Positivity constraints on chiral perturbation theory pion-pion scattering amplitudes

Petre Diţă*
National Institute of Physics & Nuclear Engineering, Bucharest, P.O. Box MG6, Romania
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We test the positivity property of the chiral perturbation theory (ChPT) pion-pion scattering amplitudes within the Mandelstam triangle. In the one-loop approximation $\mathcal{O}(p^4)$ the positivity constrains only the coefficients b_3 and b_4; namely, one obtains that b_4 and the linear combination b_3+3b_4 are positive quantities. The two-loop approximation gives inequalities involving all six arbitrary parameters entering ChPT amplitude, but the corrections to the one-loop approximation results are small. ChPT amplitudes pass unexpectedly well all the positivity tests, giving strong support to the idea that ChPT is a good theory of low-energy pion-pion scattering. [S0556-2821(99)04407-0]

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I. INTRODUCTION

Chiral perturbation theory (ChPT) is considered a low-energy effective approximation of QCD. In particular it provides a representation of the elastic pion-pion scattering amplitudes that is crossing symmetric and has good analyticity properties. In a seminal paper [1] Gasser and Leutwyler developed ChPT which allows one to compute many Green functions involving low-energy pions. It is well known that the physical pion-pion scattering amplitudes can be expressed in terms of a single function $A(s,t,u)$ whose form was obtained as a series expansion in powers of the external momenta and of the light quark masses. The first term of the series was given by Weinberg [2], the second by Gasser and Leutwyler [1], and only recently a two-loop calculation [3,4] was obtained. In this approximation the function $A(s,t,u)$ has the following form:

$$\begin{aligned}A(s,t,u)=&a(s-1)+a^2[b_1+b_2s+b_3s^2+b_4(t-u)^2]\\&+a^2[F^{(1)}(s)+G^{(1)}(s,t)+G^{(1)}(s,u)]\\&+a^3[b_5s^3+b_6s(t-u)^2]+a^3[F^{(2)}(s)+G^{(2)}(s,t)\\&+G^{(2)}(s,u)]+\mathcal{O}(a^4),\end{aligned}\tag{1.1}$$

where $a=(M_\pi/F_\pi)^2$, M_π is the mass of the physical pion, F_π the pion decay constant, s,t,u are the usual Mandelstam variables, expressed in units of the physical pion mass squared M_π^2,

$$s=(p_1+p_2)^2/M_\pi^2,\quad t=(p_1-p_2)^2/M_\pi^2,$$

$$u=(p_1-p_3)^2/M_\pi^2,$$

$F^{(i)}(s)$ and $G^{(i)}(s,t)$ are known functions, and b_i, $i=1,\ldots,6$, are arbitrary parameters which cannot be determined by ChPT [1,3,4]. In any realistic comparison with experiment we have to provide some numerical values for all these parameters obtained from other sources. One hopes that by using unitarity this can be done, although, until now, no program for implementing this property has been presented. The common belief is that imposing unitarity is not a simple matter since its implementation in one channel destroys crossing symmetry in other channels. However, there is a weak form of unitarity, the positivity of the absorptive parts, which is a linear property and which can be imposed. This property was used 30 years ago to obtain constraints on the $\pi^0\pi^0 s$-wave partial amplitude $f_0(s)$ in the unphysical region $0<s<4$ and on the d-wave scattering lengths. These constraints were useful because at that time almost nothing was known about the explicit form of the scattering amplitudes and they were used in testing models for pion-pion partial-wave amplitudes. The advantage of ChPT is that it furnishes an explicit form for the pion-pion scattering amplitudes whose unknown part is contained in a few numerical coefficients. Thus it is of certain interest to see how these properties reflect on the constraints on the b_i coefficients entering Eq. (1.1).

Beginning with Ref. [5], Martin has used the positivity, analyticity, and crossing symmetry to obtain constraints on the $\pi^0\pi^0 s$-wave partial amplitude $f_0(s)$ in the unphysical region $0<s<4$; a few of them have the following form [6]:

$$f_0(4)>f_0(0)>f_0(3.15),\quad f_0(0)>\frac{1}{2}\int_2^4 f_0(s)\,ds,$$

$$F\left[0,2\left(1+\frac{1}{\sqrt{3}}\right)\right]>f_0\left[2\left(1+\frac{1}{\sqrt{3}}\right)\right],\tag{1.2}$$

where $F(s,t)$ denotes the $\pi^0\pi^0$ elastic scattering amplitude. A more complete set is found in Ref. [7]. The most elaborate form of these constraints is the following result: the $\pi^0\pi^0 s$-wave amplitude $f_0(s)$ has a minimum located at $1.218989<s<1.696587$ [6–12] and this result can be improved only by unitarity. These results can be translated into constraints on the parameters b_i entering ChPT pion-pion scattering amplitudes. As we will see later all the above inequalities are equivalent in the one-loop approximation $\mathcal{O}(p^4)$ to a single constraint on the coefficients b_i whose typical form is

$$b_3+3b_4\geqslant\frac{37}{1920\pi^2},$$

*Email address: dita@theor1.theory.ifa.ro

the only difference being the numerical value appearing on the right hand side. This is the explanation of the inefficacy of these apparently distinct constraints which was observed from the beginning by physicists constructing models for pion-pion partial waves.

In this paper we work only in the unphysical region $|s,t,u|<4$, i.e., where the amplitude (1.1) is considered to be a very good approximation to the true amplitude. Other approaches use information from the physical region to obtain constraints on the same parameters b_i [13–15]. In the one-loop approximation $\mathcal{O}(p^4)$, the positivity property constrains only the coefficients b_3 and b_4. By taking into account the $\mathcal{O}(p^6)$ contributions one gets constraints involving all six parameters entering Eq. (1.1).

By using the unitarity bounds on $\pi^0\pi^0$ scattering amplitude in the unphysical region [16] one gets upper and lower bounds on some linear combinations of the parameters b_i. These unitarity bounds are not very constraining; to see this we give the one obtained from the bound on $F(2,0)$, where as above $F(s,t)$ denotes the $\pi^0\pi^0$ amplitude. The bound is $-3.5\leq F(2,0)\leq 2.9$ and it is equivalent to the following lower and upper bounds:

$$-3.5\times 32\pi\leq a+a^2\left(3b_1+4b_2+8b_3+8b_4+\frac{9}{32\pi^2}\right)+a^3\left[16b_5+16b_6+\frac{(4-\pi)}{\pi^2}\left(\frac{5b_1}{16}+\frac{b_2}{2}+\frac{11b_3}{12}+\frac{5b_4}{12}\right)+\frac{965}{3456\pi^4}-\frac{251}{3456\pi^3}+\frac{41}{6144\pi^2}\right]\leq 2.9\times 32\pi.$$

Because of the factor 32π appearing on the left and right hand sides, the bounds are not very strong and we will not consider them here.

The physical isospin amplitudes F^I can be expressed in terms of the single function $A(s,t,u)$ as follows:

$$F^0(s,t,u)=3A(s,t,u)+A(t,u,s)+A(u,s,t),$$

$$F^1(s,t,u)=A(t,u,s)-A(u,s,t),$$

$$F^2(s,t,u)=A(t,u,s)+A(u,s,t),$$

where $A(s,t,u)$ is given by Eq. (1.1).

Having only three independent amplitudes one gets only three independent constraints since the crossing symmetry is an exact symmetry for the ChPT amplitudes. The construction of our positivity constraints is outlined in the next section where we present an overdetermined system of constraints. Their implications on the coefficients b_i are discussed in Sec. III. The paper concludes in Sec. IV.

II. POSITIVITY CONSTRAINTS

Let $F^I(s,t)$ denote the $\pi\pi$ scattering amplitude with isotopic spin I in the s channel. In matrix notation $\mathbf{F}(s,t)$ satisfies the following crossing relation [17]:

$$\mathbf{F}(s,t)=C_{st}\mathbf{F}(t,s)=C_{su}\mathbf{F}(u,t),$$

where the notation is

$$\mathbf{F}(s,t)=\begin{pmatrix}F^0(s,t)\\F^1(s,t)\\F^2(s,t)\end{pmatrix},$$

$$C_{st}=\begin{pmatrix}1/3 & 1 & 5/3\\1/3 & 1/2 & -5/6\\1/3 & -1/2 & 1/6\end{pmatrix},$$

$$C_{su}=\begin{pmatrix}1/3 & -1 & 5/3\\-1/3 & 1/2 & 5/6\\1/3 & 1/2 & 1/6\end{pmatrix}.$$

From the results of axiomatic field theory we know that the amplitudes $F^I(s,t)$ satisfy fixed-t dispersion relations with two subtractions [18] for $|t|<4$. We may write them as

$$\mathbf{F}(s,t)=C_{st}[\mathbf{a}(t)+(s-u)\mathbf{b}(t)]+\frac{1}{\pi}\int_4^\infty\frac{dx}{x^2}\left(\frac{s^2}{x-s}+\frac{u^2}{x-u}C_{su}\right)\mathbf{A}(x,t),\quad (2.1)$$

where $\mathbf{A}(x,t)$ is the absorptive part of $\mathbf{F}(s,t)$ and the subtraction constants are of the form

$$\mathbf{a}(t)=\begin{pmatrix}a^0(t)\\0\\a^2(t)\end{pmatrix},\quad \mathbf{b}(t)=\begin{pmatrix}0\\b^1(t)\\0\end{pmatrix},$$

as a result of crossing symmetry.

In the following we shall consider that s,t,u take values in the unphysical region $|s,t,u|<4$. We calculate the difference

$$\mathbf{F}(s,t)-\mathbf{F}(s_1,t),$$

and we are looking for those combinations of isospin amplitudes for which this difference does not depend on the subtraction constants. From Eq. (2.1) we find

$$\frac{1}{s-s_1}[\mathbf{F}(s,t)-\mathbf{F}(s_1,t)]=2C_{st}\mathbf{b}(t)+f(A),\quad (2.2)$$

where $f(A)$ denotes a complicated term containing the integration over the absorptive parts. The first term on the right hand side of Eq. (2.2) is

$$C_{st}\mathbf{b}(t)=\begin{pmatrix}b(t)\\\frac{1}{2}b(t)\\-\frac{1}{2}b(t)\end{pmatrix}.$$

The last relation shows that there are three combinations of isospin amplitudes for which the difference (2.2) has no dependence on the subtraction constants. They are F^0+2F^2, F^1+F^2, and F^0-2F^1 and we shall denote them as F_i, $i=1,2,3$, in this order. The first one is the well-known $\pi^0\pi^0$ elastic amplitude. One easily obtains from Eq. (2.2) the relation

$$F_i(s,t)-F_i(s_1,t)=\frac{(s-s_1)(s-u_1)}{\pi}\times\int_4^\infty\frac{(2x+t-4)A_i(x,t)\,dx}{(x-s)(x-s_1)(x-u)(x-u_1)},$$

$i=1,2,3$. From this relation we get

$$\frac{\partial F_i(s,t)}{\partial s}=\frac{s-u}{\pi}\int_4^\infty\frac{(2x+t-4)A_i(x,t)\,dx}{(x-s)^2(x-u)^2}.$$

Because the absorptive parts A_1 and A_2 are positive, we find that

$$\frac{1}{s-u}\,\frac{\partial F_i(s,t)}{\partial s}\geqslant 0,\quad i=1,2.$$

The third combination involves the absorptive parts $A^0(x,t)-2A^1(x,t)$ whose sign is not defined and we cannot say anything about the sign of the derivatives of $F_3(s,t)$. The preceding relations show us that on the line $s=u$, $F_i(s,t)$, $i=1,2$, attain their minimum values. Indeed we obtain from them the second derivatives

$$\frac{\partial^2 F_i(s,t)}{\partial s^2}=\frac{2}{\pi}\int_4^\infty\left(\frac{1}{(x-s)^3}+\frac{1}{(x-u)^3}\right)A_i(x,t)dx\geqslant 0,\quad i=1,2, \tag{2.3}$$

which are positive definite, implying that the functions $F_i(s,t)$ have a minimum on the line $s=u$. From the last relation we obtain also

$$\frac{\partial^{2n-1}F_i(s,t)}{\partial s^{2n-1}}=\frac{(2n-1)!}{\pi}\int_4^\infty\left(\frac{1}{(x-s)^{2n}}-\frac{1}{(x-u)^{2n}}\right)A_i(x,t)\,dx=\frac{(2n-1)!(s-u)}{\pi}\int_4^\infty[(x-s)^n+(x-u)^n]\left[\frac{(x-s)^{n-1}+(x-s)^{n-2}(x-u)+\cdots+(x-u)^{n-1}}{(x-s)^{2n}(x-u)^{2n}}\right]A_i(x,t)dx.$$

In this way we obtain the set of positivity constraints

$$\frac{1}{s-u}\,\frac{\partial^{2n-1}F_i(s,t)}{\partial s^{2n-1}}\geqslant 0,\qquad \frac{\partial^{2n}F_i(s,t)}{\partial s^{2n}}\geqslant 0, \tag{2.4}$$

$i=1,2$, $n=1,2,\ldots$.

A first remark is the following: if the positivity constraints have to be satisfied, it is sufficient to test them only on the line $s=u$, i.e., $2s+t-4=0$, where the functions $F_i(s,t)$ attain their minimum values. In this way we have only one free parameter $0<|s|<4$, and on this line the odd and even derivatives give the same information. From the point of view of computation it is simpler to work with even derivatives.

Up to now we have obtained two constraints given by Eq. (2.3). Because we have three independent isospin amplitudes, it follows that we can obtain another one at most.

The positivity constraints can be imposed even on the isospin amplitudes themselves. This can be easily seen from the relation (2.1) for $F^2(s,t)$ which after derivation gives

$$\frac{\partial^2 F^2(s,t)}{\partial s^2}=\frac{2}{\pi}\int_4^\infty dx\left[\left(\frac{1}{(x-s)^3}+\frac{1}{6}\,\frac{1}{(x-u)^3}\right)A^2(x,t)+\frac{1}{3}\,\frac{1}{(x-u)^3}A^0(x,t)+\frac{1}{2}\,\frac{1}{(x-u)^3}A^1(x,t)\right].$$

The right hand side of the previous relation is a positive quantity and by iteration we obtain that

$$\frac{\partial^{2n}F^2(s,t)}{\partial s^{2n}}\geqslant 0,\quad n=1,2,\ldots. \tag{2.5}$$

Unfortunately numerical calculations show that this relation is not independent of the previous two ones. Another way to obtain them is to make use of the Gribov-Froissart representation for the partial-wave amplitudes. One writes dispersion relations for the isospin amplitudes, the subtraction constants being given by the *s*- and *p*-wave partial amplitudes, and one finds

$$F^I(s,t)=f_0^I(s)+\frac{1}{\pi}\int_0^\infty A^I(x,s)g(x,s,t)\,dx,\quad I=0,2, \tag{2.6}$$

where

$$g(x,s,t)=\frac{1}{x-t}+\frac{1}{x-u}+\frac{2}{4-s}\ln\left(1+\frac{s-4}{x}\right).$$

For $I=1$ we can write similarly

$$F^1(s,t)=3f_1^1(s)\left(1+\frac{2t}{s-4}\right)+\frac{1}{\pi}\int_0^\infty A^1(x,s)h(x,s,t)\,dx, \tag{2.7}$$

where

$$h(x,s,t)=\frac{1}{x-t}-\frac{1}{x-u}+\frac{6(2t+s-4)}{(4-s)^2}\times\left[\frac{2x+s-4}{4-s}\ln\left(1+\frac{s-4}{x}\right)+2\right].$$

The absorptive parts entering Eqs. (2.6),(2.7) are the t-channel ones and in the following we make use of the $t\leftrightarrow u$ crossing symmetry. From the relation (2.6) we find formulas analogous to Eq. (2.4), namely,

$$\frac{1}{t-u}\frac{\partial^{2n-1}F^I(s,t)}{\partial t^{2n-1}}\geqslant 0,\quad \frac{\partial^{2n}F^I(s,t)}{\partial t^{2n}}\geqslant 0, \tag{2.8}$$

$n=1,2,\ldots,I=0,2$, which are not numerically independent of the previous ones. More interesting is the relation (2.7) which can be written as

$$\frac{F^1(s,t)}{t-u}=\widehat{F^1}(s,t)=\frac{3f_1^1(s)}{s-4}+\frac{1}{\pi}\int_0^\infty A^1(x,s)\times\left\{\frac{1}{(x-t)(x-u)}+\frac{6}{(4-s)^2}\left[\frac{2x+s-4}{4-s}\times\ln\left(1+\frac{s-4}{x}\right)+2\right]\right\}dx.$$

This relation provides us with another independent relation. Because $F^1(s,t)$ is antisymmetric in $t\leftrightarrow u$, $\widehat{F^1}(s,t)$ is an analytic function within the Mandelstam triangle. Deriving it with respect of t we obtain

$$\frac{\partial\widehat{F^1}(s,t)}{\partial t}=\frac{t-u}{\pi}\int_0^\infty\frac{A^1(x,s)}{(x-t)^2(x-u)^2}dx.$$

From this relation we obtain formulas similar to Eq. (2.4), namely,

$$\frac{1}{t-u}\frac{\partial^{2n-1}\widehat{F^I(s,t)}}{\partial t^{2n-1}}\geqslant 0,\quad \frac{\partial^{2n}\widehat{F^I(s,t)}}{\partial t^{2n}}\geqslant 0. \tag{2.9}$$

Finally the positivity may also be expressed as the positivity of the partial-wave amplitudes

$$f_l^I(s)=\frac{1}{4-s}\int_4^\infty A^I(x,s)Q_l\left(\frac{2x}{4-s}-1\right)dx \tag{2.10}$$

for $l\geqslant 2$ inside the unphysical region $0\leqslant s\leqslant 4$, but the numerical calculations show that these last constraints are weaker than those derived above.

III. NUMERICAL RESULTS

In the previous section we derived a complete set of positivity constraints. In an exact theory many of them are a consequence of the others as we will see later. Because ChPT does not completely specify the amplitude and on the other hand the power of different constraints is not the same, it is useful to derive as many constraints as possible. Since all of them have to be satisfied, we will select the strongest one in every case.

To test the method we worked first in the one-loop approximation $\mathcal{O}(p^4)$; i.e., we retained terms up to a^2 in Eq. (1.1). In this order the obtained constraints do not depend on the value taken by a; in the two-loop approximation the constraints will be linear in a. We consider first the constraints on the $\pi^0\pi^0$ scattering amplitude.

The first two constraints (1.2) on the $\pi^0\pi^0,s$-wave $f_0(4)>f_0(0)>f_0(3.15)$ are equivalent to

$$b_3+3b_4\geqslant-\frac{9}{1024}-\frac{29}{384\pi^2}\approx-1.64\times10^{-2}$$

and

$$b_3+3b_4\geqslant-1.47\times10^{-2},$$

respectively. The third relation $f_0(0)>\frac{1}{2}\int_2^4 f_0(s)\,ds$ is equivalent to

$$b_3+3b_4\geqslant-4.64\times10^{-4},$$

the strongest result being the last one. Already the last relation (1.2) furnishes a better result; the above combination of coefficients becomes positive:

$$b_3+3b_4\geqslant 7.36\times10^{-4}.$$

The derivative of the s wave has the form

$$f_0'(s)=\frac{2}{3}(b_3+3b_4)(5s-8)+h(s),$$

where the function $h(s)$ has a long expression which we do not write here. Since $5s-8<0$, for $s<1.6$, the upper and lower bounds on the derivative $f_0'(s)$ describing the position of the minimum are equivalent to a single lower bound on the combination b_3+3b_4. The best result is obtained at $s=1.696587$ and is

$$b_3+3b_4\geqslant 6.67\times10^{-3}.$$

The last result is the strongest constraint upon the combination b_3+3b_4 obtained from inequalities satisfied by the $\pi^0\pi^0 s$-wave amplitudes within the unphysical region $0\leqslant s\leqslant 4$. We have given all the above results to understand why these constraints were easily satisfied by the phenomenological models for partial-wave amplitudes constructed in the past years; satisfying a few of them, the others are automatically fulfilled.

A similar analysis with analogous results was done in Ref. [19], including in the analysis the s and d waves.

Stronger constraints are obtained from the positivity of the second derivative of the full amplitudes. From Eq. (2.3) we find for $i=1$, i.e., the $\pi^0\pi^0$, amplitude again, a relation of the form

$$b_3+3b_4+h_1(s)\geqslant 0,$$

where $h_1(s)$ is a decreasing function for $s<4$. Thus a problem arises: at what point does the above relation have to be considered. We decided to limit the range of s within the interval $|s|<4$ since we are aware that the amplitude (1.1) is only an approximation of the true amplitude, an approximation truly not valid for values $s>16$ in the physical region. $s=-4$ is equivalent to $t=12$ in the physical region of the t channel. One gets

$$b_3+3b_4\geqslant\frac{37}{1920\pi^2}\approx 1.92\times 10^{-3}\quad \text{for}\quad s=0$$

and

$$b_3+3b_4\geqslant 8.28\times 10^{-3}\quad \text{for}\quad s=-4,$$

which is stronger than the previous inequality. In the following we will list numerical values at $s=0$ and $s=-4$, the last ones being a little better such as the previous relations show.

We will work now in the two-loop approximation $\mathcal{O}(p^6)$ and consider only the constraints derived in Sec. II which give the strongest results. The constraints have the form

$$\sum_{i=1}^{6} c_i^k(s,a)b_i+f_k(s,a)\geqslant 0, \tag{3.1}$$

where $k=1,2,3$ labels the independent constraints. As we said before there are only three possible constraints from the positivity of the absorptive parts, because we have only three independent amplitudes and the crossing symmetry is an exact symmetry for the ChPT amplitudes. In the previous section we derived six constraints but only three are numerically independent. The above inequalities are obtained for every value of s within the unphysical region $0\leqslant|s|\leqslant 4$. Each inequality defines a half space where the parameters b_i can reside. Thus the true result would be that obtained by constructing the intersection of these half spaces. Unfortunately the envelope cannot be found in an analytic form, the functions $f_k(s,a)$ being very complicated.

The constraint (2.3) for $i=1$ and $s=0$ is equivalent to

$$b_3+3b_4-\frac{37}{1920\pi^2}+a\left[\frac{7b_1}{320\pi^2}-\frac{b_2}{60\pi^2}-\frac{2b_3}{45\pi^2}+\frac{b_4}{180\pi^2}+16b_6-\frac{367}{552960\pi^2}+\frac{6869}{1658880\pi^4}\right]\geqslant 0. \tag{3.2}$$

For $s=-4$ one obtains

$$b_3+3b_4-8.28\times 10^{-3}+a[1.25\times 10^{-3}+9.7\times 10^{-4}b_1-1.56\times 10^{-2}b_2+0.13b_3+5.03\times 10^{-2}b_4-12b_5+52b_6]\geqslant 0. \tag{3.2$'$}$$

Taking $a\to 0$ one gets the one-loop result.

For $i=2$ we find the relations

$$b_4-\frac{31}{5760\pi^2}-a\left[\frac{b_1}{240\pi^2}+\frac{43b_2}{2880\pi^2}+\frac{b_3}{24\pi^2}+\frac{23b_4}{180\pi^2}-4b_6-\frac{67}{276480\pi^2}-\frac{707}{331776\pi^4}\right]\geqslant 0\quad \text{for}\quad s=0 \tag{3.3}$$

and

$$b_4-1.24\times 10^{-3}+a[2.85\times 10^{-4}-2.04\times 10^{-4}b_1-2.69\times 10^{-3}b_2-1.85\times 10^{-2}b_3-0.143b_4+12b_6]\geqslant 0 \tag{3.3$'$}$$

for $s=-4$, respectively.

The inequality (2.5) for $n=2$ gives for $s=0$ and $s=-4$, respectively,

$$b_3+5b_4-\frac{173}{5760\pi^2}+a\left[\frac{13b_1}{960\pi^2}-\frac{67b_2}{1440\pi^2}-\frac{23b_3}{180\pi^2}-\frac{b_4}{4\pi^2}+24b_6-\frac{11}{61440\pi^2}+\frac{13939}{1658880\pi^4}\right]\geqslant 0, \tag{3.4}$$

$$b_3+5b_4-1.08\times 10^{-2}+a[5.65\times 10^{-4}b_1-2.09\times 10^{-2}b_2+9.15\times 10^{-2}b_3-0.236b4-12b_5+76b_6]\geqslant 0, \tag{3.4$'$}$$

and it is easily seen that both are a consequence of the previous two being the sum of the first and of the second one multiplied by 2. The relations (2.6)–(2.8) give us, in principle, three new inequalities, but only one will be independent of the first two already obtained. For $I=0$ we obtain

$$b_3+7b_4-\frac{47}{1152\pi^2}+a\left[\frac{b_1}{192\pi^2}-\frac{11b_2}{144\pi^2}-\frac{19b_3}{90\pi^2}-\frac{91b_4}{180\pi^2}+32b_6+\frac{169}{552960\pi^2}+\frac{7003}{550960\pi^4}\right]\geqslant 0,$$

$$b_3+7b_4-1.33\times 10^{-2}+a[1.58\times 10^{-4}b_1-2.68\times 10^{-2}b_2+5.45\times 10^{-2}b_3-0.522b_4-12b_5+100b_6-1.1\times 10^{-4}]\geqslant 0,$$

which again are linear combinations of the first two.

For $I=1$ the relations have the form

$$\frac{11}{2688\pi^2}+a\left[\frac{b_1}{224\pi^2}+\frac{17b_2}{1344\pi^2}-\frac{151b_3}{2016\pi^2}-\frac{653b_4}{2016\pi^2}+b_5+b_6+\frac{37}{215040\pi^2}+\frac{4111}{290304\pi^4}\right]\geqslant 0, \tag{3.5}$$

$$1.6\times 10^{-4}+a[b_5+b_6+10^{-4}b_1+4.2\times 10^{-4}b_2-2.9\times 10^{-2}b_3-8.4\times 10^{-2}b_4+2.78\times 10^{-4}]\geqslant 0. \tag{3.5$'$}$$

This is the third independent relation as can easily be seen because the one-loop approximation gives a positive number independent of b_i. More important is the fact that the function $f_3(t,a)$ appearing in the relation (3.1) in the one-loop approximation is positive over (presumably) the entire negative real axis which proves that the positivity is very well satisfied even by the lowest approximation of the ChPT amplitudes.

For $I=2$ the inequalities are

$$b_3+b_4-\frac{49}{5760\pi^2}+a\left[\frac{29b_1}{960\pi^2}+\frac{19b_2}{1440\pi^2}+\frac{7b_3}{180\pi^2}+\frac{47b_4}{180\pi^2}+8b_6-\frac{127}{110592\pi^2}-\frac{67}{552960\pi^4}\right]\geqslant 0, \tag{3.6}$$

$$b_3+b_4-5.8\times 10^{-3}+(-1.82\times 10^{-3}+1.38\times 10^{-3}b_1-1.02\times 10^{-2}b_2+0.165b_3+0.336b_4-12b_5+28b_6)\geqslant 0. \tag{3.6'}$$

We have written all the inequalities since they were useful in checking the calculations.

A first remark is the following: at $s=0$ the coefficient b_5 appears only in the inequality (3.3). Thus we can say that it is unimportant around $s=0$ and make it vanish also in the amplitudes. It is true that the above inequalities have been obtained by an extremal property; they are calculated on the line where the corresponding amplitudes are taking their minimal values. This may be a suggestion that the physical partial waves satisfy also an extremal principle which has to be found. This is also supported by the findings of Wanders, who could not obtain a reliable value for this parameter [15]. What the above results suggest is that a good determination of b_5 can be obtained only from $I=1$ data.

We have tested how the parameters b_i found in the literature compare with the inequalities. Unfortunately there are only two papers that give values for all b_i [14,20]. The values given by Bijnens *et al.* are almost good since all but the last one inequality (3.6) are satisfied. The values of the left hand side of Eq. (3.6) are 0.00329 at $s=0$ and -0.0058 at $s=-4$ for the first set of the parameters b_i (corresponding to the scale $\mu=1$ GeV), while for the second set corresponding to the scale $\mu=500$ MeV the values are 0.00435 and -0.00256, respectively. The values obtained in Ref. [20] strongly violate three of the above inequalities. Thus the left hand side of Eq. (3.2) is 0.0025 at $s=0$ and -0.0168 at $s=-4$ and the corresponding values for Eq. (3.4) are 0.1244 and -0.0054, respectively. The last inequality Eq. (3.6) is violated both at $s=0$ and $s=-4$, the numbers being -0.0074 and -0.0028. This shows that something is wrong with these numerical values and an independent checking of them is welcome. The values obtained by Knecht *et al.* [3] for the last four parameters can be used to obtain constraints on the b_1 and b_2 but the allowed domain is rather large; the same is true for the values given by Wanders [15].

Using a Roy equation analysis of the available $\pi\pi$ phase shift data Ananthanaryan and Büttiker [21] obtained values for the chiral coupling constants $\overline{l_1},\overline{l_2}$ and they can be translated into constraints on b_i. Since the parameters b_i are linear combinations of all four $\overline{l_i}$, the constraints become constraints on $\overline{l_3}$ and $\overline{l_4}$; so an independent determination of these last ones is necessary to check if the inequalites are satisfied or not. Unfortunately our results, being expressed by inequalities, cannot be directly used for predicting numerical values for b_i, but they can be used, such as the above numerical analysis shows, for testing values for b_i obtained by other methods. We have tested also the inequalities (2.4), (2.8), (2.9), and (2.10) for a few values $n\geqslant 3$ and $l\geqslant 2$, respectively, and we found that they are very well fulfilled.

IV. CONCLUSION

We have tested the positivity properties of the ChPT pion-pion amplitudes and we have obtained a number of inequalities which express this property. We conclude that the pion-pion amplitudes given by the relation (1.1) satisfy this property very well. In the $\mathcal{O}(p^4)$ approximation the positivity implies two constraints: b_4 is a positive quantity and so is the combination b_3+3b_4. As concerns the positivity of the $I=1$ amplitude this was tested numerically up to $t=-10^5$ where the derivative is still positive. Including $\mathcal{O}(p^6)$ contributions one get constraints involving all six parameters b_i, but the corrections to the $\mathcal{O}(p^4)$ results are small which support the idea that the expansion (1.1) is the best candidate for the true amplitude. It might seem surprising but we consider that the main result of this study is the conclusion that the most important parameters to be determined are b_3 and b_4, these parameters appearing in the one-loop results. This is a consequence of the good properties near threshold of the Weinberg approximation together with the very powerful property of the positivity of the scattering amplitudes. Let us recall that this property was essential in deriving the analyticity domain of pion-pion amplitudes [22]. This means that the amplitude (1.1) in which all but b_3,b_4 are zero will give a fair description of the low-energy phenomenology up to about 6–700 MeV and the contribution of the other coefficients will be seen at higher energies. In conclusion a good determination of the above two parameters will be a good starting point in the comparison of theory with experiment. Work along this line is in progress.

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[1] J. Gasser and H. Leutwyler, Ann. Phys. (N.Y.) **158**, 142 (1984); Nucl. Phys. **B250**, 465 (1985).
[2] S. Weinberg, Phys. Rev. Lett. **17**, 616 (1966).
[3] M. Knecht, B. Moussallam, J. Stern, and H. Fuchs, Nucl. Phys. **B457**, 513 (1995).
[4] J. Bijnens, G. Colangelo, G. Ecker, J. Gasser, and M. E. Sainio, Phys. Lett. B **374**, 210 (1996); Nucl. Phys. **B508**, 263 (1997).
[5] A. Martin, Nuovo Cimento A **42**, 930 (1966).
[6] A. Martin, Nuovo Cimento A **47**, 265 (1967); **58**, 303 (1968).
[7] A. Martin, Nuovo Cimento A **63**, 167 (1969).
[8] A. K. Common, Nuovo Cimento A **53**, 946 (1967); **56**, 524 (1968).
[9] G. Auberson, Nuovo Cimento A **68**, 281 (1970).
[10] A. K. Common and M. K. Pidcock, Nucl. Phys. **B42**, 194 (1972).
[11] P. Grassberger, Nucl. Phys. **B42**, 481 (1972).
[12] P. Dita, Nuovo Cimento A **15**, 327 (1973).
[13] M. Knecht, B. Moussallam, J. Stern, and H. Fuchs, Nucl. Phys. **B471**, 445 (1996).
[14] J. Bijnens, G. Collangelo, G. Ecker, J. Gasser, and M. I. Saino, Nucl. Phys. **B508**, 263 (1997).
[15] G. Wanders, Phys. Rev. D **56**, 4328 (1997).
[16] C. Lopez and G. Mennessier, Nucl. Phys. **B118**, 426 (1977).
[17] S. M. Roy, Phys. Lett. **36B**, 353 (1971).
[18] Y. S. Jin and A. Martin, Phys. Rev. **135**, B1375 (1964).
[19] B. Ananthanarayan, D. Toublan, and G. Wanders, Phys. Rev. D **51**, 1093 (1995).
[20] B. Ananthanarayan, Phys. Rev. D **58**, 036002 (1998).
[21] B. Ananthanarayan and P. Büttiker, Phys. Rev. D **54**, 1125 (1996).
[22] A. Martin, *Scattering Theory: Unitarity, Analyticity and Crossing*, Lecture Notes in Physics (Springer, Heidelberg, 1969).