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Adiabatic string shape for nonuniform rotation

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It is well known that a straight Nambu-Goto string is an exact solution of the equations of motion when its end moves in a circular orbit. In this paper we investigate the shape of a confining relativistic string for a general motion of its end. We determine analytically the shape of the curved string to leading order in deviation from straightness, and show that it reduces to an expected non-relativistic result. We also demonstrate numerically that in realistic meson models this deviation is always small. We further find that the angular momentum and energy are the same as for the straight string, but that the curved string has a small radial momentum not present in a straight string. Our results justify the common assumption of straight strings usually made in hadron models. [S0556-2821(99)04309-X]

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I. INTRODUCTION

For some time flux tube models [1,2] and Wilson loop expansion calculations [3] have been used to understand hadron states. The usual assumption is that the color field can be taken to lie on straight lines connecting the quarks. This ''straight line'' or ''rigid'' flux configuration is in the spirit of an adiabatic or Born-Oppenheimer approximation that is assumed to hold at least for slowly moving quarks.

Nesterenko [4] points out that this adiabatic approximation cannot be valid if the quark has angular acceleration. The argument is based on a classic theorem of ruled surfaces. The Nambu-Goto string action requires the string to sweep out a surface of minimal area. Catalan's theorem [5] states that if the string is straight there is only one minimal surface swept out; a helicoid. This surface describes a meson state consisting of uniformly rotating quarks. As we will observe, a straight string with the end moving radially also sweeps out a minimal surface as long as the angular velocity is constant. This solution does not correspond to the motion of an actual meson since the quark plus string angular momentum is not conserved. However, one may take the point of view that the string shape can be studied with arbitrary end motion, such as caused by an arbitrary external force acting on the ends. The motion of the string with quarks at the ends is then a special case. In this paper we use the rigidly rotating solution as our starting point and consider angular acceleration of the end points as a perturbation.

A rigidly rotating string with its end point quarks moving in perfectly circular orbits is not a realistic model for mesons because quantum mechanics requires there to be some radial motion, and by angular momentum conservation, some angular acceleration as well. The string must be curved if it is to sweep out a minimal area while its end points undergo angular acceleration. We will discuss here the shape of the curved string and conclude that small curvatures do not change its dynamics. This result holds even for relativistic string motion.

In Sec. II we define our notation for the Nambu-Goto action and obtain the string wave equation, angular momentum and energy relations. We also establish exact straight string solutions with arbitrary radial motion but constant angular velocity. An intuitive picture of a non-relativistic string is developed in Sec. III. The relativistic shape equation and solutions for small angular acceleration are established in Sec. IV. We demonstrate numerically in Sec. V that the size of the string deformations is small and therefore that the perturbative approach is sound. The angular momentum and energy for the curved string are considered in Sec. VI and our conclusions are given in Sec. VII.

II. THE NAMBU-GOTO-POLYAKOV STRING

The string action is proportional to the string tension a and to the area swept out by the string. It is conventionally written in Polyakov [1] form as

$$S=-\frac{a}{2}\int d\tau\int_{\sigma_1}^{\sigma_2}d\sigma\sqrt{-h}\,h^{ab}X^{\mu}_{,a}X^{\nu}_{,b}\eta_{\mu\nu}, \qquad (2.1)$$

where h_{ab} is a two-dimensional metric auxiliary field whose indices run over τ and σ, and $h=\det(h_{ab})$. $X^{\mu}(\tau,\sigma)$ is the string position and $X^{\mu}_{,a}\equiv\partial_a X^{\mu}$. The action is invariant under reparametrization of the coordinates $\tau\rightarrow\tilde{\tau}(\tau,\sigma)$ and $\sigma\rightarrow\tilde{\sigma}(\tau,\sigma)$, and under Weyl scaling of the metric

$$h_{ab}(\tau,\sigma)\rightarrow e^{\phi(\tau,\sigma)}h_{ab}(\tau,\sigma). \qquad (2.2)$$

Because the metric h_{ab} has no dynamics, we may solve its field equations and substitute the result back into the action without changing the equations of motion for the other fields. The equations of motion for h_{ab},

$$X^{\mu}_{,a}X_{\mu,b}=\frac{1}{2}h_{ab}h^{cd}X^{\mu}_{,c}X_{\mu,d}, \qquad (2.3)$$

and invariance under Weyl scaling (2.2) allow us to set h_{ab} equal to the induced metric

$$h_{ab}=X^{\mu}_{,a}X_{\mu,b}, \qquad (2.4)$$

from which we see that Eq. (2.1) reduces to, and is therefore classically equivalent to, the standard square-root Nambu-Goto action.

We will fix the coordinates τ and σ by choosing τ to be equal to coordinate time t, and σ to run linearly along the length of the string and have a fixed interval (σ_1,σ_2) regardless of the physical length of the string. This simplifies the description of the end points of the string, but unfortunately precludes the use of orthonormal coordinates. A more detailed explanation of the Nambu-Goto-Polyakov string, its geometrical aspects, and its relation to QCD strings can be found in Ref. [6].

The boundary conditions we take on $X^{\mu}(\tau,\sigma)$ are that one end is fixed and the other is arbitrarily forced:

$$X^{\mu}(\tau,\sigma_1)=(t,\mathbf{0}), \quad X^{\mu}(\tau,\sigma_2)=(t,\mathbf{x}(t)), \qquad (2.5)$$

where $\mathbf{x}(t)$ is a prescribed function of time. This simplifies our presentation because we are only examining the piece of string from the center of mass to one of the quarks. It is a trivial generalization to include quarks at both ends. To include the dynamics of a (spinless) quark on the end of the string, we would use its equation of motion,

$$\dot{p}^{\mu}=-a\sqrt{-h}\,h^{\sigma a}X^{\mu}_{,a}, \qquad (2.6)$$

which is essentially Newton's second law with the tension in the string providing the force. The equation of motion (2.6) arises from the variation of the boundary of the string when quarks are included from the beginning.

Our purpose here is not to solve for the actual motion of the string–quark system, which is a difficult problem, but rather to find the shape of a string given a prescribed motion of its end. Knowing the shape of the string, and using Eq. (2.6), we are in principle able to find the motion of the string–quark system. In practice, we use the expressions for the total energy and angular momentum and quark momentum in terms of the quark velocity to find the spectrum for the quantized string–quark system numerically. We demonstrate in Appendix A that Eq. (2.6) is equivalent to the conservation of energy, angular momentum and the quark mass-shell relation.

Planar motion is best described in terms of complex coordinates instead of vector notation. In complex coordinates, the string position X^{μ} can be written as

$$X^{\mu}(\tau,\sigma)=(\tau,X^{+}(\tau,\sigma),X^{-}(\tau,\sigma),0), \qquad (2.7)$$

with

$$X^{\pm}=\frac{1}{\sqrt{2}}(X^1\pm iX^2). \qquad (2.8)$$

The metric h_{ab}, from Eq. (2.4), is

$$\begin{aligned}h_{ab}&=-X^0_{,a}X^0_{,b}+X^1_{,a}X^1_{,b}+X^2_{,a}X^2_{,b}\\&=-X^0_{,a}X^0_{,b}+2\,\mathrm{Re}(X^{*}_{,a}X_{,b}),\end{aligned} \qquad (2.9)$$

where we adopt the notation $X\equiv X^{+}$.

For simplicity, we consider here the fixed end at $\sigma_1=0$ and the moving end at $\sigma_2=1$. Variation of the action with respect to the position of the string yields the equations of motion

$$(\sqrt{-h}\,h^{ab}X^{\mu}_{,b})_{,a}=0. \qquad (2.10)$$

Once $X^{\mu}(\tau,\sigma)$ is known, the string four-momentum and angular momentum are

$$P_{\mu}=\int d\sigma\,\Pi_{\mu}(\tau,\sigma)=-a\int_0^1 d\sigma\sqrt{-h}\,h^{\tau a}X_{\mu,a} \qquad (2.11)$$

$$J^3=\int d\sigma\,X^{[1}\Pi^{2]}=-2a\int_0^1 d\sigma\sqrt{-h}\,h^{\tau a}\ \mathrm{Im}(X^{*}X_{,a}). \qquad (2.12)$$

Equations (2.10) have an exact solution in which a straight string rotates uniformly (constant angular velocity), but has an arbitrarily changing length. This is the solution that we will perturb to find the string shape when its end undergoes angular acceleration. Our ansatz is

$$X^0(\tau,\sigma)=t=\tau,$$

$$X(\tau,\sigma)=\frac{\sigma R(t)}{\sqrt{2}}e^{i\omega t}. \qquad (2.13)$$

From Eq. (2.9), we find the metric tensor

$$h_{ab}=\begin{pmatrix}-\gamma^{-2} & \sigma R\dot{R}\\ \sigma R\dot{R} & R^2\end{pmatrix}$$

$$\sqrt{-h}=R/\gamma_{\perp}, \qquad (2.14)$$

$$\sqrt{-h}\,h^{ab}=\gamma_{\perp}\begin{pmatrix}-R & \sigma\dot{R}\\ \sigma\dot{R} & R\gamma^{-2}\end{pmatrix},$$

where we define

$$v_{\perp}=\omega R,$$

$$\gamma_{\perp}^{-2}=1-\sigma^2v_{\perp}^2, \qquad (2.15)$$

$$\gamma^{-2}=1-\sigma^2(v_{\perp}^2+\dot{R}^2).$$

Substituting the above into the string equations (2.10), we find that they are exactly satisfied. This result is the realization of Catalan's theorem mentioned earlier, where the straight line to the quark sweeps out a helicoid of fixed pitch but arbitrarily varying radius. Again we emphasize that this is a solution of the string equations (2.10) only. In a realistic meson, for which Eq. (2.6) also holds for the quarks on the ends, we must perturb Eq. (2.13) to find a solution that takes into account the necessary angular acceleration of the ends.

For future reference, we give here expressions for J^3 and P_μ of the straight string. From Eq. (2.12) we find that the angular momentum is given as

$$J^3_{\text{straight}} = -2\frac{a}{2}\int_0^1 d\sigma\, \gamma_\perp[-R\,\text{Im}(i\omega\sigma^2R^2+\sigma\dot{R}) + \sigma\dot{R}\,\text{Im}(\sigma R^2)]$$

$$= a\omega R^3\int_0^1 \frac{\sigma^2 d\sigma}{\sqrt{1-\sigma^2v_\perp^2}}$$

$$= \frac{aR^2}{2v_\perp}\left[\frac{\arcsin(v_\perp)}{v_\perp} - \sqrt{1-v_\perp^2}\right]. \tag{2.16}$$

The corresponding energy and spatial momentum from Eq. (2.11) are

$$E_{\text{straight}} = -a\int_0^1 d\sigma\,(-R\gamma_\perp) = aR\frac{\arcsin v_\perp}{v_\perp}, \tag{2.17}$$

$$P_{\text{straight}} = \frac{i}{\sqrt{2}}P_{\perp\ \text{straight}}, \tag{2.18}$$

where

$$P_{\perp\ \text{straight}} = aR^2\omega\int_0^1 \frac{\sigma d\sigma}{\sqrt{1-\sigma^2v_\perp^2}}$$

$$= \frac{aR}{v_\perp}[1-\sqrt{1-v_\perp^2}]. \tag{2.19}$$

We observe that P_{straight} is purely transverse in direction.

It is also interesting to note the small velocity limits for these quantities. If we define the moment of inertia of a uniform string of ''mass'' aR rotating about one end as

$$I = \frac{1}{3}(aR)R^2, \tag{2.20}$$

then the low velocity limits of J^3, E, and $P_\perp$ are

$$J^3 \to I\omega,$$

$$E \to aR + \frac{1}{2}I\omega^2, \tag{2.21}$$

$$P_\perp \to \frac{1}{2}(aR)v_\perp .$$

As one might expect, the string can be thought of as a rod of mass aR.

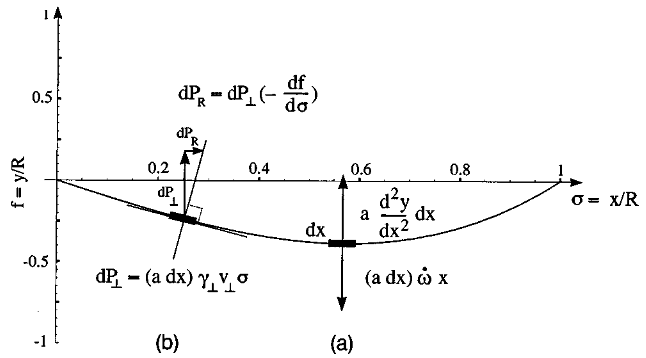

FIG. 1. The solution to the non-relativistic string shape. (a) Force balance on a small element dx. (b) Transverse and radial momentum of an element.

III. NON-RELATIVISTIC STRING SHAPE

In the above we have demonstrated that a uniformly rotating string can remain straight even if the length changes. When the end has angular acceleration the string must curve. To gain intuition we consider a quasi-Newtonian string of ''mass'' density a which rotates with instantaneous angular velocity ω about one end. In the rotating frame an element at distance $x=\sigma R$ is assumed to be in equilibrium under two forces, the tension force

$$F_{\text{tension}} = a\frac{d^2y}{dx^2}dx, \tag{3.1}$$

and the angular acceleration fictitious force

$$F_{\text{fictitious}} = -(a\,dx)\dot{\omega}x, \tag{3.2}$$

as shown in Fig. 1. The only other possible transverse force is the Coriolis force due to the radial motion of the right end. However, the element does not experience a Coriolis force since motion of the end only creates more string and the notion of longitudinal velocity has no meaning. This type of motion can be thought of as ''adiabatic'' since the resulting shape depends only on the end acceleration. The force equilibrium condition yields

$$\frac{d^2y}{dx^2} = \dot{\omega}x. \tag{3.3}$$

In terms of dimensionless variables, $f \equiv y/R$ and $\sigma = x/R$, the above equation is

$$\frac{d^2f}{d\sigma^2} = \dot{\omega}R^2\sigma. \tag{3.4}$$

Using the end condition $f(\sigma=0)=f(\sigma=1)=0$, we find the non-relativistic string shape

$$f(\sigma) = -\frac{\dot{\omega}R^2}{6}\sigma(1-\sigma^2), \tag{3.5}$$

which is illustrated in Fig. 1.

IV. RELATIVISTIC STRING SHAPE

In order to find the relativistic string shape, in this section we consider the nature of small string deflections. We again consider adiabatic solutions generalizing the non-relativistic concept of the previous section to the Nambu-Goto case. We straightforwardly perturb about the straight string solution (2.13) to take

$$X^0(\tau,\sigma)=t,$$

$$X(\tau,\sigma)=\frac{1}{\sqrt{2}}(\sigma R(t)+F(\sigma))\exp[i(\omega t+\phi(t))], \tag{4.1}$$

where the function $F(\sigma)$ is assumed complex. We also assume that $F(\sigma)$, $\dot{\phi}(t)$, $\ddot{\phi}\equiv\dot{\omega}$, and $\dot{R}$ are small, and therefore we drop terms like $F\dot{\omega}$, $F\dot{R}$, $\dot{\phi}\dot{R}$, etc. Using Eq. (2.9), we find in this approximation

$$\begin{aligned} h_{tt}&=-\frac{1}{\gamma^2}+2\sigma\omega^2 R\ \mathrm{Re}F-2\sigma^2\omega^2R^2\dot{\phi},\\ h_{t\sigma}&=h_{\sigma t}=\sigma R\dot{R}-\omega R\ \mathrm{Im}F+\sigma\omega\ \mathrm{Im}F',\\ h_{\sigma\sigma}&=R^2+2R\ \mathrm{Re}F'. \end{aligned} \tag{4.2}$$

The assumed string position (4.1) and the metric h_{ab} above must satisfy the equations of motion (2.10) for the time and spatial components

$$(\sqrt{-h}\,h^{at})_{,a}=0, \tag{4.3}$$

$$(\sqrt{-h}\,h^{ab}X_{,b})_{,a}=0. \tag{4.4}$$

Upon substitution, and with considerable but straightforward algebra, we find that each equation is satisfied when $F(\sigma)$ satisfies

$$(1-\sigma^2 v_\perp^2)\frac{d^2(\mathrm{Im}F)}{d\sigma^2}+\sigma v_\perp^2\frac{d(\mathrm{Im}F)}{d\sigma}-v_\perp^2\ \mathrm{Im}F=\sigma R^3\dot{\omega}. \tag{4.5}$$

There are no constraints on the real part of F, which is a consequence of the reparametrization invariance of the Nambu-Goto-Polyakov action (2.1). In terms of a dimensionless quantity

$$f\equiv \mathrm{Im}F/R, \tag{4.6}$$

we find that the displacement from the straight string satisfies

$$(1-\sigma^2 v_\perp^2)\frac{d^2 f}{d\sigma^2}+\sigma v_\perp^2\frac{df}{d\sigma}-v_\perp^2 f=\sigma R^2\dot{\omega}. \tag{4.7}$$

For small rotational velocities $v_\perp$, the string shape equation (4.7) reduces to

$$\frac{d^2 f}{d\sigma^2}=\sigma R^2\dot{\omega}, \tag{4.8}$$

which is identical to the non-relativistic result, Eq. (3.4). The Nambu-Goto shape must then reduce to the previous result, Eq. (3.5).

With the choice of independent variable

$$\xi=\sigma v_\perp, \tag{4.9}$$

the shape equation (4.7) reduces to the simpler form

$$(1-\xi^2)\frac{d^2 f}{d\xi^2}+\xi\frac{df}{d\xi}-f=\xi\left(\frac{\dot{\omega}R^2}{v_\perp^3}\right), \tag{4.10}$$

whose exact analytic solution,

$$\begin{aligned} f(\sigma)=&\frac{\dot{\omega}R^2}{v_\perp^3}\left[\frac{1}{2}\xi\arcsin\xi+\sqrt{1-\xi^2}\right]\arcsin\xi+C_1\xi\\ &+C_2(\sqrt{1-\xi^2}+\xi\arcsin\xi), \end{aligned} \tag{4.11}$$

is discussed in Appendix B. The constants C_1 and C_2 are fixed by the end conditions that $f(\sigma)$ vanish at $\sigma=0$ and $\sigma=1$. The final result is

$$f(\sigma)=\frac{\dot{\omega}R^2}{6}\ \mathrm{shape}(\sigma), \tag{4.12}$$

$$\begin{aligned} \mathrm{shape}(\sigma)=-\frac{6}{v_\perp^3}\Bigg[&\frac{1}{2}\sigma v_\perp((\arcsin v_\perp)^2\\ &-(\arcsin\sigma v_\perp)^2)\\ &+\sigma\sqrt{1-v_\perp^2}\arcsin v_\perp\\ &-\sqrt{1-\sigma^2 v_\perp^2}\arcsin\sigma v_\perp\Bigg]. \end{aligned} \tag{4.13}$$

By comparison of the above expression with Eq. (3.5) we see that the only difference from the non-relativistic result is in the shape function. From the power series expansion it is straightforward to verify that

$$\mathrm{shape}(\sigma)\overset{v_\perp\ll 1}{\to}-\sigma(1-\sigma^2), \tag{4.14}$$

and, hence, $f(\sigma)$ reduces to the non-relativistic limit. In Fig. 2 we show $shape(\sigma)$ for non-relativistic, intermediate, and fully relativistic speeds. One can observe that even for a very rapid rotation, $v_\perp=\omega R\to 1$, the string shape does not change dramatically with respect to the non-relativistic result.

V. VALIDITY OF THE PERTURBATIVE APPROACH

As we have seen from Eq. (4.12), the actual size of the displacement from the straight string is controlled by a factor of $\frac{1}{6}\dot{\omega}R^2$. Its magnitude for meson states can be estimated using the heavy-light version of the relativistic (straight) flux tube (RFT) model [2]. Since the numerical solution of the

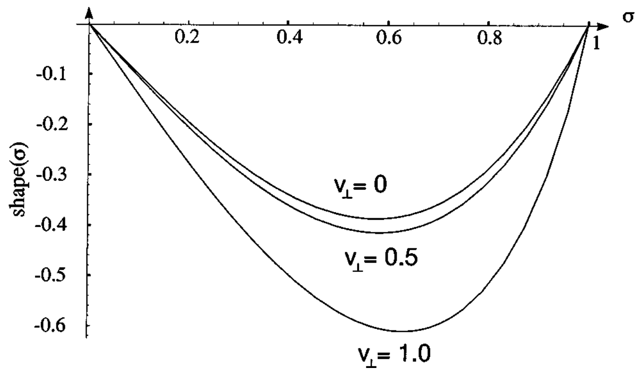

FIG. 2. The relativistic string shape function given in Eq. (4.13) for various transverse velocities $v_\perp = \omega R$.

model provides us with a matrix representation for the $v_\perp$ operator in a particular basis, it is convenient to rewrite $\dot{\omega} R^2$ as

$$\dot{\omega} R^2 = \dot{v}_\perp R - v_\perp \dot{R}. \tag{5.1}$$

The above classical expression can be quantized by the appropriate symmetrization procedure, and by promoting $v_\perp$ and R to quantum-mechanical operators, for which one can use $\dot{\Omega} = -i[\Omega, \mathcal{H}]$. In this way, once the model is solved and matrix representations for $v_\perp$ and the Hamiltonian $\mathcal{H}$ are found, it is straightforward to compute the expectation value of $\dot{\omega} R^2$ for a given quantum state.

Figures 3 and 4 show the dependence of the quantity $\frac{1}{6}\sqrt{\langle(\dot{\omega} R^2)^2\rangle}$ on the light quark mass and angular momentum, respectively. These results indicate that the actual size of the displacement from the straight string in real mesons is smaller than 10%, which illustrates the validity of the straight string approximation. Note that numerical estimates shown in those figures were obtained using only the confining part of the heavy-light RFT model with string tension $a = 0.2\ \text{GeV}^2$. Addition of the short-range one gluon exchange interaction would further reduce our results.

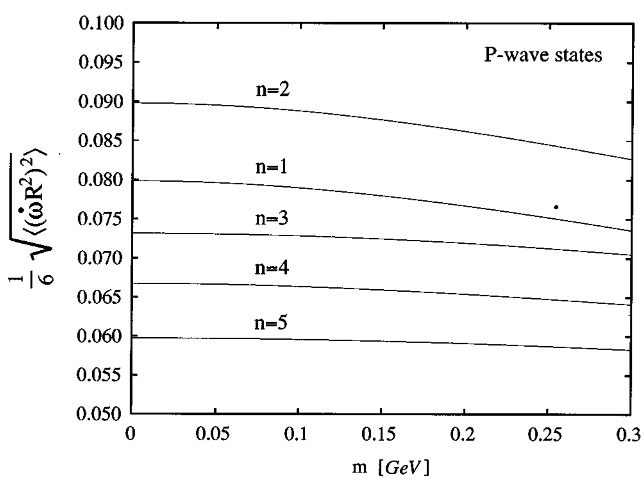

FIG. 3. The dependence of $\frac{1}{6}\sqrt{\langle(\dot{\omega} R^2)^2\rangle}$ on the light quark mass m. These results were obtained for the P-wave states in the heavy-light RFT model, with string tension $a = 0.2\ \text{GeV}^2$. The $n=1$ line denotes the ground state, while the other lines correspond to the first four radially excited states.

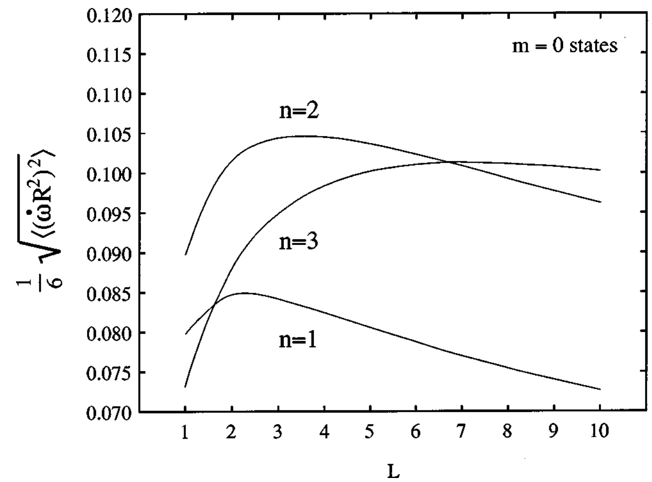

FIG. 4. The dependence of $\frac{1}{6}\sqrt{\langle(\dot{\omega} R^2)^2\rangle}$ on the angular momentum. These results were obtained in the heavy-light RFT model, with string tension $a = 0.2\ \text{GeV}^2$ and light quark mass $m = 0$. The $n=1$ line corresponds to the ground state. Also shown are the first two radially excited states.

VI. STRING ANGULAR MOMENTUM, ENERGY, AND LINEAR MOMENTUM

We now compute the effects of the deformation away from straightness on string dynamics. To this end we compare the angular momentum, energy, and linear momentum of the actual string with that of the straight string. The similarities and differences are quite interesting. We proceed in each case by substituting the perturbed string form (4.1) and the consequent perturbed metric into the desired dynamical quantities.

A. Angular momentum

The expression (2.12) for the angular momentum to first order in small quantities becomes

$$J^3 = J^3_{\text{straight}} + \frac{aRv_\perp}{\sqrt{1-v_\perp^2}}\ \mathrm{Re}\, F(1) + \frac{a\dot{\phi} R^3}{v_\perp^2}\left(\frac{1}{\sqrt{1-v_\perp^2}} - \frac{\arcsin v_\perp}{v_\perp}\right), \tag{6.1}$$

where we refer to Eq. (2.16) for the straight string result. To compare with the straight string angular momentum we must have strings of the same length and with the same end velocity which requires that both $\mathrm{Re}\, F(1)$ and $\dot{\phi}$ vanish. We then conclude that for small deviations from straightness the angular momentum of the curved string is the same as that of the straight string,

$$J^3 = J^3_{\text{straight}}. \tag{6.2}$$

B. Energy

Similarly, when we evaluate Eq. (2.11) with $\mu=0$ we obtain

$$E = E_{\text{straight}} + \frac{a\,\mathrm{Re}\,F(1)}{\sqrt{1-v_\perp^2}} - a\,\mathrm{Re}\,F(0) + \frac{aR^2\dot{\phi}}{v_\perp}\left(\frac{1}{\sqrt{1-v_\perp^2}} - \frac{\arcsin v_\perp}{v_\perp}\right). \tag{6.3}$$

Again requiring the accelerating string to have the same length and angular velocity as the straight string, $\mathrm{Re}\,F(1) = \mathrm{Re}\,F(0) = \dot{\phi} \equiv 0$, we have

$$E = E_{\text{straight}}. \tag{6.4}$$

C. Linear momentum

Finally we compute the linear momentum,

$$P \equiv \frac{1}{\sqrt{2}}(P_R + iP_\perp), \tag{6.5}$$

of the curved string using the spatial part of Eq. (2.11). The transverse momentum component is

$$P_\perp = P_{\perp\,\text{straight}} + \frac{av_\perp}{\sqrt{1-v_\perp^2}}\ \mathrm{Re}\,F(1) + \frac{a\dot{\phi}R^2}{v_\perp^2}\left(\frac{1}{\sqrt{1-v_\perp^2}} - 1\right). \tag{6.6}$$

With the usual end conditions, we obtain

$$P_\perp = P_{\perp\,\text{straight}}, \tag{6.7}$$

where $P_{\perp\,\text{straight}}$ is given in Eq. (2.19). The curved string also has radial momentum

$$P_R = -aR^2\omega\int_0^1 d\sigma\,\sigma\gamma_\perp\frac{df}{d\sigma}, \tag{6.8}$$

which, after integration by parts, becomes

$$P_R = aRv_\perp\int_0^1 d\sigma\frac{f(\sigma)}{(1-\sigma^2v_\perp^2)^{3/2}}. \tag{6.9}$$

It is worth noting that the expression for radial momentum (6.8) (and also $P_\perp$) can be directly read off of Fig. 1.

Referring back to our explicit solution (4.13) for $f(\sigma)$, we find the analytic solution P_R to be

$$P_R = -aR\frac{\dot{\omega}R^2}{v_\perp^4}\left[1-\sqrt{1-v_\perp^2} - \frac{v_\perp}{2}\arcsin v_\perp\right]\arcsin v_\perp\,. \tag{6.10}$$

For small orbital velocities we have

$$P_R \overset{v_\perp \ll 1}{\rightarrow} -\frac{aR\dot{\omega}R^2v_\perp}{24}. \tag{6.11}$$

To understand the size of the radial string momentum we will compare it to other ''relativistic corrections'' that arise in meson dynamics. In a meson with a quark mass m large enough that the quark velocity is small, the quark's angular momentum dominates that of the string. In this case $J \simeq mR^2\omega \simeq$ constant and we have $R\dot{\omega} \simeq -2\dot{R}\omega$. The radial momentum is primarily due to the non-relativistic quark with corrections from relativity and the string;

$$P_R^{\text{tot}} = m\dot{R}\left[1 + \frac{1}{2}v_\perp^2 + \frac{aR}{12m}v_\perp^2 + \cdots\right]. \tag{6.12}$$

The string radial momentum is smaller by a factor of $aR/6m$ than the first relativistic correction.

In this section we have observed that to leading order the angular momentum, energy, and transverse momentum of the curved string are unchanged by small deviations from a straight string. The bending of the string will induce a radial momentum but it is of higher order than the leading relativistic corrections. In this way, we agree with the work of Brambilla et al. [3] who showed that in the Wilson loop formalism relativistic corrections are correctly computed assuming a straight path between the quarks.

VII. CONCLUSIONS

We have considered here the shape of a QCD string with one end fixed and the other moving with arbitrary velocity, but with a small angular acceleration. We calculated the deviation from a straight string and found it to reduce to a physically reasonable non-relativistic limit. The solutions we have obtained are adiabatic in the sense that they depend only on the end condition, and would be static in the end rest frame.

To find the shape of the curved string, we perturbed an exact straight solution to the Nambu-Goto-Polyakov string equations and solved exactly the resulting equations of motion to leading order in the perturbation. As long as the angular acceleration is sufficiently small this should provide an accurate picture of the string shape. We numerically solved a straight string model to estimate the angular acceleration that occurs in actual mesons. The result was that the string deviation from a straight line is never very large, justifying the use of our perturbative approach, and showing the validity of the straight string approximation.

In addition we have computed the angular momentum, energy, and linear momentum of the curved string. In each case but one, the perturbation drops out and the perturbed string behaves identically to the straight one. Only for the radial momentum does the deviation from straightness have an effect. In the semi-relativistic approximation this radial momentum is smaller by a factor of $aR/6m$ than the first relativistic correction. The straight string approximation is then justified for heavy quark mesons as previously pointed out from a different point of view [3]. For relativistic mesons the string radial momentum may have small, but perhaps interesting and calculable consequences.

ACKNOWLEDGMENTS

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APPENDIX A: CONSERVATION OF MOMENTUM AND ANGULAR MOMENTUM

Here we consider the energy, momentum and angular momentum of a string with a quark at each end. The relevant actions are the Nambu-Goto-Polyakov action, Eq. (2.1), and the quark action

$$S_{\rm quark}=\frac{1}{2}\sum_{i=1,2}\int d\tau\,(e_i^{-1}\dot{x}_i^{\mu}\dot{x}_{i\mu}-e_i m_i^2), \tag{A1}$$

in which x_i^{μ} is the position of the i$^{\rm th}$ quark of mass m_i, and the auxiliary field e_i is a one-dimensional metric density analogous to the string's auxiliary field metric h_{ab}. The boundary condition relating the two actions (2.1) and (A1) is that the quarks sit at the ends of the string,

$$x_i^{\mu}(\tau)=X^{\mu}(\tau,\sigma_i), \quad i=1,2. \tag{A2}$$

Variation of the sum of the actions (2.1) plus (A1) with respect to X^{μ} yields the equations of motion for the quarks

$$\dot{p}_i^{\mu}=(-1)^i\mathcal{P}^{\mu\sigma}|_{\sigma=\sigma_i}, \tag{A3}$$

where p_i^{μ} is the momentum of the quark at $\sigma_i=(-1)^i$, related to the quark velocity by

$$p_i^{\mu}\equiv\frac{\delta S_{\rm quark}}{\delta\dot{x}_{i\mu}}=e_i^{-1}\dot{x}_i^{\mu}=m_i\frac{\dot{x}_i^{\mu}}{\sqrt{-\dot{x}_i^{\mu}\dot{x}_{i\mu}}}, \tag{A4}$$

and

$$\mathcal{P}^{\mu a}\equiv\frac{\delta S}{\delta X_{\mu,a}}=-a\sqrt{-h}\,h^{ab}X^{\mu}_{,b} \tag{A5}$$

is the momentum current on the string worldsheet, which is related to the canonical string momentum Π^{μ} by $\Pi^{\mu}=\mathcal{P}^{\mu\tau}$.

It is easy to show that the total momentum of the string-quark system,

$$P^{\mu}=\int_{\sigma_1}^{\sigma_2}d\sigma\,\mathcal{P}^{\mu\tau}+\sum_{i=1,2}p_i^{\mu}, \tag{A6}$$

is conserved under τ evolution,

$$\begin{aligned}\partial_{\tau}P^{\mu}&=\int_{\sigma_1}^{\sigma_2}d\sigma\,\dot{\mathcal{P}}^{\mu\tau}+\sum_{i=1,2}\dot{p}_i^{\mu}\\&=-\int_{\sigma_1}^{\sigma_2}d\sigma\,\mathcal{P}^{\mu\sigma\prime}+\sum_{i=1,2}\dot{p}_i^{\mu}\\&=\dot{p}_1^{\mu}+\mathcal{P}^{\mu\sigma}|_{\sigma=\sigma_1}+\dot{p}_2^{\mu}-\mathcal{P}^{\mu\sigma}|_{\sigma=\sigma_2}=0.\end{aligned} \tag{A7}$$

The equations of motion $\mathcal{P}^{\mu a}_{,a}=0$ are used to go from the first to the second line, and the final result vanishes by the quark equations of motion (A3).

The conservation of the angular momentum,

$$J^{\mu\nu}=\int_{\sigma_1}^{\sigma_2}d\sigma\,X^{[\mu}\mathcal{P}^{\nu]\tau}+\sum_{i=1,2}x_i^{[\mu}p_i^{\nu]}, \tag{A8}$$

is only slightly more complicated to demonstrate. The τ derivative of $J^{\mu\nu}$ is

$$\begin{aligned}\partial_{\tau}J^{\mu\nu}&=\int_{\sigma_1}^{\sigma_2}d\sigma\,(\dot{X}^{[\mu}\mathcal{P}^{\nu]\tau}+X^{[\mu}\dot{\mathcal{P}}^{\nu]\tau})\\&\quad+\sum_{i=1,2}(\dot{x}_i^{[\mu}p_i^{\nu]}+x_i^{[\mu}\dot{p}_i^{\nu]})\\&=\int_{\sigma_1}^{\sigma_2}d\sigma\,(\dot{X}^{[\mu}\mathcal{P}^{\nu]\tau}-X^{[\mu}\mathcal{P}^{\nu]\sigma\prime})+\sum_{i=1,2}x_i^{[\mu}\dot{p}_i^{\nu]}\\&=\int_{\sigma_1}^{\sigma_2}d\sigma\,(-X^{[\mu\prime}\mathcal{P}^{\nu]\sigma}-X^{[\mu}\mathcal{P}^{\nu]\sigma\prime})+\sum_{i=1,2}x_i^{[\mu}\dot{p}_i^{\nu]}\\&=-X^{[\mu}\mathcal{P}^{\nu]\sigma}|_{\sigma=\sigma_2}+X^{[\mu}\mathcal{P}^{\nu]\sigma}|_{\sigma=\sigma_1}+\sum_{i=1,2}x_i^{[\mu}\dot{p}_i^{\nu]}.\end{aligned} \tag{A9}$$

In going from the second to the third equality, we have used

$$\begin{aligned}\dot{X}^{[\mu}\mathcal{P}^{\nu]\tau}&=-a\sqrt{-h}\,h^{\tau a}\dot{X}^{[\mu}X^{\nu]}_{,a}\\&=-a\sqrt{-h}\,h^{\tau\sigma}\dot{X}^{[\mu}X^{\nu]\prime}\\&=-X^{[\mu\prime}\mathcal{P}^{\nu]\sigma}.\end{aligned}$$

Using the boundary conditions (A2) that place the quark at the end of the string, we find

$$\begin{aligned}\partial_{\tau}J^{\mu\nu}&=(x_1^{[\mu}\dot{p}_1^{\nu]}+x_1^{[\mu}\mathcal{P}^{\nu]\sigma}|_{\sigma=\sigma_1})+(x_2^{[\mu}\dot{p}_2^{\nu]}-x_2^{[\mu}\mathcal{P}^{\nu]\sigma}|_{\sigma=\sigma_2})\\&=0.\end{aligned} \tag{A10}$$

Finally, the quark mass-shell condition,

$$p_i^{\mu}p_{i\mu}+m_i^2=0, \tag{A11}$$

which follows identically from the relation of the quark momenta to quark velocities,

$$p_i^\mu = m_i \frac{\dot{x}_i^\mu}{\sqrt{-\dot{x}_i^\mu \dot{x}_{i\,\mu}}}, \tag{A12}$$

must be preserved by the equations of motion (2.6). We find that

$$\begin{aligned}\partial_\tau(p^\mu p_\mu + m^2) &= 2p^\mu \dot{p}_\mu \\ &\propto p^\mu h^{\sigma a} X_{\mu,a} \propto \dot{X}^\mu h^{\sigma a} X_{\mu,a} \\ &= h^{\sigma a} X_{\mu,a} X^\mu_{,\tau} = h^{\sigma a} h_{a\tau} = \delta^\sigma_\tau = 0.\end{aligned} \tag{A13}$$

The three conditions (A7), (A10), and (A13) together imply that the conservation of energy momentum, angular momentum, and the quark mass-shell relation are equivalent to the quark equations of motion, Eqs. (2.6) and (A3).

APPENDIX B: EXACT SOLUTION OF THE RELATIVISTIC STRING SHAPE EQUATION

We consider here the details of the solution of Eq. (4.10)

$$(1-\xi^2)\frac{d^2 f}{d\xi^2} + \xi \frac{df}{d\xi} - f = \beta\xi \tag{B1}$$

to obtain the central result (4.13) of the paper. The solution method follows standard procedures. First we define a new function, g, so that

$$f(\xi) = \xi g(\xi), \tag{B2}$$

giving

$$\xi(1-\xi^2)g'' + (2-\xi^2)g' = \beta\xi. \tag{B3}$$

Now define

$$g' \equiv H(\xi) I(\xi) \tag{B4}$$

to obtain

$$\xi(1-\xi^2)I(\xi)H' + H[\xi(1-\xi^2)I' + (2-\xi^2)I] = \beta\xi. \tag{B5}$$

The integrating factor $I(\xi)$ is chosen to make the coefficient of H vanish and by quadrature,

$$I(\xi) = \frac{\sqrt{1-\xi^2}}{\xi^2}. \tag{B6}$$

Substituting into Eq. (B5), we find

$$H(\xi) = \beta\left(\frac{\xi}{\sqrt{1-\xi^2}} - \arcsin\xi\right) + C_2, \tag{B7}$$

and, by Eq. (B4),

$$\frac{dg}{d\xi} = \beta\left(\frac{1}{\xi} - \frac{\xi}{\sqrt{1-\xi^2}}\arcsin\xi\right) + C_2(\sqrt{1-\xi^2} + \xi\arcsin\xi). \tag{B8}$$

Finally by quadrature we have $g(\xi)$ and $f = \xi g$ is

$$\begin{aligned} f(\xi) = {} & \beta\left(\sqrt{1-\xi^2} + \frac{\xi}{2}\arcsin\xi\right)\arcsin\xi + C_1\xi \\ & + C_2(\sqrt{1-\xi^2} + \xi\arcsin\xi). \end{aligned} \tag{B9}$$

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