the difference $\rho^{33}(q^2)-\rho^{88}(q^2)$ with the weight function occurring naturally (due to kinematics) in the finite energy sum rule (FESR) treatment of hadronic τ decays [1] produces a sum rule from which one can, in principle, determine the running strange quark mass, $m_s(\mu)$ [2], and (2) integrating the same difference $\rho^{33}(q^2)-\rho^{88}(q^2)$ with weight function $w(s)=1/s$ produces a sum rule from which one can extract one of the 6^{th} order low-energy constants (LEC's), Q, appearing in the 6^{th} order version of the effective chiral Lagrangian [3]. [See Ref. [4] for a discussion of chiral perturbation theory (ChPT) and the method of effective chiral Lagrangians in general, Ref. [5] for the form of the $\mathcal{O}(q^6)$ terms in the effective Lagrangian in the most general case, and Ref. [6] for both a discussion of the subset of these terms surviving when one restricts one's attention to vacuum correlators and a definition of Q.]

Of course, J_μ^3 and J_μ^8 do not couple separately in the standard model, but only in the combination

$$J_\mu^{EM}=J_\mu^3+\frac{1}{\sqrt{3}}J_\mu^8 \tag{2}$$

which gives the light quark (u,d,s) part of the electromagnetic (EM) current. Thus, what is measured in $e^+e^-\to$hadrons is not the desired quantities, ρ^{33} and ρ^{88}, separately, but the combination

$$\rho^{EM}(q^2)=\rho^{33}(q^2)+\frac{2}{\sqrt{3}}\rho^{38}(q^2)+\frac{1}{3}\rho^{88}(q^2). \tag{3}$$

In the isospin symmetry limit, ρ^{38} would vanish and, since one could then classify the final hadronic states according to their G parity, it would be straightforward to separate the isovector (33) and isoscalar (88) components of the EM spectral function.

In the presence of isospin breaking, however, this process is no longer so straightforward. The most obvious experimental signature of the presence of isospin breaking in $e^+e^-\to$hadrons is the interference shoulder in the $e^+e^-\to\pi^+\pi^-$ cross section in the ρ-ω region [7]. The $e^+e^-\to\omega\to\pi^+\pi^-$ contribution to ρ^{EM} is clearly, to leading order in isospin breaking, to be associated with ρ^{38} and hence is usually removed explicitly in analyzing the data. This removal is accomplished by (1) fitting the parameters of a model for the total $e^+e^-\to\pi^+\pi^-$ amplitude, consisting of ρ,ω and possible background contributions, to the experimental data, (2) removing the ω contribution once the fit has

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been performed and (3) squaring the modulus of the remaining ρ contribution and identifying this result with the ρ contribution to ρ^{33} [8].

While this procedure *does* remove one source of isospin-breaking contamination from the nominal ρ^{33} so extracted, it is easy to see that other such contaminations still remain. Indeed, once one allows isospin breaking, the physical ρ and ω are admixtures of pure isovector and isoscalar states, the size of the admixture of the ''wrong'' isospin component being governed by the scale of isospin breaking. As a consequence, the intermediate ρ contribution to ρ^{38}, for example, does not vanish. In fact, if one denotes the pure isovector ρ state by $\rho^{(0)}$ and the pure isoscalar ω state by $\omega^{(0)}$, one expects ρ contributions to ρ^{38} from two sources: (1) that due to $\rho^{(0)}$-$\omega^{(0)}$ mixing (a one particle reducible contribution, with coupling of the isovector current to the $\rho^{(0)}$ component and the isoscalar current to the $\omega^{(0)}$ component of the ρ) and (2) that due to the ''direct'' [one particle irreducible $(1PI)$] coupling of the $\rho^{(0)}$ component to the isoscalar current (such a coupling being unavoidable in any hadronic effective Lagrangian based on QCD). Thus, removing the contribution due to the intermediate state ω from the $e^+e^- \to \pi^+\pi^-$ cross section, while removing part of the ρ^{38} contribution, does not remove it all. One is then left with, *not* the desired quantity, ρ^{33}, but rather with a combination of ρ^{33} and the residual part of ρ^{38} associated with the intermediate ρ state (plus possible additional such contaminations from elsewhere in the spectrum). Similar isospin-breaking flavor 38 contributions exist for $e^+e^- \to \omega \to 3\pi$, complicating the extraction of the isoscalar spectral function.

Corrections for such isospin-breaking effects, *which are unavoidable as long as no process exists in which only one of the two neutral flavor currents couples*, are thus necessary if one wishes to perform phenomenological analyses of the type mentioned above. Such corrections would also be important in performing precision tests of conserved vector current (CVC), which involve a comparison of ρ^{33} and the charged isovector spectral function $\rho^{(+)}$, measured in hadronic τ decays (see, for example, Ref. [9]).

It is easy to see that, to be able to make these corrections (at least in the region below $s \sim 2$ GeV^2, where the EM spectral function is, experimentally, resonance dominated), it is sufficient to determine the isospin-breaking vector meson decay constants. Let us first clarify the notation. We define the flavor 3 and 8 vector meson decay constants via

$$\langle 0|J^a_\mu|V(k)\rangle = m_V F^a_V \epsilon_\mu(k) \tag{4}$$

where $V=\rho,\omega,\phi,\ldots$, $\epsilon_\mu(k)$ is the vector meson polarization vector, and $a=3,8$. $F^{(3)}_\rho$, $F^{(8)}_\omega$ and $F^{(8)}_\phi$ are non-zero in the limit of isospin symmetry; $F^{(8)}_\rho$, $F^{(3)}_\omega$ and $F^{(3)}_\phi$ are zero in the absence of isospin breaking. The experimentally determined EM decay constants, F^{EM}_V, are then given by

$$F^{EM}_V = F^3_V + \frac{1}{\sqrt{3}} F^8_V . \tag{5}$$

Thus, for example, the broad ρ contribution to ρ^{EM}, usually taken to be associated purely with ρ^{33}, consists not only of a flavor 33 contribution proportional to $[F^3_\rho]^2$, but also of a flavor 38 contribution proportional to $(2/\sqrt{3})F^3_\rho F^8_\rho$. The ω contribution to ρ^{EM}, similarly, contains both a flavor 88 part proportional to $\frac{1}{3}[F^8_\omega]^2$ and a flavor 38 part proportional to $(2/\sqrt{3})F^3_\omega F^8_\omega$. The flavor 38 parts, in both cases, are present only due to isospin breaking, and have to be removed from the experimental ρ and ω contributions to ρ^{EM} in order to obtain the corresponding ρ contribution to ρ^{33} and ω contribution to ρ^{88}.

It is important to stress at this point that the conventional ''few-percent'' rule of thumb for estimating the size of isospin-breaking effects, which might lead one to expect such effects to be numerically negligible, is inapplicable in the cases involving $\rho^{33}(q^2) - \rho^{88}(q^2)$ discussed above. This is true for a number of reasons. First, because the difference of spectral functions is itself flavor breaking, the *relative* importance of isospin breaking is enhanced by a factor of ~ 3, characteristic of the inverse of the scale of flavor breaking. Second, the effect of ρ-ω mixing naturally produces corrections for the ρ contribution to ρ^{33} and ω contribution to ρ^{88} which are opposite in sign; the effects therefore add when the difference is taken. Finally, there is a natural numerical enhancement which makes the size of the correction needed to remove the ρ^{38} part of the ω contribution to ρ^{EM}, and hence isolate ρ^{88}, larger than naively expected [10]. The latter two points are discussed in somewhat more detail in Sec. II below.

In what follows, we evaluate the isospin-breaking vector meson decay constants by performing a QCD sum rule analysis of the isospin-breaking vector current correlator Π^{38}. The vector meson spectral contributions are, in this case, proportional to $F^3_V F^8_V$, so that a determination of this product, in combination with the experimental determination of F^{EM}_V, given in terms of F^3_V and F^8_V above, allows a separate determination of F^3_V and F^8_V. The rest of the paper is organized as follows. In Sec. II we discuss qualitative expectations for the pattern of isospin-breaking corrections based on the structure of the leading (chiral) order terms in the vector meson effective chiral Lagrangian, as well as semi-quantitative expectations for their probable scale which can, using this perspective, be obtained from experimental data. In Sec. III, we discuss briefly the form of QCD sum rules employed (a version of the FESR) and the advantages of this approach. In Sec. IV, we discuss the input used for the hadronic and operator product expansion (OPE) sides of the sum rules employed and present our results. Some advantages of the approach, in particular in relation to the handling of the $D=6$ terms in the OPE of the 38 correlator, will also be discussed here. Finally, in Sec. V we summarize and make some brief comments on the phenomenological significance of our results.

II. CHIRAL CONSTRAINTS AND THE SCALE OF ISOSPIN-BREAKING CORRECTIONS

ChPT provides both an underlying conceptual framework and systematic procedure [11] for writing down the most

general effective Lagrangian relevant to a given set of hadronic states which fully incorporates the symmetries of QCD and implements the broken symmetries (such as chiral symmetry) with the same pattern of symmetry breaking as occurs in QCD. Although the resulting effective Lagrangian, $\mathcal{L}_{eff}$, is non-renormalizable, it is possible to formulate the theory in such a way that only a finite number of terms appears to a given order in the chiral, or low-energy, expansion. (For so-called ''heavy'' fields, those whose masses are non-vanishing in the chiral limit, this requires a reformulation in terms of velocity-dependent fields [12,13].) The leading order terms in this expansion [in which light quark masses, m_q, $q=u,d,s$, count as $\mathcal{O}(q^2)$, with q representing some soft external momentum], incorporate the leading constraints associated with either chiral symmetry or the symmetry pattern of its breaking.

The framework of the heavy field implementation of ChPT given in Ref. [13] for the vector mesons and their interactions with the members of the pseudo Goldstone boson pseudoscalar octet provides two useful pieces of information about the scale of isospin breaking in the vector meson sector. First, note that the leading (in chiral order) term in $\mathcal{L}_{eff}$ generating isospin-breaking mixing involves one power of the quark mass matrix and no derivatives [13], and hence produces no off-diagonal contributions to the wave function renormalization matrix. The *leading order* mixing effect thus results in a physical ρ and ω basis which is related by a rotation to the original pure isospin $\rho^{(0)}$, $\omega^{(0)}$ basis. At this order, therefore, the ''wrong'' isospin $\omega^{(0)}$ admixture in the physical ρ state is equal in magnitude, but opposite in sign, to the $\rho^{(0)}$ admixture in the physical ω state, a pattern which should remain approximately valid, even at higher order. The second point concerns the vector meson decay constants, which are necessarily $SU(3)_F$ symmetric in the chiral limit. When one considers the effects of flavor- and isospin-symmetry breaking (recalling that both are generated by the quark mass matrix, and hence both are produced by the same set of terms in the effective Lagrangian), there are two potential sources of such breaking. The first is that associated with higher order terms, involving at least one power of the quark mass matrix, coupling the external photon field to the vector meson nonet, and the second that induced by the leading quark-mass-dependent term, responsible for mixing, discussed above. The leading order mixing effect simply reproduces the standard leading order $SU(3)_F$ mixing analysis [13], leading to near ideal mixing in the vector meson sector. As is well known, the combination of ideal mixing and neglect of flavor breaking in the EM couplings of the unmixed states leads to the prediction that the vector meson EM decay constants, measured experimentally in $V\to e^+e^-$ [14], should be in the proportions $F_\rho^{(0)}:F_\omega^{(0)}:F_\phi^{(0)}=3:1:-\sqrt{2}$, where the superscript (0) indicates that the couplings refer to the ideally mixed, but isospin pure, vector meson states. That this prediction is borne out by experiment represents empirical evidence that, despite the potential $SU(3)_F$-breaking photon coupling contributions being of the same formal order as effects induced by mixing, the former are numerically suppressed relative to the latter. Since flavor breaking and isospin breaking are generated by the same terms in the effective Lagrangian, this implies that isospin breaking in the vector meson decay constants should also be dominated by mixing effects.

If we take this point of view then, up to sub-leading corrections, we find, for the physical ρ and ω decay constants, now including isospin breaking and taking into account the relation $F_\rho^{(0)}\simeq 3F_\omega^{(0)}$,

$$F_\rho^{EM}=F_\rho^{(0)}-\epsilon F_\omega^{(0)}\simeq F_\rho^{(0)}\left(1-\frac{\epsilon}{3}\right)$$

$$F_\omega^{EM}=F_\omega^{(0)}+\epsilon F_\rho^{(0)}\simeq F_\omega^{(0)}(1+3\epsilon), \tag{6}$$

where ϵ is the leading order mixing angle, defined via

$$\rho=\rho^{(0)}-\epsilon\,\omega^{(0)},\quad \omega=\omega^{(0)}+\epsilon\,\rho^{(0)}. \tag{7}$$

We note two relevant features of these results: (1) because of the dominance by mixing, the corrections required to convert the pure isovector $F_\rho^{(0)}$ coupling to the experimental F_ρ^{EM} coupling is opposite in sign to that required to convert the pure isoscalar $F_\omega^{(0)}$ coupling to the F_ω^{EM}, and (2) because of the pattern of ideal mixing and the numerical suppression of the isoscalar current relative to the isovector current in J_μ^{EM}, the magnitude of the correction is a factor of 9 larger in the ω than in the ρ case.

In view of the discussion above, a rough idea of the size of the isospin-breaking vector meson decay constants can be obtained by analyzing experimental data on ρ-ω interference, ignoring all non-mixing effects. Although crude, this estimate will provide a qualitative constraint for our later sum rule analysis.

In order to obtain the parameter ϵ describing ρ-ω mixing at leading order, it is sufficient to determine the off-diagonal element, $\Pi_{\rho\omega}$, of the vector meson self-energy matrix. In the past, values for $\Pi_{\rho\omega}$ around ~ -4000 MeV^2 have been quoted, based on simplified analyses of $e^+e^-\to\pi^+\pi^-$ data in the interference region which effectively assume that the one-particle irreducible $\omega^{(0)}\pi^+\pi^-$ vertex is zero, even in the presence of isospin breaking. Since effective operators which generate such a coupling exist in the vector meson effective Lagrangian, however, this assumption is unphysical (in the sense of being incompatible with QCD). Once one includes contributions to the $\omega\to\pi\pi$ amplitude generated both by ρ-ω mixing *and* the 1PI vertex (whose strength we will denote by $g^{(0)}_{\omega\pi\pi}$), the analysis of the experimental data is somewhat more complicated but, in principle, allows a separate determination of both $\Pi_{\rho\omega}$ and the isospin-breaking ratio of couplings of the isospin pure states $G=g^{(0)}_{\omega\pi\pi}/g^{(0)}_{\rho\pi\pi}$ [15,16].

An important general conclusion, which follows from the analysis framework developed in Ref. [15], is that the smallness of previously quoted errors for $\Pi_{\rho\omega}$ is an artifact of the unphysical assumption $G=0$, and does not survive the more general treatment. It is worth outlining why this is the case since, in so doing, the reason for the difficulty in improving the experimental situation sufficiently to really pin down the mixing contribution will become clear.

The contribution of the *physical* (i.e., mixed-isospin) ω to the amplitude for $e^+e^-\to\pi^+\pi^-$ is obtained experimentally by determining the timelike pion form factor, $F_\pi(q^2)$, in the interference region and fitting it to a form

$$F_\pi(q^2)\propto\left[\frac{1}{q^2-m_\rho^2}+\frac{A^{i\phi}}{q^2-m_\omega^2}\right]+\ \text{background} \tag{8}$$

where m_V^2 are the complex pole positions, $m_V^2=\hat{m}_V^2-i\hat{m}_V\Gamma_V$, and the fit parameter, ϕ, is known as the ''Orsay phase.'' The ω contribution in Eq. (8) is generated by the coupling of the physical ω to $\pi^+\pi^-$ which, as discussed above, has two sources: 1PR ($\rho^{(0)}$-$\omega^{(0)}$ mixing) and 1PI (associated with the $\omega^{(0)}\pi\pi$ vertex). The physical coupling is given, in terms of these contributions, by

$$g_{\omega\pi\pi}=g^{(0)}_{\omega\pi\pi}+\epsilon\, g^{(0)}_{\rho\pi\pi}, \tag{9}$$

where, as usual, the superscript (0) indicates couplings of the isospin-pure states. In the (physically plausible) approximation in which one assumes saturation of the imaginary part of $\Pi_{\rho\omega}$ by $\pi\pi$ intermediate states, one finds

$$\text{Im}\ \ \Pi_{\rho\omega}(m_\rho^2)=-G\hat{m}_\rho\Gamma_\rho \tag{10}$$

and hence, in the narrow ρ-ω interference region,

$$\Pi_{\rho\omega}\simeq\tilde{\Pi}_{\rho\omega}-iG\hat{m}_\rho\Gamma_\rho \tag{11}$$

where $\tilde{\Pi}_{\rho\omega}$ is now real. The mixing angle ϵ is then given by [15]

$$\epsilon=\frac{\Pi_{\rho\omega}(m_\rho^2)}{m_\omega^2-m_\rho^2}\equiv -iz\tilde{T}-zG, \tag{12}$$

where

$$z\equiv\frac{i\hat{m}_\rho\Gamma_\rho}{m_\omega^2-m_\rho^2},\quad \tilde{T}\equiv\frac{\tilde{\Pi}_{\rho\omega}(m_\rho^2)}{\hat{m}_\rho\Gamma_\rho}. \tag{13}$$

One then finds, upon substitution of Eq. (12) into Eq. (9), that

$$g_{\omega\pi\pi}=\left[G(1-z)+\frac{\tilde{\Pi}_{\rho\omega}(m_\rho^2)}{i\hat{m}_\rho\Gamma_\rho}\right]g^{(0)}_{\rho\pi\pi}. \tag{14}$$

In many places in the literature, the approximation $m_\omega^2-m_\rho^2\simeq im_\rho\Gamma_\rho(z=1)$ is employed. Since Re $z\simeq 1$ and Im z is small ($\sim$.2–.3), this approximation (which was, in fact, made uniformly in analyses previous to the discussion of Ref. [15]) might appear rather safe. If this were true, then the effect of G in Eq. (14) would cancel exactly [17], and the experimental data would determine the real part of $\Pi_{\rho\omega}$ in the interference region with the usually quoted errors [$\tilde{\Pi}_{\rho\omega}(m_\rho^2)=-3844\pm 271$ MeV^2; see Refs. [16,18,19] and earlier references cited therein]. Unfortunately, it turns out that the approximation is both misleading and unreliable.

TABLE I. Results for the $D=6$ VSA-violation parameter, ρ_{red}, and the spectral strength parameters, f_ρ, f_ω, f_ϕ and $f_{\rho'\omega'}$, as a function of the isospin-breaking mass ratio, r. The first line, for each value of r, corresponds to the results obtained using the single-pinch weight family, the second line to those obtained using the double-pinch family. The units of f_V are GeV^2.

r	ρ_{red}	$f_\rho(\times 10^3)$	$f_\omega(\times 10^3)$	$f_\phi(\times 10^3)$	$f_{\rho'\omega'}(\times 10^3)$
0.251	1.02	2.3	1.7	−0.28	−0.020
		2.3	1.7	−0.28	−0.020
0.288	1.15	2.6	2.0	−0.32	−0.026
		2.6	2.0	−0.32	−0.026
0.325	1.28	2.9	2.2	−0.36	−0.032
		2.9	2.2	−0.36	−0.032

The reason is that, although z is approximately real and near 1,$(1-z)$ is dominantly imaginary. Since the denominator of the second term in Eq. (14) is also dominantly imaginary, the two terms add nearly constructively. Were the phases of these terms to be actually identical, it would be impossible to separate them experimentally, regardless of how precise the data. Fortunately, there is a small phase difference which, at least in principle, means that a determination, with sufficient accuracy, of both the magnitude, A, and phase, ϕ, of the ω contribution to F_π, would allow separate determination of G and $\tilde{\Pi}_{\rho\omega}$. From this, one would be able to reconstruct $\Pi_{\rho\omega}$ and hence determine ϵ. The smallness of the phase difference, however, turns out to severely limit the accuracy attainable using current experimental information. If one takes the updated numerical analyses of Ref. [16], for example, one finds that values of $\tilde{\Pi}_{\rho\omega}$ between -4000 and -8000 MeV^2 are allowed (with a central value ~ -6800 MeV^2), and that, while the central extraction for G is moderately large, $\sim$.1,$G=0$ is only 2.5σ distant. A significant improvement in this situation would require a significant reduction of the errors in the determination of the Orsay phase. The prospects for such an improvement at any time in the near future are remote, at present.

Although present experimental accuracy allows one to place only rather weak constraints on ϵ, we can, nonetheless, use the range of values obtained in Refs. [15,16] to set a rough scale for the size of those corrections required to go from F_ρ^{EM} to F_ρ^3 and from F_ω^{EM} to F_ω^8. Using the central values for the four fits given in Table I of Ref. [16], one finds that F_ρ^3 is *less than* F_ρ^{EM} by between 0.3% and 3.8% (the former corresponding to fixing $G=0$ by hand, the latter to the MOW and A solutions contained in Table I of Ref. [16]) and F_ω^8 *greater than* F_ω^{EM} by between 2.6% and 24.6%. We will see that the solutions obtained below via the sum rule analysis satisfy these rather loose constraints.

III. QCD SUM RULES AND THE CHOICE OF THE FESR METHOD

As is well known, the properties of unitarity and analyticity lead to the existence of (appropriately subtracted) dispersion relations for typical hadronic correlators, $\Pi(q^2)$. The

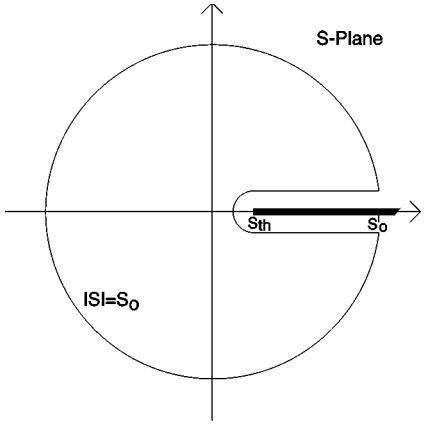

FIG. 1. The FESR ''Pac-man'' contour.

term ''QCD sum rules'' describes those versions of these relations in which kinematic restrictions allow one to take advantage of the asymptotic freedom of QCD, and hence techniques based on the OPE/perturbative QCD (PQCD).

The most common [Shifman-Vainshtein-Zakharov (SVZ)] implementation of this approach [20] involves Borel transformation of the original Cauchy representation, which is of the form (up to possible subtractions)

$$\Pi(q^2)=\int_{s_{th}}^{\infty} ds\,\frac{\rho(q^2)}{s-q^2}, \tag{15}$$

with s_{th} the lowest physical threshold, and ρ the relevant spectral function. The left-hand side (LHS), for q^2 large and spacelike, is to be computed using the OPE/PQCD, the RHS using measured spectral data and/or some spectral ansatz. The effect of the Borel transform is to (1) replace the weight $1/(s-q^2)$ on the RHS of Eq. (15) with $\exp(-q^2/M^2)$ (where M, the Borel mass, is a parameter of the transformation), (2) destroy subtraction terms and (3) create a factorial suppression of the contributions of higher dimensional operators on the OPE side of the equation $[c/(Q^2)^n \to c/(n-1)!M^{2n}]$. On the hadronic side one would thus prefer to work with small M, in order to suppress contributions from the poorly known large-s part of the spectral function, and on the OPE side, to work with large M, in order to suppress the contributions of unknown higher dimension condensates. Since one cannot simultaneously satisfy both conditions, one must hope to find a ''stability window'' in M, i.e. a range of values for which neither requirement is too badly violated. Typically, as a result of this compromise, neither the contribution from the large-s part of the spectrum nor that from the highest dimension operator retained on the OPE side is negligible [20–23].

An alternate approach, based on Cauchy's theorem, is the method of FESR's. A convenient integration contour is that of Fig. 1, where the radius, s_0, is taken large enough that the OPE, to the order available, is reliable in the spacelike region of the circle. The resulting sum rule is then generically of the form

$$\frac{-1}{2\pi \mathrm{i}}\oint_C dq^2\, w(q^2)\Pi(q^2)=\int_{s_{th}}^{s_0} dq^2\, w(q^2)\rho(q^2), \tag{16}$$

where $w(q^2)$ is any function analytic in the integration region, and C denotes the circular part of the contour, traversed counterclockwise (from above to below the cut). The OPE is to be used on the LHS, spectral data and/or a spectral ansatz on the RHS. The most common choice of weight has been $w(s)=s^k$, with $k=0,1,2,\ldots$ [24,25], though the standard theoretical treatment of hadronic τ decays involves a more complicated weight determined by kinematics [1].

For the discussion which follows, it is important to maintain the distinction between ''local'' and ''semi-local'' duality. The OPE for a typical hadronic correlator is expected to be reliable, not only for q^2 large and spacelike, but also for q^2 on any circle of sufficiently large radius in the complex q^2-plane, apart possibly from a region of hadronic size about the timelike real axis (where confinement is expected to become important) [26]. ''Local duality'' is the postulate that, at scales $q^2 \sim s_0$ for which resonance separation is small compared to the typical resonance widths, the region of validity of the OPE extends all the way down to the real timelike axis. The hadronic spectral function in this region is thus identical to that obtained using the OPE. ''Semi-local duality'' refers to the idea that, at somewhat lower (''intermediate'') scales, where local duality is no longer valid, nonetheless, averaged over some range of (timelike) momenta, the mean values given by using either the actual hadronic spectral function or the OPE version thereof should be the same. It is important to understand that, empirically, the condition that resonance spacing be *much* smaller than typical resonance widths is crucial to the validity of local duality. Indeed, one can test local duality using various FESR's in the case of the isovector vector channel, for which the hadronic spectral function is very accurately measured in hadronic τ decays [9]. One finds that, even at scales as large as $m_\tau^2 \simeq 3.2$ GeV^2 (where resonance widths and separations are comparable), and even though the experimental spectral function appears rather featureless in this region, nonetheless, local duality is rather poorly satisfied [27].

As noted above, in the SVZ approach, the location of the stability window for most analyses is such that one cannot avoid non-trivial contributions from the intermediate- and high-s part of the spectrum. This is a problem because, typically, in the intermediate region, the qualitative form of the spectral function is either not known or, if known, involves too many free parameters to be tractable, given the limited amount of information available in the truncated OPE [23]. Conventionally, this problem is dealt with by employing a spectral ansatz in which (1) the low-s region is assumed to be dominated by one or two low-lying resonance contributions and (2) the intermediate- and high-s region is approximated using local duality, which one assumes to start at some ''continuum threshold,'' s_0. It is well known that this form of ''continuum ansatz'' represents a rather crude approximation, and hence can create significant uncertainties in the analysis if the continuum contributions are large for M values in the stability window.

A similar problem exists for the integer-power weighted FESR's. In principle, one would like to choose s_0 as large as possible, in order to improve convergence on the OPE side. In practice, however, the spectral ansatz will be intractable unless the s_0 chosen lies not too far into the intermediate region. The possibility of working at such intermediate scales can, even so, represent a practical advantage in cases where the stability window of the SVZ analysis lies at relatively low M (e.g., $M \sim 1$ GeV^2, as found for many applications in the literature). Unfortunately, this advantage is usually more than offset by increased uncertainties associated with the use of local duality in the intermediate region. This increase results from the fact that, on the circular part of the contour, the region near the timelike real axis does not have the exponential suppression present for ''continuum'' contributions in the SVZ approach. The errors that result can be quantified in the case of the isovector vector channel, where the hadronic spectral function is known experimentally. As shown in Ref. [27], the errors in integer-power weighted FESR's, even at scales as high as m_τ^2, can be very large, despite the fact that the OPE at this scale is both dominated by the leading ($D=0$) perturbative term and rather rapidly converging.

This problem, however, is not intrinsic to the FESR approach. Indeed, at least one non-integer-power-weighted FESR is known to be very well satisfied: that giving the hadronic τ decay widths in terms of an integral, over the circle of radius $s_0=m_\tau^2$, of the product of the OPE for the isovector vector current correlator and the weight function $w_\tau(s)=(1-s/m_\tau^2)^2(1+2s/m_\tau^2)$ [where the dominant input parameter in the OPE representation is $a(m_\tau^2)$, which can be taken as obtained by running the value measured at the Z mass down to the τ scale]. The reason for the success of this sum rule is simple: the juncture of the cut and circular portions of the contour corresponds to the edge of hadronic phase space and hence, because of kinematics, the weight function $w_\tau(s)$ has a (double) zero at $s=m_\tau^2$, which suppresses contributions from that portion of the circle, C, near the real timelike axis for which the OPE representation of the correlator is unreliable (at intermediate scales like m_τ^2) [1]. This suggests that, in implementing FESR's in other channels, one should restrict one's attention to weight functions having a zero at $s=s_0$. In Ref. [27] it was shown that, in the isovector vector channel, where one can check the procedure explicitly, weight functions of of the forms

$$w_s(s)=\left(1-\frac{s}{s_0}\right)\left(1+A\frac{s}{s_0}\right), \tag{17}$$

$$w_d(s)=\left(1-\frac{s}{s_0}\right)^2\left(1+A\frac{s}{s_0}\right), \tag{18}$$

having, respectively, single and double zeros at $s=s_0$, both produce extremely well-satisfied FESR's, for a wide range of values of s_0 and the continuous parameter, A. In addition, using *only* the OPE representation, for a range of A and s_0, and fitting the parameters of a sum-of-resonances ansatz to this representation, results in a very good reconstruction of the hadronic spectral function, including a determination of the ρ decay constant accurate to within a few percent [27]. In what follows, we will refer to the families $w_s(s)$ and $w_d(s)$, as single-pinch and double-pinch weights, respectively. The freedom to vary A plays a role analogous to that of the variation of M within the stability window in a SVZ-style analysis. An additional advantage of the FESR approach, at least if one wishes to determine not just the parameters of the lowest resonance in the channel but also those of higher resonances, is that the weight function can be arranged to be larger in the second resonance region than in the first.

In what follows, in light of its success in the isovector vector channel, we will investigate the isospin-breaking vector current correlator, Π^{38}, defined above, in the FESR framework. As usual, we will work at scales as high as possible, compatible with the constraint of having a tractable and physically sensible spectral ansatz for $s<s_0$. Since little is known about the vector meson resonance spectrum beyond the second excited resonance region, and since including even the second excited resonance region would lead to a spectral ansatz with more parameters than are generally tractable for the present analysis, we are forced to work at scales no higher than ~ 2.8 GeV^2. Since the separation of the first and second excited vector meson resonance regions is comparable to the resonance widths (the ρ' and ω' lie at 1419 and 1452 MeV, the ρ'' and ω'' at 1723 MeV and 1649 MeV, respectively [14]), it is clear that, at these scales, we are not yet in the region of the validity of local duality, making use of the single- and double-pinch families crucial to the reliability of the analysis. In order to maintain as good a convergence as possible on the OPE side of the two sum rule families, while at the same time allowing enough variation in s_0 to get a good determination of the parameters of the spectral ansatz, we also restrict our attention to scales, s_0, greater than 2 GeV^2.

IV. DETAILS OF THE ANALYSIS

Since the general framework to be employed in the analysis has been outlined in the previous section, it remains only to discuss the input required on the hadronic and OPE sides of the various sum rules.

We begin with the hadronic side. We take, as our ansatz for the hadronic spectral function, a sum of resonance contributions. For the scales used in the analysis, the resonances present in the region of the hadronic spectral integral are the ρ, ω, ϕ, ρ' and ω'. (Although the tails of the ρ'' and ω'' intrude slightly into the hadronic integration region for s_0 near 2.8 GeV^2, their contributions are strongly suppressed by the zeros in the weight functions. We have checked that including an effective, combined ρ''-ω'' contribution in the spectral ansatz has negligible effect on the extracted ρ, ω and ϕ spectral strength parameters.) We thus include contributions, written in terms of Breit-Wigner resonance forms, for all these resonances. Because the separation of the ρ' and ω' is much smaller than either of their widths, and also to reduce the number of free parameters in the spectral ansatz, we have combined the latter two contributions. The strong overlap of the two resonances would, in any case, prevent

one from being able to sensibly extract separate ρ' and ω' strength parameters by means of any sum rule analysis of Π^{38}.

The spectral ansatz then has the form

$$\rho^{38}(q^2)=\frac{1}{4\sqrt{3}}[f_\rho\hat{\delta}(q^2-m_\rho^2)-f_\omega\hat{\delta}(q^2-m_\omega^2) + f_\phi\hat{\delta}(q^2-m_\phi^2)+f_{\rho'\omega'}\delta(q^2-\overline{m^2}_{\rho'\omega'})], \quad (19)$$

where

$$\hat{\delta}(q^2-m^2)\equiv\frac{1}{\pi}\frac{m\Gamma}{(q^2-m^2)^2+m^2\Gamma^2} \quad (20)$$

(with Γ the width of the resonance in question). This expression reduces to $\delta(q^2-m^2)$ in the narrow width approximation (NWA). The minus sign in front of f_ω and the factor of $1/4\sqrt{3}$ are conventional; inclusion of the former ensures that f_ω and f_ρ become equal in the limit that the spectral contributions in the ρ-ω region are generated entirely by leading order ρ-ω mixing. For the combined ρ'-ω' contribution we have taken average values for the effective mass and width. f_ρ, f_ω, f_ϕ and $f_{\rho'\omega'}$ are free parameters, to be determined from the matching of hadronic and OPE sides of the single- and double-pinch sum rules, for a range of s_0, A values.

A few comments are in order concerning the form of the ansatz above and the physical meaning of the parameters to be extracted from the analysis which follows.

The first concerns the need for the inclusion of a ϕ contribution. Note that the correlator Π^{38} is very closely related to that, $\Pi^{\rho\omega}$, obtained by dropping the strange part of the hypercharge current from Π^{38} (the OPE's are, in fact, identical to three-loop order). The latter correlator has been studied in a number of earlier SVZ-style analyses [28–31]. In the earliest of these, the NWA was employed for all resonances, and no ϕ contribution included in the spectral ansatz [28,29]. As pointed out in Ref. [30], however, the existence of significant cancellations between the NWA ρ and ω contributions (which would be exact in the limit of mixing dominance and equality of ρ and ω masses) means that a ϕ contribution, even if significantly smaller than the *individual* ρ and ω contributions, could nonetheless be important. Performing the sum rule analysis with a ϕ contribution included shows that this is indeed the case [30]. Including the ϕ contribution in the spectral ansatz also cured an unphysical feature of the solutions obtained earlier, which did not include it [30]. The analysis of Ref. [30], however, still employed the NWA for all resonances.

The second point concerns the need to incorporate the ρ width into the analysis. Because, again, of the high degree of cancellation between the NWA ρ and ω contributions, it was pointed out that the precise degree of this cancellation might well be sensitive to whether or not the difference between the ρ and ω widths was retained in the spectral ansatz [31]. The subsequent analysis of Ref. [31] showed that this is, indeed, the case: the spectral parameters, f_V, decrease by factors ~ 6 when one employs the physical widths in place of the NWA.

The third point concerns the interpretation of the higher resonance strength parameters, f_ϕ and $f_{\rho'\omega'}$. It is, of course, very natural to take the spectral function to be resonance dominated. Moreover, the near-threshold region of the spectral function has been computed to two loops in ChPT [32], and one can see from this result that the corresponding low-s background contribution to the relevant spectral integrals is tiny compared to that from the ρ-ω region. The case of the ϕ, however, is less clear, since background contributions above the ρ-ω region are not amenable to being reliably estimated, and hence might not be similarly negligible. In the ansatz as written, such physical contributions, if present, could only be mocked up (approximately) by additional effective contributions to the ϕ and ρ'-ω' strengths. Thus, one must use some caution in interpreting, for example, the extracted f_ϕ in terms of the physical resonance parameters F_ϕ^3 and F_ϕ^8—some portion of f_ϕ could actually correspond to an averaged version of background contributions in the region between ρ-ω and ρ'-ω'. The quality of the agreement between the hadronic and OPE sides of our sum rules is, however, *post facto* evidence in favor of resonance dominance and, hence, also in favor of the possibility of interpreting f_ϕ in terms of F_ϕ^3 and F_ϕ^8.

Let us turn, then, to the input on the OPE side of the sum rules. We will discuss the contributions, in turn, by operator dimension.

Since the correlator in question is isospin breaking, the only dimension $D=0$ contribution to Π^{38} is electromagnetic (we adhere, here, to common usage, according to which the leading mass-dependent perturbative terms are labelled $D=2$). We retain only the leading order (2-loop) graph in this case.

The $D=2$ contributions are dominated by the strong interaction terms proportional to $(m_d-m_u)^2$. To 3-loop order, the results for these terms follow from the 3-loop expressions for the correlator involving a flavor-non-diagonal current and its conjugate [33], since the perturbative contributions involving two quark loops and a purely gluonic intermediate state (present for flavor diagonal currents but not for flavor-non-diagonal currents) do not enter until 4-loop order. The resulting expressions are given in the Appendix. To evaluate them, we require the running masses, $m(Q^2)$, and running strong coupling, $\alpha_s(Q^2)$. These can be obtained once the values are determined at any fixed scale, μ_0. Since the 4-loop γ [34] and β [35] functions for QCD are now known, we have employed these when running the masses and coupling [explicitly, we solve the renormalization group (RG) equations exactly, using the truncated 4-loop γ and β functions as input].

As input for the running coupling, we take $\mu_0=m_\tau$ and use the latest (1998) value for $\alpha_s(m_\tau^2)$ obtained by the ALEPH Collaboration in their analysis of non-strange hadronic τ decays [36]. [The analysis of the strange decays employed previous theoretical results for the $D=2$ terms, proportional to $(m_s-m_u)^2$, which turn out to be in error [37–39]; the value obtained in the global ALEPH analysis must, therefore, be excluded.]

The situation with the light quark mass difference $\delta m \equiv (m_d - m_u)$ is somewhat more complicated. We first write

$$\delta m = \left(\frac{m_d - m_u}{m_d + m_u} \right) (m_d + m_u) \equiv r(m_d + m_u). \tag{21}$$

The isospin-breaking mass ratio, r, is known, from a number of ChPT analyses, to be $r = 0.288 \pm 0.037$ [40], which would allow one to determine δm if $m_d + m_u$ were known. The most recent determination of $m_d + m_u$ is that based on an integer-power-weighted FESR analysis of the isovector pseudoscalar channel [25,41]. In this analysis, the pion pole contribution to the spectral function is known experimentally but the continuum contribution is not. The authors of Refs. [25,41], therefore, constructed an ansatz for the unmeasured continuum contribution. It turns out that the continuum portion of the resulting model spectral function provides roughly 3/4 of the contribution to the extracted value of $(m_d + m_u)^2$. Unfortunately, it has recently been shown, using the FESR framework discussed above, that this continuum ansatz is unphysical [27]; so one cannot employ the values of Refs. [25,41].

If m_s were known (at some scale), then one could straightforwardly determine $m_d + m_u$ (at that same scale) using the known (scale-independent) ratio of masses, $r_s = 2m_s/(m_d + m_u) = 24.4 \pm 1.5$, obtained by Leutwyler [40] using ChPT. Unfortunately, the situation is also somewhat complicated for m_s. A number of recent analyses produce values of $m_s(1\ \mathrm{GeV}^2)$ [in the modified minimal subtraction ($\overline{\mathrm{MS}}$) scheme] ranging from ~ 110 MeV to ~ 210 MeV, often with rather large errors [42–46,37,38,47]. Because the analyses based either on flavor breaking in hadronic τ decays [37,38] or Narison's τ-decay-like sum rule for $\Pi^{33} - \Pi^{88}$ [45,46] have rather large errors resulting from experimental uncertainties which are unlikely to be significantly improved in the near future, the most favorable approach would appear to be that based on various sum rule treatments of the strange scalar channel, where the dominant $K\pi$ part of the spectral function is in principle determined, via the Omnes representation of the timelike scalar $K\pi$ form factor, in terms of experimental $K\pi$ phase shifts and K_{e3} data [42]. The most recent analyses of this channel [44,47] employ the SVZ framework, and produce values $m_s(1\ \mathrm{GeV}^2) = 125$–$160$ MeV [44], and 160 ± 30 MeV [47]. (The same low-s part of the spectral function is used in both analyses; the only difference between the two lies in the treatment of the ''continuum.'' The results of Ref. [47], in addition, show no stability window for m_s.) Preliminary work using the FESR framework discussed above indicates that residual errors associated with the use of the local duality approximation in the continuum region remain, for this channel, when one uses the SVZ approach. [See, e.g., the results of Ref. [27]. From these one can see (1) that using the central values for the parameters describing the fit to the $K\pi$ phases from Refs. [42,44], together with the central value from the m_s range from Ref. [44], one obtains rather poorly satisfied families of FESR's, and (2) that using the spectral function of Refs. [42,44,47], again with central values for the fit parameters, the FESR analysis, in fact, produces m_s values larger by ~ 20 MeV than those obtained in the analysis of Ref. [44].] Although work on the extraction of m_s is still in progress [48], we conclude already from the preliminary results noted above that $m_s(1\ \mathrm{GeV}^2) \sim 165$ MeV, probably with errors $\sim \pm 20$ MeV or less. The ChPT ratio then produces $(m_d + m_u)(1\mathrm{GeV}^2) \simeq 13.5$ MeV, with errors $\sim \pm 2$ MeV. For any value in this range it turns out that the $D=2$ contributions are at the $\sim 15\%$ (or less) level of the $D=4$ contributions, and the resulting errors lead, therefore, to very small (percent level) uncertainties in the final results. Since these uncertainties are much smaller than those generated by the uncertainty in the isospin-breaking mass ratio r, we have employed the central value $(m_d + m_u)(1\ \mathrm{GeV}^2) = 13.5$ MeV, and retained only the uncertainty in r, in the analysis which follows.

The $D=4$ contributions are much more straightforward. Although in principle both those $D=4$ terms proportional to the isospin-breaking mass difference, δm, and those proportional to the isospin-breaking condensate difference, $\langle \bar{d}d \rangle - \langle \bar{u}u \rangle$, appear in the OPE of Π[38], the latter are numerically tiny compared to the former. The dominant $D=4$ contribution can then be written in terms of r and the combination $(m_d + m_u)\langle \bar{q}q \rangle$, which we can take from the Gell-Mann Okubo (GMO) relation

$$(m_d + m_u)\langle \bar{q}q \rangle = -m_\pi^2 f_\pi^2. \tag{22}$$

The dominant uncertainties for the $D=4$ terms thus result from those in r.

The phenomenological situation is not so favorable in the case of the $D=6$ condensates. Usually, in the absence of pre-existing determinations of the relevant condensates, one makes estimates based on the vacuum saturation approximation (VSA). It is well known that, in situations where it has been possible to perform phenomenological checks by extracting the total $D=6$ contribution from data, the VSA has proved to significantly underestimate these contributions [49]. Usually one simply replaces the factor $\alpha_s \langle \bar{q}q \rangle^2$, which is produced by the VSA, by an effective scale-independent factor, written $\rho' \alpha_s \langle \bar{q}q \rangle^2$. The parameter, ρ', then represents the deviation from the VSA. Ideally, it should either be possible to determine ρ' from data, or the $D=6$ contributions should be small, for the sum rule in question. In our case, neither of these conditions holds. In particular, because we are forced to work at scales as low as $2\ \mathrm{GeV}^2$ in order to constrain the spectral parameters, the $D=6$ contributions can, for $s_0 \sim 2\ \mathrm{GeV}^2$, and certain values of A employed in our analysis, approach $\sim 40\%$ of the leading $D=4$ term. Fortunately, it turns out, as we will see explicitly below, that by working with both the single- and double-pinch weight families, we can actually obtain a rather good determination of the $D=6$ contribution to the correlator (albeit it as a function of r) by insisting on the consistency of the results obtained from the two different sum rule families.

In the Appendix, it is shown that the VSA leads to an expression for the $D=6$ contribution to Π[38] proportional to

$$\alpha_s(\langle \bar{d}d \rangle^2 - \langle \bar{u}u \rangle^2) = \gamma(\alpha_s \langle \bar{q}q \rangle^2), \tag{23}$$

where $\langle\bar{q}q\rangle$ is the average of the u and d condensates, and

$$\gamma \equiv \frac{\langle\bar{d}d\rangle}{\langle\bar{u}u\rangle} - 1 \tag{24}$$

describes isospin breaking in the light quark condensates. In order to compare the deviation from the VSA in the isospin-breaking channel with that in the analogous isospin-conserving isovector vector ($ab=33$) channel, we write the re-scaled version of the RHS of Eq. (23) in the form

$$\rho' \alpha_s(\langle\bar{d}d\rangle^2 - \langle\bar{u}u\rangle^2) = \rho_{red}\gamma(\rho\alpha_s\langle\bar{q}q\rangle^2), \tag{25}$$

where ρ is the parameter describing the deviation from the VSA in the 33 channel, and one has, phenomenologically, [50]

$$\rho\alpha_s\langle\bar{q}q\rangle^2 = (5.8 \pm 0.9)\times 10^{-4}\ \ \mathrm{GeV}^6. \tag{26}$$

With this definition, ρ_{red} reduces to 1 in the limit that the deviation from the VSA is the same in the 33 and 38 channels.

The consistency procedure for fixing the $D=6$ contribution to Π^{38}, together with the phenomenological input of Eq. (26), of course, determines only the product $\rho_{red}\gamma$. In presenting our results for ρ_{red} below, we have taken $\gamma \simeq -0.008$, which represents an average of the previous determinations listed in Ref. [29], except one. [We omit the value based on an analysis of baryon splittings because it implies (via the 1-loop ChPT relations between flavor breaking and isospin breaking in the light quark condensate [4]) $\langle\bar{s}s\rangle/\langle\bar{u}u\rangle > 1$, which appears unphysical.] We will discuss the determination of ρ_{red} in more detail below when we present the results of the analysis.

The last point in need of discussion concerns the way in which we handle the integrals on the OPE side of the various FESR's. Two options exist in the literature. The first, sometimes called the ''fixed order expansion,'' involves first expanding $\alpha_s(Q^2)$ and the mass factors, generically $m(Q^2)$, in terms of $\alpha_s(s_0)$ and $m(s_0)$. The coefficients of the perturbative expansions in powers of $\alpha_s(s_0)$ are then polynomials in $\log(s/s_0)$ [51], and the desired contour integrals can thus be written in terms of elementary integrals involving logarithms and powers of s. The integrated OPE expressions which result involve powers of $m(s_0)$, each multiplied by a power series in $\alpha_s(s_0)$. There is, of course, in this expression, the usual residual dependence on the choice of scale s_0 for the expansions discussed above, which results from truncating the full perturbative series at fixed order. The second alternative, often referred to as ''contour improvement,'' involves numerically integrating the factors $[m(Q^2)]^k[\alpha_s(Q^2)]^j s^l$ around the circular contour in the $s=-Q^2$ plane [52]. It is known that this has the effect of simultaneously improving the convergence of the perturbative series and reducing the residual scale dependence [52,25,41]. As a result, we have evaluated all the integrals on the OPE sides of our sum rules using this approach.

Let us now turn to the results, which are presented in Table I. As explained above, the dominant uncertainty is due to that in the ChPT determination of r. We have, therefore, tabulated the results for the range of values corresponding to the errors on r quoted by Leutwyler [40]. All results are based on matching the hadronic and OPE sides of the two sum rule families for s_0 in the range 2.0–2.8 GeV^2 and with A in the range 2–5 for the single-pinch case and 3–6 in the double-pinch case. The choice of range of A in each case has been made so as to keep the convergence of the perturbative series for the $D=2$ term under control. It is worth mentioning that the quality of the match between the OPE and hadronic sides which results after the fitting of the spectral parameters is significantly better for Leutwyler's central value of r.

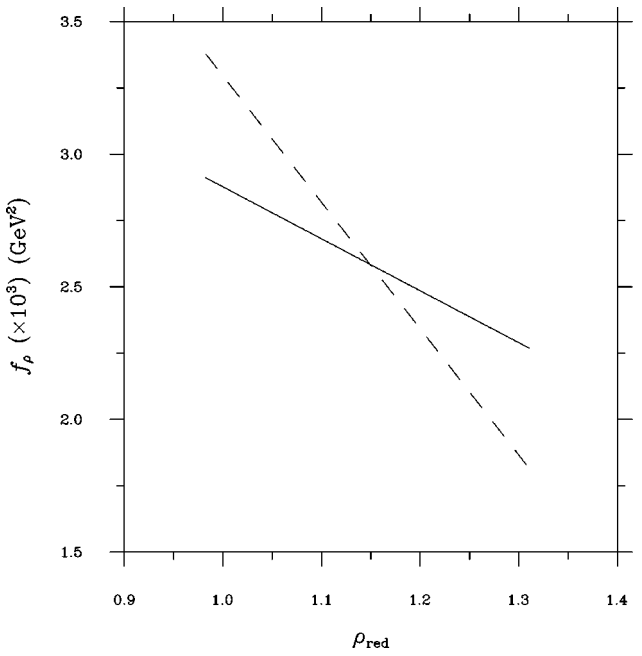

FIG. 2. The variation of f_ρ with ρ_{red} for the single- and double-pinch weight families. Results are displayed here for the central value $r=0.288$. The solid line corresponds to the single-pinch weight analysis, the dashed line to the double-pinch analysis. The intersection point determines the value of ρ_{red} quoted in Table I.

The value of the $D=6$ VSA-violating parameter, ρ_{red}, given in Table I, is determined by requiring that the values of f_ρ obtained using the single- and double-pinch weight families be the same. The sensitivity of f_ρ to variations in ρ_{red} (true also of the other f_V), as well as the difference in the ρ_{red} dependence of f_ρ for the single- and double-pinch analyses, is shown in Fig. 2. The fact that, once ρ_{red} has been determined by the requirement of the consistency of the two output f_ρ values, all the rest of the spectral parameters, determined using either the single- or double-pinch weights, also become consistent is strong evidence in favor of the reliability of the analysis. Note that (1) the possibility of determining the correction to the VSA for the $D=6$ operators and (2) the inclusion of both the $D=2$ terms and the $\mathcal{O}(\alpha_s, \alpha_s^2)$ contributions to the Wilson coefficient of the $D=4$ term are features not present in previous analyses of the analogous isospin-breaking $\Pi^{\rho\omega}$ correlator. Although the value of ρ_{red}, determined as just described, depends some-

what on r, this dependence is not strong, and we obtain $\rho_{red}=1.15\pm0.15\pm0.2$. The first error corresponds to that in Eq. (26), the second to that on r. We see that the violation of the VSA is very similar in both the 33 and 38 channels. The importance, in reducing the errors in the determinations of the spectral parameters, f_V, of being able to determine ρ_{red} is also evident from Fig. 2.

Having determined the $D=6$ contributions by self-consistency, the errors on the extracted values of f_V are determined solely by those on r, and are $\sim 10\%-15\%$, completely correlated with r.

Having extracted the parameters f_V, it is straightforward to determine the isospin-violating decay constants. One finds

$$F_\rho^8=2.4\pm0.3 \ \text{MeV}$$

$$F_\omega^3=-3.4\pm0.4 \ \text{MeV}$$

$$F_\phi^3=0.33\pm0.02 \ \text{MeV}, \tag{27}$$

where the errors reflect those in the input isospin-breaking mass ratio, r.

V. SUMMARY AND DISCUSSION OF PHENOMENOLOGICAL CONSEQUENCES

A number of useful general observations follow from the analysis above. First, we have found that the violation of the VSA for the $D=6$ condensates is very similar in the isospin-breaking (38) and isospin-conserving (33) vector current channels. Second, we have seen that, although the analysis can be performed successfully for any r in the range given by Leutwyler, the central value of that range is preferred, in the sense of giving the best match between OPE and hadronic sides of both the single- and double-pinch families of sum rules. Finally, we have demonstrated that the FESR method, particularly when implemented using both the single- and double-pinch weight families, is very effective, allowing a determination of the ρ^{38} spectral parameters, f_V, with rather small errors. These errors (between 10% and 15% for f_ρ, f_ω and f_ϕ) are a factor of 3 smaller than those obtained in the earlier analysis of Ref. [10] based on results of an SVZ analysis of $\Pi^{\rho\omega}$ [31]. In order to obtain this level of reduction, the ability to self-consistently determine ρ_{red} was crucial.

Let us now turn to the phenomenological consequences of our results. First note that the corrections required to convert the measured contribution of the vector meson, V, to the EM spectral function, ρ^{EM}, into the corresponding contribution to either ρ^{33} (for $V=\rho$) or ρ^{88} (for $V=\omega,\phi$) are given by the ratios

$$\left[\frac{F_\rho^3}{F_\rho^{EM}}\right]^2=0.982\pm0.0021$$

$$\left[\frac{F_\omega^8}{\sqrt{3}\,F_\omega^{EM}}\right]^2=1.154\pm0.017$$

$$\left[\frac{F_\phi^8}{\sqrt{3}\,F_\phi^{EM}}\right]^2=1.009\pm0.001, \tag{28}$$

where the numerical values follow from those in Eq. (27). The size of the deviations of the ρ and ω corrections from 1 is reduced by $\sim 15\%-20\%$ from those obtained in the earlier analysis [10]; that for the ϕ is increased, but remains small. In all cases the errors have been reduced by a factor of 3 or more. Note that the first of these corrections is the one relevant to precision tests of CVC. Note also that, as claimed above, the corrections given in Eqs. (28), for both the ρ and ω, lie in the corresponding ranges produced by the estimate of Sec. II.

With the results given in Eq. (28), it is now possible to correct the EM data used as input to the inverse moment chiral sum rule for the 6^{th} order LEC, Q. The sum rule is given by [6,53]

$$\begin{aligned}\int_{4m_\pi^2}^\infty ds\,\frac{(\rho^{33}-\rho^{88})(s)}{s} &= \frac{16(m_K^2-m_\pi^2)}{3F^2}Q(\mu^2)+\frac{1}{48\pi^2}\log\left(\frac{m_K^2}{m_\pi^2}\right)\\ &+\left(\frac{L_9^r(\mu^2)+L_{10}^r(\mu^2)}{2\pi^2F^2}\right)\left[m_\pi^2\log\left(\frac{m_\pi^2}{\mu^2}\right)\right.\\ &\left.-m_K^2\log\left(\frac{m_K^2}{\mu^2}\right)\right],\end{aligned} \tag{29}$$

where μ is the renormalization scale of the effective chiral theory and L_k^r are the usual renormalized 4^{th} order LEC's of Gasser and Leutwyler [4]. This sum rule was evaluated in Ref. [53] using as input EM data for the isoscalar spectral function and both EM and τ decay data for the isovector spectral function. The corrections above, required for the EM data, were not considered in this analysis. It is not clear, from our reading of the discussion of Ref. [53], exactly what the relative weightings of τ decay and EM data in the determination of the ρ^0 contribution to the LHS above actually were. Since, however, τ decay data is considerably more precise than electroproduction data, we have assumed in what follows that the determination is dominated by τ decay data. To the extent that this is true, we need only make corrections to the (nominally) isoscalar ω and ϕ contributions. The result of this exercise is a shift of $Q(m_\rho^2)$ from $(3.7\pm2.0)\times10^{-5}$ to

$$Q(m_\rho^2)=(2.4\pm2.0)\times10^{-5}. \tag{30}$$

[For reference, making, instead, the somewhat perverse assumption that the determination of the ρ^0 contribution was dominated by EM data, one would find $Q(m_\rho^2)=(2.0\pm2.0)\times10^{-5}$. The full correction is dominated by that to the ω contribution. The reason this correction is so much larger than the others has been discussed above.]

One should bear in mind, in interpreting these results, that (1) there are, in principle, additional corrections to be made

to the nominal isovector and isoscalar contributions at higher s, and (2) the $\bar{K}K2\pi$ contributions were taken to be purely isoscalar in the analysis of Ref. [53]. Because the separations within the higher isovector and isoscalar resonance pairs, ρ'-ω' and ρ''-ω'', are much smaller than the resonance widths, it is not possible to use sum rule methods to extract the individual isospin-breaking decay constants of these resonances. As such, we are unable to estimate the size of the former corrections. Were the $\bar{K}K2\pi$ states to have a significant isovector component, the effect would be to raise $Q(m_\rho^2)$.

An alternate method for determining Q is based on the observation that Q occurs not only in the inverse chiral moment sum rule above, but also in the 2-loop ChPT expression for $\Pi^{38}(0)$ [32]. It is thus possible to make an independent estimate by using the fitted spectral ansatz for Π^{38} to compute $\Pi^{38}(0)$, assuming a negligible contribution from the portion of the spectrum above 2.8 GeV^2. One obtains, from this exercise,

$$Q(m_\rho^2)=(3.3\pm 0.4\pm 0.2)\times 10^{-5}, \tag{31}$$

where the first error is that associated with the uncertainties (shown in Table I) in the extraction of the spectral parameters, f_V, and the second is our estimate of the error resulting from truncating the perturbative series for the Wilson coefficients in the OPE. The numerical value quoted for this latter error was obtained by doubling the difference in $Q(m_\rho^2)$ values obtained using the $\mathcal{O}(\alpha_s)$ and $\mathcal{O}(\alpha_s^2)$ versions of the $D=2$ and $D=4$ coefficients in our analysis. Since there is no positivity constraint on $\rho^{38}(s)$, one does not know in which direction this result would be changed by corrections due to the small higher-s part of the spectral integral. The results of a study of the effect of including two combined spectral contributions, one for the ρ'-ω' and one for the ρ''-ω'', however, shows negligible change in $Q(m_\rho^2)$, suggesting that such corrections are unlikely to be numerically significant. This conclusion is supported by an estimate obtained using the local duality approximation for the spectral function in the region $s>2.8$ GeV^2. Performing the relevant spectral integral, one finds that the high-s portion of the spectrum, in this approximation, contributes less than 1% to $Q(m_\rho^2)$, making the resulting uncertainties totally negligible on the scale of those quoted above.

Since the two independent determinations of Q are completely consistent, within errors, the conclusion that $Q(m_\rho^2)\simeq 3\times 10^{-5}$ is considerably strengthened.

The last phenomenological application of our results concerns the effect on Narison's τ-decay-like sum rule for m_s. Since a detailed discussion of the way in which one implements the isospin-breaking corrections is given in Ref. [46], we report here only the results of employing the improved determinations of the correction factors determined above. In doing so we will also take the opportunity to update the input parameters to the analysis of Ref. [46], employing the newer (1998) ALEPH value of $\alpha_s(m_\tau^2)$ [36]. One finds, for example, using τ decay data for the isovector input, that the average over the values of $m_s(1\ \text{GeV}^2)$ extracted using the set of scales $s_0=1.4$,1.5 and 1.6 GeV^2 in the sum rule analysis is shifted from 138 MeV to 146 MeV (147 MeV if one retains the 1997 ALEPH value of α_s as input). Unfortunately, the errors in this value associated with uncertainties in the experimental input are still very large, $\sim\pm 50$ MeV at least, and this uncertainty cannot be appreciably reduced without a significant improvement in the accuracy of the determination of the experimental $\omega\to e^+e^-$ and $\phi\to e^+e^-$ widths. As such, although the central value is brought into better agreement with that discussed above, little more can learned from the Narison sum rule, at the present time.

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APPENDIX: THE OPE FOR Π^{38}

The explicit form of the OPE for Π^{38}, keeping terms only up to dimension 6, and to $\mathcal{O}(\alpha_s^2, m_q^2)$, can be obtained, as explained in the text, from the relevant expressions for the flavor-non-diagonal case given in the literature [see Ref. [33] and the paper by Braaten, Narison and Pich (BNP) in Ref. [1]]. We list the results by operator dimension.

Dimension 0: The only isospin-breaking contribution at dimension 0 is that due to EM, and is given by [28]

$$[\Pi^{38}_{1\gamma E}]_{D=0}=-\frac{\alpha}{16\pi^3}\frac{1}{4\sqrt{3}}\ln(Q^2) \tag{A1}$$

where α is the usual EM coupling.

Dimension 2: The $D=2$ term consists of the leading mass-dependent part of the perturbative contribution to the OPE, and follows from the expression given in Ref. [33]. One finds

$$\begin{aligned} Q^2[\Pi^{38}(Q^2)]_{D=2} &= \frac{3}{2\pi^2}\frac{1}{4\sqrt{3}}[(m_d^2-m_u^2)(Q^2)]\left[1+\frac{8}{3}a(Q^2)\right. \\ &\left.+\left(\frac{17981}{432}+\frac{62}{27}\zeta(3)-\frac{1045}{54}\zeta(5)\right)a^2(Q^2)\right] \end{aligned} \tag{A2}$$

where $a(Q^2)=\alpha_s(Q^2)/\pi$, and $\zeta(n)$ is the Riemann zeta function. Further details on how the running of the coupling and the masses is handled can be found in Sec. IV.

Dimension 4: Our expression for the $D=4$ contribution also follows from that given in Ref. [33]. Only the m_q^4 and quark condensate terms survive once one takes the relevant isospin-breaking difference. The former are numerically tiny compared to the latter, and hence have not been written down explicitly. We then find

$$Q^4[\Pi^{38}(Q^2)]_{D=4}=\frac{2(\langle m_u\bar{u}u\rangle-\langle m_d\bar{d}d\rangle)}{4\sqrt{3}}\times\left[1+\frac{1}{3}a(Q^2)+\frac{11}{2}a^2(Q^2)\right]. \tag{A3}$$

The scale-invariant $\langle m\bar{q}q\rangle$ difference can be written in terms of the m_d-m_u, $\langle\bar{d}d-\bar{u}u\rangle$, and the averages of the u and d quark masses and condensates. Since isospin breaking in the condensates is much smaller than in the masses, the term proportional to m_d-m_u dominates numerically. It can be recast in terms of the isospin-breaking quark mass ratio, r, and f_π, m_π, as explained in the text.

Dimension 6: The 4-quark operators are the dominant operators at dimension 6. Their contribution to Π^{38} can be obtained from the expressions given in the Appendix of Braaten, Narison and Pich (BNP) [1]. Since a lack of phenomenological information on the various condensates forces one to work with the rescaled version of the VSA, one must, for consistency, drop the terms of $\mathcal{O}(\alpha_s^2)$ contained there. (See the discussion of this point contained in BNP.) One then finds

$$\begin{aligned}4\sqrt{3}Q^6[\Pi^{38}(Q^2)]_{D=6}&=-8\pi^2a(Q^2)(\langle\bar{u}\gamma_\mu\gamma_5T^a u\bar{u}\gamma^\mu\gamma_5T^a u\rangle\\&\quad-\langle\bar{d}\gamma_\mu\gamma_5T^a d\bar{d}\gamma^\mu\gamma_5T^a d\rangle) \qquad \text{(A4)}\\&\quad-\frac{16\pi^2}{9}a(Q^2)\sum_k(\langle\bar{u}\gamma_\mu T^a u\bar{q_k}\gamma^\mu T^a q_k\rangle\\&\quad-\langle\bar{d}\gamma_\mu T^a d\bar{q_k}\gamma^\mu T^a q_k\rangle), \qquad \text{(A5)}\end{aligned}$$

where T^a is an $SU(3)$ generator. Implementing the re-scaled VSA, this expression reduces to

$$4\sqrt{3}Q^6[\Pi^{38}(Q^2)]_{D=6}=\frac{448\pi}{81}\rho_{red}\gamma(\rho\alpha_s\langle\bar{q}q\rangle^2), \tag{A6}$$

where γ, ρ and ρ_{red} are as defined in the text.

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