Breaking of the flavor permutational symmetry: Mass textures and the CKM matrix

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Different *Ansätze* for the breaking of flavor permutational symmetry according to $S_L(3) \otimes S_R(3) \supset S_L(2) \otimes S_R(2)$ give different Hermitian mass matrices of the same modified Fritzsch type, which differ in the symmetry breaking pattern. In this work we obtain a clear and precise indication on the preferred symmetry breaking scheme from a fit of the predicted $|\mathbf{V}^{th}|$ to the experimentally determined absolute values of the elements of the CKM matrix. The preferred scheme leads to simple mass textures and allows us to compute the CKM mixing matrix, the Jarlskog invariant J, and the three inner angles of the unitarity triangle in terms of four quark mass ratios and only one free parameter: the CP violating phase Φ. Excellent agreement with the experimentally determined absolute values of the entries in the CKM matrix is obtained for $\Phi = 90°$. The corresponding computed values of the Jarlskog invariant and the inner angles are $J = 3.00 \times 10^{-5}$, $\alpha = 84°$, $\beta = 24°$, and $\gamma = 72°$ in very good agreement with current data on CP violation in the neutral kaon-antikaon system and oscillations in the B_s^0-$\bar{B}_s^0$ system. [S0556-2821(99)01309-0]

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I. INTRODUCTION

Recent interest in flavor or horizontal symmetry building (mass textures) has been spurred mainly by the large top quark mass and hence, the strong hierarchy in the quark masses [1–8]. A permutational flavor symmetry has been advocated by many authors in order to constrain the fermion mass matrices and mixing parameters [9–14]. Recently, various symmetry breaking schemes have been proposed based on the discrete non-Abelian group $S_L(3) \otimes S_R(3)$, which is broken according to $S_L(3) \otimes S_R(3) \supset S_L(2) \otimes S_R(2) \supset S_{diag}(2)$. The group $S(3)$ treats three objects symmetrically while the hierarchical nature of the Yukawa matrices is a consequence of the representation structure, $\mathbf{1} \oplus \mathbf{2}$, of $S(3)$ which treats the generations differently. Different *Ansätze* for the breaking of the permutational symmetry give rise to different Hermitian mass matrices $\mathbf{M}_q$ of the same modified Fritzsch type which, in a symmetry adapted basis, differ in the numerical value of the ratio $Z^{1/2} = M_{23}/M_{22}$. In the absence of a physically motivated argument to fix the value of $Z^{1/2}$, different values for $Z^{1/2}$ have been proposed by various authors [1–3,6,7,15–19].

In this paper, different symmetry breaking schemes are classified in terms of the irreducible representations of an auxiliary $\tilde{S}(2)$ group. Then, diagonalizing the mass matrices, we obtain exact explicit expressions for the elements of the mixing matrix, $\mathbf{V}_{CKM}$, the Jarlskog invariant J, and the inner angles of the unitarity triangle in terms of the quark mass ratios, the symmetry mixing parameter, and one CP violating phase. A χ^2 fit of the theoretical expressions to the experimentally determined absolute values of the elements of the $\mathbf{V}_{CKM}^{exp}$ mixing matrix gives a clear and precise indication on the preferred pattern for the breaking of the $S_L(3) \otimes S_R(3)$ symmetry. Simple, explicit expressions for the corresponding best mass textures are obtained from the best value of the mixing parameter $Z^{1/2}$. In this way we obtain an explicit parametrization of the Cabibbo-Kobayashi-Maskawa (CKM) matrix in terms of the four quark mass

ratios $m_u/m_t, m_c/m_t, m_d/m_b, m_s/m_b$ and one CP violating phase in good agreement with the experimental information about quark mixings and CP violation in the K^0-$\bar{K}^0$ system and the most recent data on oscillations of the B_s^0-$\bar{B}_s^0$ system.

The plan of this paper is as follows: In Sec. II we review some previous work on the breaking of the permutational flavor symmetry. A brief group theoretical analysis of the modified Fritzsch texture is made in Sec. III. The next section is devoted to the derivation of explicit expressions for the elements of the CKM mixing matrix and the Jarlskog invariant J in terms of the quark mass ratios and the mixing parameter $Z^{1/2}$. In Sec. V, we extract the best value of $Z^{1/2}$ from a χ^2 fit of our theoretical expressions to the experimentally determined absolute values of the entries in $|V_{CKM}^{exp}|$. The interpretation of the best value of $Z^{1/2}$ in terms of the analysis made in Secs. II and III and the derivation of the corresponding best mass textures is made in Sec. VI. The resulting parametrization of the CKM matrix in terms of four mass ratios and one CP violating phase is compared with the relevant experimental information in Sec. VII. Our paper ends with a summary of results and some conclusions.

II. FLAVOR PERMUTATIONAL SYMMETRY

In this section, we review some previous work on the breaking of the permutational flavor symmetry.

In the standard model, analogous fermions in different generations, say u, c, and t or d, s, and b, have completely identical couplings to all gauge bosons of the strong, weak, and electromagnetic interactions. Prior to the introduction of the Higgs boson and mass terms, the Lagrangian is chiral and invariant with respect to any permutation of the left and right quark fields. The introduction of a Higgs boson and the Yukawa couplings give mass to the quarks and leptons when the gauge symmetry is spontaneously broken. The quark mass term in the Lagrangian, obtained by taking the vacuum expectation value of the Higgs field in the quark Higgs coupling, gives rise to quark mass matrices $\mathbf{M_d}$ and $\mathbf{M_u}$:

$$\mathcal{L}_Y = \bar{\mathbf{q}}_{d,L}\mathbf{M}_d\mathbf{q}_{d,R} + \bar{\mathbf{q}}_{u,L}\mathbf{M}_u\mathbf{q}_{u,R} + \text{H.c.} \qquad (2.1)$$

In this expression, $\mathbf{q}_{d,L,R}(x)$ and $\mathbf{q}_{u,L,R}(x)$ denote the left and right quark d and u fields in the current or weak basis, $\mathbf{q}_q(x)$ is a column matrix, its components $\mathbf{q}_{q,k}(x)$ are the quark Dirac fields, k is the flavor index. In this basis, the charged hadronic currents are

$$J_\mu \sim \bar{q}_{u,L}\gamma_\mu q_{d,L}, \qquad (2.2)$$

where

$$\mathbf{q}_{u,W} = \begin{pmatrix} u_1(x) \\ u_2(x) \\ u_3(x) \end{pmatrix}_W, \quad \mathbf{q}_{d,W} = \begin{pmatrix} d_1(x) \\ d_2(x) \\ d_3(x) \end{pmatrix}_W, \quad (2.3)$$

and the subindex W means weak basis.

As is evident from Eq. (2.2), the charged hadronic currents are not changed if both, the d-type and the u-type fields are transformed with the same unitary matrix.

A number of authors [9–14,20] have pointed out that realistic quark mass matrices result from the flavor permutational symmetry $S_L(3) \otimes S_R(3)$ and its spontaneous or explicit breaking. The group $S(3)$ treats three objects symmetrically, while the hierarchical nature of the mass matrices is a consequence of the representation structure $\mathbf{1} \oplus \mathbf{2}$ of $S(3)$, which treats the generations differently. Under exact $S_L(3) \otimes S_R(3)$ symmetry, the mass spectrum, for either up or down quark sectors, consists of one massive particle (top and bottom quarks) in a singlet irreducible representation and a pair of massless particles in a doublet irreducible representation. In the weak basis, the mass matrix with the exact $S_L(3) \otimes S_R(3)$ symmetry reads

$$\mathbf{M}'_{3q,W} = \frac{m_{3q}}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}_W, \qquad (2.4)$$

where m_{3q} denotes the mass of the third family quark, t or b.

To generate masses for the second family, one has to break the permutational symmetry $S_L(3) \otimes S_R(3)$ down to $S_L(2) \otimes S_R(2)$. This may be done by adding to $\bar{\mathbf{q}}_L(\mathbf{M}'_{3q,W})\mathbf{q}_R$ a term $\bar{\mathbf{q}}_L(\mathbf{M}'_{2q,W})\mathbf{q}_R$ which is invariant under $S_L(2) \otimes S_R(2)$ but breaks $S_L(3) \otimes S_R(3)$. The most general form of a matrix $\mathbf{M}'_{2q,W}$ which is invariant under the permutations of the first two rows or two columns is

$$\mathbf{M}'_{2q,W} = m_{3q}\begin{pmatrix} \alpha' & \alpha' & \beta' \\ \alpha' & \alpha' & \beta' \\ \beta' & \beta' & \gamma \end{pmatrix}_W. \qquad (2.5)$$

Without loss of generality, this matrix may be decomposed in the sum of a $S_L(3) \otimes S_R(3)$ invariant term plus a traceless matrix $\mathbf{M}_{2q,W}$ invariant under $S_L(2) \otimes S_R(2)$:

$$\mathbf{M}'_{2q,W} = \frac{m_{3q}}{3}\left\{ (2\alpha' + \gamma)\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}_W + \begin{pmatrix} \alpha' - \gamma & \alpha' - \gamma & 3\beta' - 2\alpha' - \gamma \\ \alpha' - \gamma & \alpha' - \gamma & 3\beta' - 2\alpha' - \gamma \\ 3\beta' - 2\alpha' - \gamma & 3\beta' - 2\alpha' - \gamma & -2(\alpha' - \gamma) \end{pmatrix}_W \right\}. \qquad (2.6)$$

The first term in the right hand side of Eq. (2.6) is added to the term $\mathbf{M}'_{3q,W}$:

$$\mathbf{M}_{3q,W} = \frac{m_{3q}}{3}(1 - \Delta_q)\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}_W, \qquad (2.7)$$

where Δ_q stands for the factor $-(2\alpha' + \gamma)$.

The second term on the right hand side of Eq. (2.6) gives the most general form of the traceless matrix $\mathbf{M}_{2q,W}$ that breaks $S_L(3) \otimes S_R(3)$ down to $S_L(2) \otimes S_R(2)$ and gives mass to the second family:

$$\mathbf{M}_{2q,W} = \frac{m_{3q}}{3}\begin{pmatrix} \alpha & \alpha & \beta \\ \alpha & \alpha & \beta \\ \beta & \beta & -2\alpha \end{pmatrix}_W. \qquad (2.8)$$

In this expression we have simplified the notation by calling α and β in Eq. (2.8), the terms $(\alpha' - \gamma)$ and $(3\beta' - 2\alpha' - \gamma)$ in Eq. (2.6).

From expression (2.8) it is evident that $\mathbf{M}_{2q,W}$ is a linear combination of two linearly independent numerical matrices, $\mathbf{M}^A_{2q}$ and $\mathbf{M}^S_{2q}$,

$$\mathbf{M}_{2q,W} = \frac{m_{3q}}{3}(\sqrt{8}\,\alpha\mathbf{M}^A_{2q,W} + 2\beta\mathbf{M}^S_{2q,W}), \qquad (2.9)$$

where

$$\mathbf{M}^A_{2q,W} = \frac{1}{\sqrt{8}}\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}_W$$

$$\text{and} \quad \mathbf{M}^S_{2q,W} = \frac{1}{2} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}_W . \quad (2.10)$$

Later on, this property will be used to characterize the symmetry breaking pattern.

We may now turn our attention to the question of breaking the $S_L(2) \otimes S_R(2)$ symmetry. In order to give mass to the first family, we add another term $\mathbf{M}_{q1}$ to the mass matrix. It will be assumed that $\mathbf{M}_{q1}$ transforms as the mixed symmetry term of the doublet complex tensorial representation of the $S(3)_d$ diagonal subgroup of $S_L(3) \otimes S_R(3)$. Putting the first family in a complex representation will allow us to have a CP violating phase in the mixing matrix. Then, in the weak basis, $\mathbf{M}_{q1}$ is given by

$$\mathbf{M}_{q1,W} = \frac{m_{3q}}{\sqrt{3}} \begin{pmatrix} A_1 & iA_2 & -A_1 - iA_2 \\ -iA_2 & -A_1 & A_1 + iA_2 \\ -A_1 + iA_2 & A_1 - iA_2 & 0 \end{pmatrix}_W . \quad (2.11)$$

Finally, adding the three mass terms, (2.7), (2.8), and (2.11), we get the mass matrix $\mathbf{M}_q$ in the weak basis.

III. MODIFIED FRITZSCH TEXTURE

To make explicit the assignment of particles to irreducible representations of $S_L(3) \otimes S_R(3)$, it will be convenient to make a change of basis from the weak basis to a symmetry adapted or hierarchical basis. In this basis, the quark fields are

$$q_{1q,H}(x) = \frac{1}{\sqrt{2}}(q_{1q,W}(x) - q_{2q,W}(x)), \quad (3.1)$$

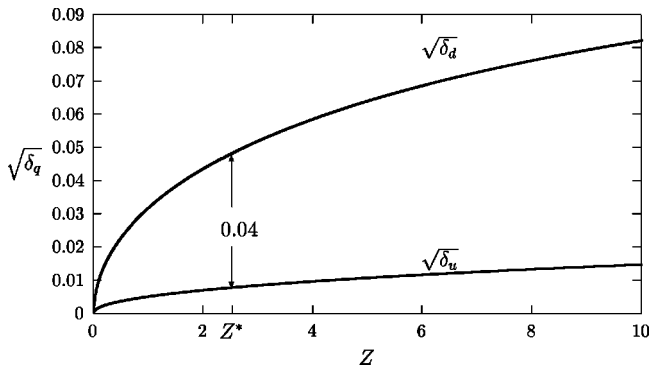$$q_{2q,H}(x) = \frac{1}{\sqrt{6}}(q_{1q,W}(x) + q_{2q,W}(x) - 2q_{3q,W}(x)), \quad (3.2)$$

FIG. 1. The square root of the parameters δ_u, δ_d is shown as a function of the ratio Z_q. The value $Z \approx 5/2$ which satisfies the constraining condition (5.11) may be read from the figure.

$$q_{3q,H}(x) = \frac{1}{\sqrt{3}}(q_{1q,W}(x) + q_{2q,W}(x) + q_{3q,W}(x)), \quad (3.3)$$

the subindex H denotes the hierarchical basis. In the hierarchical basis the third family quarks, t or b, are assigned to the invariant singlet irreducible representation $q_{3q,H}(x)$, the other two families are assigned to $q_{2q,H}(x)$ and $q_{1q,H}(x)$, the two components of the doublet irreducible representation of $S_{diag}(3)$.

The mass matrix $\mathbf{M}_{q,H}$ in the symmetry adapted basis is related to the mass matrix in the weak basis by the unitary transformation

$$\mathbf{M_{q,H}} = \mathbf{U}^\dagger \mathbf{M_{q,W}} \mathbf{U}, \quad (3.4)$$

where

$$\mathbf{U} = \frac{1}{\sqrt{6}} \begin{pmatrix} \sqrt{3} & 1 & \sqrt{2} \\ -\sqrt{3} & 1 & \sqrt{2} \\ 0 & -2 & \sqrt{2} \end{pmatrix}. \quad (3.5)$$

Then, in this basis, $\mathbf{M}_q$ takes the form

$$\mathbf{M}_{qH} = m_{3q} \left[\begin{pmatrix} 0 & A_q e^{-i\phi_q} & 0 \\ A_q e^{i\phi_q} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}_H \right.$$
$$+ \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\Delta_q + \delta_q & B_q \\ 0 & B_q & \Delta_q - \delta_q \end{pmatrix}_H \right]$$
$$+ m_{3q} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 - \Delta_q \end{pmatrix}_H$$
$$= m_{3q} \begin{pmatrix} 0 & A_q e^{-i\phi_q} & 0 \\ A_q e^{i\phi_q} & -\Delta_q + \delta_q & B_q \\ 0 & B_q & 1 - \delta_q \end{pmatrix}_H, \quad (3.6)$$

where

$$\delta_q = \Delta_q - \frac{2}{9}(\alpha + 2\beta) \quad \text{and} \quad B_q = \frac{2}{9}\left(\sqrt{8}\alpha - \frac{1}{\sqrt{8}}\beta\right). \quad (3.7)$$

From the strong hierarchy in the masses of the quark families, $m_{3q} \gg m_{2q} > m_{1q}$, we expect $1 - \delta_q$ to be very close to unity.

The entries in the mass matrix may be readily expressed in terms of the mass eigenvalues $(m_{1q}, -m_{2q}, m_{3q})$ and the small parameter δ_q. Computing the invariants of M_q, tr M_q, tr M_q^2, and det M_q, we get

$$A_q{}^2 = \tilde{m}_{1q}\tilde{m}_{2q}(1 - \delta_q)^{-1}, \quad \Delta_q = \tilde{m}_{2q} - \tilde{m}_{1q}, \quad (3.8)$$

$$B_q^2 = \delta_q((1 - \tilde{m}_{1q} + \tilde{m}_{2q} - \delta_q) - \tilde{m}_{1q}\tilde{m}_{2q}(1 - \delta_q)^{-1}), \tag{3.9}$$

where $\tilde{m}_{1q} = m_{1q}/m_{3q}$ and $\tilde{m}_{2q} = m_{2q}/m_{3q}$.

If each possible symmetry breaking pattern is now characterized by the ratio

$$Z_q^{1/2} = B_q/(-\triangle_q + \delta_q), \tag{3.10}$$

the small parameter δ_q is obtained as the solution of the cubic equation

$$\delta_q[(1 + \tilde{m}_{2q} - \tilde{m}_{1q} - \delta_q)(1 - \delta_q) - \tilde{m}_{1q}\tilde{m}_{2q}]$$
$$- Z_q(1 - \delta_q)(-\tilde{m}_{2q} + \tilde{m}_{1q} + \delta_q)^2 = 0, \tag{3.11}$$

which vanishes when Z_q vanishes.

Equation (3.11) may be written as

$$\delta_q^3 - \frac{1}{(Z_q + 1)}[Z_q(2(\tilde{m}_{2q} - \tilde{m}_{1q}) + 1) + (\tilde{m}_{2q} - \tilde{m}_{1q}) + 2]\delta_q^2$$

$$+ \frac{1}{Z_q + 1}[Z_q(\tilde{m}_{2q} - \tilde{m}_{1q})(\tilde{m}_{2q} - \tilde{m}_{1q} + 2)$$

$$+ (1 - \tilde{m}_{1q})(1 + \tilde{m}_{2q})]\delta_q - \frac{Z_q(\tilde{m}_{2q} - \tilde{m}_{1q})^2}{Z_q + 1} = 0. \tag{3.12}$$

The last term in the left hand side of Eq. (3.12) is equal to the product of the three roots of Eq. (3.11). Therefore, the root of Eq. (3.11) which vanishes when Z_q vanishes may be written as

$$\delta_q(Z_q) = \frac{Z_q}{Z_q + 1}\frac{(\tilde{m}_{2q} - \tilde{m}_{1q})^2}{W(Z_q)}, \tag{3.13}$$

where $W(Z_q)$ is the product of the two roots of Eq. (3.12) or Eq. (3.11) which do not vanish when Z_q vanishes.

The product $W(Z_q)$ is given by

$$W(Z_q) = \{[2q^2 + p^3 + 2q\sqrt{q^2 + p^3}]^{1/2} + [2q^2 + p^3 - 2q\sqrt{q^2 + p^3}]^{1/3}\} + \frac{1}{3}\frac{1}{Z_q + 1}[Z_q(2(\tilde{m}_{2q} - \tilde{m}_{1q}) + 1) + (\tilde{m}_{2q} - \tilde{m}_{1q}) + 2]$$

$$\times \{[q + \sqrt{q^2 + p^3}]^{1/3} + [q - \sqrt{q^2 + p^3}]^{1/3}\} - |p| + \frac{1}{9}\frac{1}{(Z_q + 1)^2}[Z_q(2(\tilde{m}_{2q} - \tilde{m}_{1q}) + 1) + (\tilde{m}_{2q} - \tilde{m}_{1q}) + 2]^2, \tag{3.14}$$

where

$$2q = -\frac{2}{27}\frac{1}{(Z_q + 1)^3}[Z_q(2(\tilde{m}_{2q} - \tilde{m}_{1q}) + 1) + (\tilde{m}_{2q} - \tilde{m}_{1q}) + 2]^3 + \frac{1}{3}\frac{1}{(Z_q + 1)^2}[Z_q(2(\tilde{m}_{2q} - \tilde{m}_{1q}) + 1) + (\tilde{m}_{2q} - \tilde{m}_{1q}) + 2]$$

$$\times \{Z_q(\tilde{m}_{2q} - \tilde{m}_{1q})((\tilde{m}_{2q} - \tilde{m}_{1q}) + 2) + (1 - \tilde{m}_{1q})(1 + \tilde{m}_{2q})\} - \frac{Z_q}{Z_q + 1}(\tilde{m}_{2q} - \tilde{m}_{1q})^2, \tag{3.15}$$

and

$$3p = -\frac{1}{3}\frac{1}{(Z_q + 1)^2}[Z_q(2(\tilde{m}_{2q} - \tilde{m}_{1q}) + 1) + (\tilde{m}_{2q} - \tilde{m}_{1q}) + 2]^2 + \frac{1}{(Z_q + 1)}[Z_q(\tilde{m}_{2q} - \tilde{m}_{1q})(\tilde{m}_{2q} - \tilde{m}_{1q} + 2)$$

$$+ (1 - \tilde{m}_{1q})(1 + \tilde{m}_{2q})]. \tag{3.16}$$

Then, the vanishing of Z_q implies that $\delta_q(Z_q)$ vanishes and so does B_q, or equivalently, there is no mixing of singlet and doublet irreducible representations of $S_L(3) \otimes S_R(3)$ and the heaviest quark in each sector, t or b, is in a pure singlet representation.

In Fig. 1, $\delta_q^{1/2}$ is shown as function of Z_q. It may be seen that, as Z_q increases, $\sqrt{\delta_q(Z_q)}$ increases with decreasing curvature. For very large values of Z_q, $\sqrt{\delta_q(Z_q)}$ goes to the asymptotic limit $\tilde{m}_{2q} - \tilde{m}_{1q}$,

$$\lim_{z_q \to \infty} \delta_q^{1/2}(Z_q) = \tilde{m}_{2q} - \tilde{m}_{1q}. \tag{3.17}$$

Hence, $\delta_q(Z_q)$ is a small parameter

$$\delta_q(Z_q) \ll 1, \tag{3.18}$$

for all values of Z_q. For large values of Z_q, say $Z_q \gtrsim 20$, $\delta_q(Z_q)$ is not sensitive to small changes in Z_q.

From Eqs. (3.11) or (3.12) we derive an approximate solution for $\delta_q(Z_q)$ valid for small values of Z_q ($Z_q \lesssim 10$). Computing in the leading order of magnitude we obtain

$$\delta_q(Z_q) \approx \frac{Z_q(\tilde{m}_{2q} - \tilde{m}_{1q})^2}{(1 - \tilde{m}_{1q})(1 + \tilde{m}_{2q}) + 2Z_q(\tilde{m}_{2q} - \tilde{m}_{1q})\left(1 + \frac{1}{2}(\tilde{m}_{2q} - \tilde{m}_{1q})\right)}. \tag{3.19}$$

A. Symmetry breaking pattern

In the symmetry adapted basis, $\mathbf{M}_{3q,H}$ is a singlet tensorial irreducible representation of $S_L(3) \otimes S_R(3)$,

$$\mathbf{M}_{3q,H} = m_{3q}(1 - \Delta_q) \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}_H. \tag{3.20}$$

In this same basis, the term $\mathbf{M}_{2q,H}$ which breaks $S_L(3) \otimes S_R(3)$ down to $S_L(2) \otimes S_R(2)$ is given by

$$\mathbf{M}_{2q,H} = m_{3q}(-\Delta_q + \delta_q(Z_q^{1/2})) \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & Z_q^{1/2} \\ 0 & Z_q^{1/2} & -1 \end{pmatrix}_H. \tag{3.21}$$

The symmetry breaking pattern is characterized by the parameter $Z_q^{1/2}$ which is a measure of the mixing of singlet and doublet irreducible representations of $S_L(3) \otimes S_R(3)$. The decomposition of $\mathbf{M}_{2q,W}$ in a linear combination of two numerical matrices, given in Eqs. (2.9) and (2.10), now takes the form

$$\mathbf{M}_{2q,H} = m_{3q}(-\Delta_q + \delta_q)\left[3\sqrt{2}N_{Aq}M_H^A + \frac{3}{2}N_{Sq}M_H^S\right], \tag{3.22}$$

where the matrices

$$\mathbf{M}_{2,H}^A = \frac{1}{3\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & -\sqrt{8} \\ 0 & -\sqrt{8} & -1 \end{pmatrix}_H, \tag{3.23}$$

$$\mathbf{M}_{2,H}^S = \frac{2}{3} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & \frac{1}{\sqrt{8}} \\ 0 & \frac{1}{\sqrt{8}} & -1 \end{pmatrix}_H, \tag{3.24}$$

are of the same form as $\mathbf{M}_{2q,H}$ with mixing parameters $-\sqrt{8}$ and $1/\sqrt{8}$, respectively. The coefficients N_A and N_S are given by

$$N_A = \frac{2\sqrt{2}}{9}\left(\frac{1}{\sqrt{8}} - Z_q^{1/2}\right) \tag{3.25}$$

and

$$N_S = \frac{2\sqrt{2}}{9}(\sqrt{8} + Z_q^{1/2}). \tag{3.26}$$

From Eqs. (3.21)–(3.24), it is evident that there is a corresponding decomposition of the mixing parameter $Z_q^{1/2}$

$$Z_q^{1/2} = N_{Aq}Z_A^{1/2} + N_{Sq}Z_S^{1/2}, \tag{3.27}$$

with

$$1 = N_{Aq} + N_{Sq}, \tag{3.28}$$

where $Z_A^{1/2} = -\sqrt{8}$ is the mixing parameter in the matrix $\mathbf{M}_{2,H}^A$, and $Z_S^{1/2} = 1/\sqrt{8}$ is the mixing parameter in $\mathbf{M}_{2,H}^S$. In this way, a unique linear combination of $Z_A^{1/2}$ and $Z_S^{1/2}$ is associated to the symmetry breaking pattern characterized by $Z_q^{1/2}$.

We notice that the symmetry breaking term in the Yukawa Lagrangian $\bar{\mathbf{q}}_L \mathbf{M}_{2q} \mathbf{q}_R$ depends only on two fields. According to Eqs. (3.22), (3.23), and (3.24), the term $\bar{\mathbf{q}}_L \mathbf{M}_{2q,H} \mathbf{q}_R$ splits into the sum of a term proportional to $\bar{\mathbf{q}}_L \mathbf{M}_2^A \mathbf{q}_R$, which changes sign under the exchange of those two fields, and a term proportional to $\bar{\mathbf{q}}_L \mathbf{M}_2^S \mathbf{q}_R$, which remains invariant under the same exchange. Therefore, the decomposition of $\mathbf{M}_{2q,H}$ given in Eq. (3.22) is equivalent to a decomposition of the term $\bar{\mathbf{q}}_L \mathbf{M}_{2q} \mathbf{q}_R$ into its symmetric and antisymmetric parts under the exchange of those two fields. Thus, the characterization of $\mathbf{M}_{2q}$ and $Z_q^{1/2}$ as a linear combination, of $\mathbf{M}_2^A$ and $\mathbf{M}_2^S$, and $Z_A^{1/2}$ and $Z_S^{1/2}$, respectively [given in Eqs. (3.22)–(3.28)], is equivalent to a classification of the symmetry breaking pattern defined by $\mathbf{M}_{2q,H}$ in terms of the irreducible representations of the group $\tilde{S}(2)$ of permutations of the two fields in $\bar{\mathbf{q}}_L \mathbf{M}_{2q} \mathbf{q}_R$.

Lehmann, Newton, and Wu [3] observed that in the case of only two families (the first two generations), the term that breaks the $S_L(2) \otimes S_R(2)$ symmetry changes sign when permuting the two quark fields. By analogy, they extended this observation to the case of three families and postulated that the symmetry breaking term $\mathbf{M}_{2q}$ should change sign under the exchange of the two fields in $\bar{\mathbf{q}}_L \mathbf{M}_{2q} \mathbf{q}_R$. This assumption amounts to choosing a fixed value for the mixing parameter $Z_q^{1/2}$ equal to $-\sqrt{8}$. In this paper, the pair of numbers (N_A, N_S) enters as a convenient mathematical label of the

symmetry breaking pattern without introducing any assumption about the actual pattern of $S_L(3) \otimes S_R(3)$ symmetry breaking realized in nature.

IV. THE CKM MIXING MATRIX

The Hermitian mass matrix $\mathbf{M}_q$ may be written in terms of a real symmetric matrix $\bar{\mathbf{M}}_q$ and a diagonal matrix of phases $\mathbf{P}_q$ as follows:

$$\mathbf{M}_q = \mathbf{P}_q \bar{\mathbf{M}}_q \mathbf{P}_q^\dagger. \tag{4.1}$$

The real symmetric matrix $\bar{\mathbf{M}}_q$ may be brought to diagonal form by means of an orthogonal transformation

$$\bar{\mathbf{M}}_q = \mathbf{O}_q \mathbf{M}_{q\ diag} \mathbf{O}_q^T, \tag{4.2}$$

where

$$\mathbf{M}_{q\ diag} = m_{3q}\, \mathrm{diag}[\tilde{m}_{1q}, -\tilde{m}_{2q}, 1], \tag{4.3}$$

with subscripts 1,2,3 referring to u,c,t in the u-type sector and d,s,b in the d-type sector. After diagonalization of the mass matrices $\mathbf{M}_q$, one obtains the CKM mixing matrix as

$$\mathbf{V}_{CKM} = \mathbf{O}_u^T \mathbf{P}^{(u-d)} \mathbf{O}_d, \tag{4.4}$$

where $\mathbf{P}^{(u-d)}$ is the diagonal matrix of the relative phases.

In the hierarchical basis, where $\mathbf{M}_q$ is given by Eqs. (3.6)–(3.8), $\mathbf{P}^{(u-d)}$ is

$$\mathbf{P}^{(u-d)} = \mathrm{diag}[1, e^{i\Phi}, e^{i\Phi}], \tag{4.5}$$

where

$$\Phi = \phi_u - \phi_d, \tag{4.6}$$

and the orthogonal matrix $\mathbf{O}_q$ is given by [21]

$$\mathbf{O}_q = \begin{pmatrix} (\tilde{m}_{2q}\mathrm{f}_1/D_1)^{1/2} & -(\tilde{m}_{1q}\mathrm{f}_2/D_2)^{1/2} & (\tilde{m}_{1q}\tilde{m}_{2q}\mathrm{f}_3/D_3)^{1/2} \\ ((1-\delta_q)\tilde{m}_{1q}\mathrm{f}_1/D_1)^{1/2} & ((1-\delta_q)\tilde{m}_{2q}\mathrm{f}_2/D_2)^{1/2} & ((1-\delta_q)\mathrm{f}_3/D_3)^{1/2} \\ -(\tilde{m}_{1q}\mathrm{f}_2\mathrm{f}_3/D_1)^{1/2} & -(\tilde{m}_{2q}\mathrm{f}_1\mathrm{f}_3/D_2)^{1/2} & (\mathrm{f}_1\mathrm{f}_2/D_3)^{1/2} \end{pmatrix}, \tag{4.7}$$

where

$$\mathrm{f}_1 = 1 - \tilde{m}_{1q} - \delta_q, \quad \mathrm{f}_2 = 1 + \tilde{m}_{2q} - \delta_q, \quad \mathrm{f}_3 = \delta_q, \tag{4.8}$$

$$D_1 = (1-\delta_q)(1-\tilde{m}_{1q})(\tilde{m}_{2q} + \tilde{m}_{1q}), \tag{4.9}$$

$$D_2 = (1-\delta_q)(1+\tilde{m}_{2q})(\tilde{m}_{2q} + \tilde{m}_{1q}), \tag{4.10}$$

$$D_3 = (1-\delta_q)(1+\tilde{m}_{2q})(1-\tilde{m}_{1q}). \tag{4.11}$$

From Eqs. (4.1)–(4.11), all entries in the $\mathbf{V}_{CKM}$ matrix may be written in terms of four mass ratios: $(\tilde{m}_u, \tilde{m}_c, \tilde{m}_d, \tilde{m}_s)$ and three free real parameters: δ_u, δ_d, and $\Phi = \phi_u - \phi_d$. The phase Φ measures the mismatch in the $S_L(2) \otimes S_R(2)$ symmetry breaking in the u- and d-sectors. In this picture of the $\mathbf{V}_{CKM}$ matrix, it is this phase, and consequently, that mismatch, which is responsible for the violation of CP.

The Jarlskog invariant, J, may be computed directly from the commutator of the mass matrices [22]:

$$J = -\frac{\det\{-i[\mathbf{M}_{u,H}, \mathbf{M}_{d,H}]\}}{F}, \tag{4.12}$$

where

$$F = (1+\tilde{m}_c)(1-\tilde{m}_u)(\tilde{m}_c + \tilde{m}_u)(1+\tilde{m}_s)(1-\tilde{m}_d)(\tilde{m}_s + \tilde{m}_d). \tag{4.13}$$

Substitution of the expression (3.6) for $\mathbf{M}_u$ and $\mathbf{M}_d$, in Eq. (4.12), with $Z_u^{1/2} = Z_d^{1/2} = Z^{1/2}$, gives

$$J = \frac{Z\sqrt{\dfrac{\tilde{m}_u/\tilde{m}_c}{1-\delta_u}}\sqrt{\dfrac{\tilde{m}_d/\tilde{m}_s}{1-\delta_d}}\sin\Phi}{(1+\tilde{m}_c)(1-\tilde{m}_u)(1+\tilde{m}_u/\tilde{m}_c)(1+\tilde{m}_s)(1-\tilde{m}_d)(1+\tilde{m}_d/\tilde{m}_s)} \left\{ [(-\Delta_u + \delta_u)(1-\delta_d) - (-\Delta_d + \delta_d)(1-\delta_u)]^2 \right.$$
$$\left. -\left(\frac{\tilde{m}_u\tilde{m}_c}{1-\delta_u}\right)(-\Delta_d + \delta_d)^2 - \left(\frac{\tilde{m}_d\tilde{m}_s}{1-\delta_d}\right)(-\Delta_u + \delta_u)^2 + 2\sqrt{\frac{\tilde{m}_u\tilde{m}_c}{1-\delta_u}}\sqrt{\frac{\tilde{m}_d\tilde{m}_s}{1-\delta_d}}(-\Delta_u + \delta_u)(-\Delta_d + \delta_d)\cos\Phi \right\}. \tag{4.14}$$

Explicit expressions for Δ_q and δ_q in terms of the quark masses are given in Eqs. (3.8) and (3.13)–(3.16).

In this way, an exact closed expression for J in terms of the quark masses, the symmetry breaking parameter Z and the CP violating phase Φ is obtained. Let us recall that the non-vanishing of J is a necessary and sufficient condition for the violation of CP [22]. From Eq. (4.14), it is apparent that J vanishes when Z, $\sin\Phi$ and $\tilde{m}_u$ or $\tilde{m}_d$ vanish. Therefore, the violation of CP and the consequent non-vanishing of Z necessarily implies a mixing of singlet and doublet representations of $S_L(3)\otimes S_R(3)$.

V. THE BEST VALUE OF $Z_q^{1/2}$

At this stage in our argument, a question comes naturally to mind. Does a comparison of the theoretical mixing matrix $\mathbf{V}_{CKM}^{th}$ with the experimentally determined $\mathbf{V}_{CKM}^{exp}$ give any clue about the actual pattern of $S_L(3)\otimes S_R(3)$ symmetry breaking realized in nature? or phrased differently: What are the best values for Z_u and Z_d? What is the best value for Φ? Do these values correspond to some well defined symmetry breaking pattern?

As a first step in the direction of finding an answer to these questions, we made a χ^2 fit of the exact expressions for the absolute value of the entries in the mixing matrix, that is $|V_{CKM}^{th}|$ and the Jarlskog invariant J^{th} to the experimentally determined values of $|V_{CKM}^{exp}|$ and J^{exp}. Since the value of the observed CKM matrix parameters $|\mathbf{V}_{CKM}^{exp}|$ are given at $\mu=m_t$, in the calculation we used the values of the running quark masses evaluated at m_t. These values were taken from the work of Fritzsch [20], see also Fusaoka and Koide [23]

and Ynduráin [24]. We kept the mass ratios $\tilde{m}_c=m_c/m_t$ and $\tilde{m}_s=m_s/m_b$ fixed at their central values

$$\tilde{m}_c=0.0048 \quad \text{and} \quad \tilde{m}_s=0.03437, \tag{5.1}$$

but, for reasons which will be apparent later, we took the values

$$\tilde{m}_u=0.000042 \quad \text{and} \quad \tilde{m}_d=0.00148, \tag{5.2}$$

which are close to the upper and lower bounds of $\tilde{m}_u=m_u/m_t$ and $\tilde{m}_d=m_d/m_b$ respectively, and we looked for the best values of the three parameters δ_u,δ_d, and Φ. We found the following results [25]: (I) excellent fits of similar quality, $\chi^2\leqslant 0.33$, were obtained for a continuous family of values of the parameters (δ_u,δ_d); (II) in each good quality fit, the best value of Φ was fixed without ambiguity; (III) the best value of Φ was nearly stable against large changes in the values of (δ_u,δ_d) which produced fits of the same good quality; (IV) in all good quality fits, the difference $\sqrt{\delta_d}-\sqrt{\delta_u}$ takes the same value

$$\sqrt{\delta_d}-\sqrt{\delta_u}\simeq 0.040. \tag{5.3}$$

These results may be understood if we notice that not all entries in $\mathbf{V}_{CKM}^{th}$ are equally sensitive to variations of the different parameters. Some entries, like V_{us}, are very sensitive to changes in Φ but are almost insensitive to changes in (δ_u,δ_d) while, some others, like V_{cb} are almost insensitive to changes in Φ but depend critically on the parameters δ_u and δ_d.

From Eqs. (4.1)–(4.11), we obtain

$$V_{us}=-\left(\frac{\tilde{m}_c\tilde{m}_d}{(1-\tilde{m}_u)(\tilde{m}_c+\tilde{m}_u)(1+\tilde{m}_s)(\tilde{m}_s+\tilde{m}_d)}\right)^{1/2}\left(\frac{(1-\tilde{m}_u-\delta_u)(1+\tilde{m}_s-\delta_d)}{(1-\delta_u)(1-\delta_d)}\right)^{1/2}+\left\{\left(\frac{(1-\tilde{m}_u-\delta_u)(1+\tilde{m}_s-\delta_d)}{(1+\tilde{m}_s)}\right)^{1/2}\right.$$
$$\left.+\left(\frac{(1+\tilde{m}_c-\delta_u)\delta_u}{1-\delta_u}\right)^{1/2}\left(\frac{(1-\tilde{m}_d-\delta_d)\delta_d}{(1-\delta_d)(1+\tilde{m}_s)}\right)^{1/2}\right\}\left(\frac{\tilde{m}_u\tilde{m}_s}{(1-\tilde{m}_u)(\tilde{m}_c+\tilde{m}_u)(\tilde{m}_d+\tilde{m}_s)}\right)^{1/2}e^{i\Phi}. \tag{5.4}$$

In the leading order of magnitude,

$$|V_{us}|\approx|\sqrt{\tilde{m}_d/\tilde{m}_s}-\sqrt{\tilde{m}_u/\tilde{m}_c}e^{i\Phi}|(1+\tilde{m}_u/\tilde{m}_c+\tilde{m}_d/\tilde{m}_s)^{-1/2}. \tag{5.5}$$

Hence,

$$\cos\Phi\approx\frac{\tilde{m}_d/\tilde{m}_s+\tilde{m}_u/\tilde{m}_c-|V_{us}|^2(1+\tilde{m}_u/\tilde{m}_c+\tilde{m}_d/\tilde{m}_s)}{2\sqrt{(\tilde{m}_d/\tilde{m}_s)(\tilde{m}_u/\tilde{m}_c)}}. \tag{5.6}$$

Substitution of $|V_{us}^{exp}|^2$ for $|V_{us}|^2$ and the numerical value of the mass ratios, (5.1) and (5.2), in Eq. (5.6) gives

$$87°\leqslant\Phi\leqslant 92° \tag{5.7}$$

with a mean value

$$\bar{\Phi}=89.5°, \tag{5.8}$$

in good agreement with the best value extracted from the preliminary χ^2 fits [25].

Similarly, V_{cb}^{th} is given by

$$V_{cb}^{th} = -\left(\frac{\tilde{m}_u(1+\tilde{m}_c-\delta_u)}{(1-\delta_u)(1+\tilde{m}_c)(\tilde{m}_c+\tilde{m}_u)}\frac{\tilde{m}_d\tilde{m}_s\delta_d}{(1-\delta_d)(1+\tilde{m}_s)(1-\tilde{m}_d)}\right)^{1/2} + \left\{\left(\frac{\tilde{m}_c(1+\tilde{m}_c-\delta_u)}{(\tilde{m}_c+\tilde{m}_u)(1+\tilde{m}_c)}\frac{\delta_d}{(1+\tilde{m}_s)(1-\tilde{m}_d)}\right)^{1/2}\right.$$

$$\left. -\left(\frac{\tilde{m}_c(1-\tilde{m}_u-\delta_u)\delta_u(1-\tilde{m}_d-\delta_d)(1+\tilde{m}_s-\delta_d)}{(1-\delta_u)(1+\tilde{m}_c)(\tilde{m}_c+\tilde{m}_u)(1-\delta_d)(1+\tilde{m}_s)(1-\tilde{m}_d)}\right)^{1/2}\right\}e^{i\Phi}. \tag{5.9}$$

Therefore, in the leading order of magnitude, $|V_{cb}|$ is independent of Φ and given by

$$|V_{cb}| \approx \sqrt{\delta_d} - \sqrt{\delta_u}. \tag{5.10}$$

Hence, good agreement with $|V_{cb}^{exp}| \approx 0.039$ [26] requires that

$$\sqrt{\delta_d} - \sqrt{\delta_u} \approx 0.040, \tag{5.11}$$

at least for one pair of values (δ_u, δ_d).

Finally, let us notice that the matrix elements V_{ub} and V_{dt}, as well as the Jarlskog invariant [see Eq. (4.14)], are sensitive to small changes in the masses of the light quarks $\tilde{m}_u$ and $\tilde{m}_d$. For instance,

$$V_{ub} = \left(\frac{\tilde{m}_c(1-\tilde{m}_u-\delta_u)}{(1-\delta_u)(1-\tilde{m}_u)(\tilde{m}_c+\tilde{m}_u)}\frac{\tilde{m}_d\tilde{m}_s\delta_d}{(1-\delta_d)(1+\tilde{m}_s)(1-\tilde{m}_d)}\right)^{1/2} + \left\{\left(\frac{\tilde{m}_u(1-\tilde{m}_u-\delta_u)\delta_d}{(1-\tilde{m}_u)(\tilde{m}_c+\tilde{m}_u)(1+\tilde{m}_s)(1-\tilde{m}_d)}\right)^{1/2}\right.$$

$$\left. -\left(\frac{\tilde{m}_u(1+\tilde{m}_c-\delta_u)\delta_u(1-\tilde{m}_d-\delta_d)(1+\tilde{m}_s-\delta_d)}{(1-\delta_u)(1-\tilde{m}_u)(\tilde{m}_c+\tilde{m}_u)(1-\delta_d)(1+\tilde{m}_s)(1-\tilde{m}_d)}\right)^{1/2}\right\}e^{i\Phi} \tag{5.12}$$

computing in the leading order of magnitude, we get

$$V_{ub} \approx \sqrt{\frac{\tilde{m}_u}{\tilde{m}_c}}(\sqrt{\delta_d} - \sqrt{\delta_u})e^{i\Phi}. \tag{5.13}$$

A similar computation gives for V_{td}

$$V_{td} \approx -\sqrt{\frac{\tilde{m}_d}{\tilde{m}_s}}(\sqrt{\delta_d} - \sqrt{\delta_u})e^{i\Phi}. \tag{5.14}$$

However, since the masses of the light quarks are the less well determined, and the moduli $|V_{ub}^{exp}|$ and $|V_{td}^{exp}|$ have the largest error bars, relatively large changes in the values of $\tilde{m}_u$ and $\tilde{m}_d$ produce only very small variations in the goodness of fit of the thoretical matrix of moduli $|V^{th}|$ to the experimentally determined $|V^{exp}|$. The sensitivity of the matrix elements $|V_{ub}|$ and $|V_{td}|$ to changes in $\tilde{m}_u$ and $\tilde{m}_d$ is reflected in the shape of the unitarity triangle which changes appreciably when the masses of the light quarks change within their uncertainty bounds, as may be seen from the following expressions:

$$\alpha = \arg\left(-\frac{V_{ub}^*V_{ud}}{V_{tb}^*V_{td}}\right) \approx \arg\left\{\left(\frac{\tilde{m}_u}{\tilde{m}_c}\frac{\tilde{m}_s}{\tilde{m}_d}\right)^{1/2}e^{-i\Phi}\right\} = \Phi, \tag{5.15}$$

$$\beta = \arg\left(-\frac{V_{tb}^*V_{td}}{V_{cb}^*V_{cd}}\right) \approx \arctan\left\{\frac{\sqrt{\frac{\tilde{m}_u}{\tilde{m}_c}}\sin\Phi}{\sqrt{\frac{\tilde{m}_d}{\tilde{m}_s}} - \sqrt{\frac{\tilde{m}_u}{\tilde{m}_c}}\cos\Phi}\right\}, \tag{5.16}$$

and

$$\gamma = \arg\left(-\frac{V_{cb}^*V_{cd}}{V_{ub}^*V_{ud}}\right) \approx \arctan\left\{\frac{\sqrt{\frac{\tilde{m}_d}{\tilde{m}_s}}\sin\Phi}{\sqrt{\frac{\tilde{m}_u}{\tilde{m}_c}} - \sqrt{\frac{\tilde{m}_d}{\tilde{m}_s}}\cos\Phi}\right\}, \tag{5.17}$$

α, β, and γ are the inner angles of the unitarity triangle. When the central values of $\tilde{m}_u = 0.000018$ and $\tilde{m}_d = 0.0019$ [20] are used in the fitting procedure, the agreement of $|\mathbf{V}^{th}|$ with $|\mathbf{V}^{exp}|$ is very good, $\chi^2 = 0.33$, but we systematically obtain $\gamma^{th} > \alpha^{th}$ in stark disagreement with the most recent data on the K^0-$\bar{K}^0$ system and the most recent data on the $B_{s,d}^0$ oscillations [26] and [27]. We could not change the values of $\sqrt{\delta_d} - \sqrt{\delta_u}$ without spoiling the good overall agreement of $|\mathbf{V}^{th}|$ to $|\mathbf{V}^{exp}|$. Therefore, we let the masses of the light quarks vary within their uncertainty ranges. The best

simultaneous χ^2 fit of $|V^{th}|$, J^{th} and α^{th}, β^{th} and γ^{th} to the experimentally determined quantities $|V^{exp}|$, J^{exp} and α^{exp}, β^{exp} and γ^{exp} [26,27] was obtained when the value of $\tilde{m}_u$ is taken close to its upper bound, $\tilde{m}_u \approx 0.000042$, and the value of $\tilde{m}_d \approx 0.00148$, which is close to its lower bound. Notice that, the chosen high value of $\tilde{m}_u$ gives for the ratio $|V_{ub}|/|V_{cb}|$ the value

$$\frac{|V_{ub}|}{|V_{cb}|} \approx \sqrt{\frac{\tilde{m}_u}{\tilde{m}_c}} = 0.093 \qquad (5.18)$$

in very good agreement with its latest world average [27].

We may now return to our discussion of the determination of the best pattern of symmetry breaking. As explained above, in the preliminary χ^2 fit to the data it was found that $\sqrt{\delta_d} - \sqrt{\delta_u} \approx 0.04$, Eq. (5.11), is satisfied almost exactly even when we let the masses of the light quarks vary, not just for one pair of values (δ_u, δ_d) but for a continuous range of values of δ_u and δ_d in which these parameter change by more than one order of magnitude.

Therefore, Eq. (5.11) may be used as a constraining condition on the possible values of (δ_u, δ_d). In this way, we eliminate one free parameter in $\mathbf{V}_{CKM}^{th}$ without spoiling the good quality of the fit. However, fixing the numerical value of this free parameter is not enough to get a clear indication on what is the actual pattern of flavor symmetry breaking realized in nature. This is so because according to Eqs. (3.11)–(3.16), δ_q is a function of the mass ratios $(\tilde{m}_{1q}, \tilde{m}_{2q})$ and the parameter $Z_q^{1/2}$ which characterizes the pattern of $S_L(3) \otimes S_R(3)$ symmetry breaking in the q sector. Hence, a convenient way to isolate the information about the pattern of symmetry breaking carried by the constraining conditions (5.11) from the information on the numerical values of the quark mass ratios, is to change the parametrization of $\mathbf{V}_{CKM}^{th}$ by writing δ_q as function of $Z_q^{1/2}$ with fixed values of $(\tilde{m}_{1q}, \tilde{m}_{2q})$. In this way $\mathbf{V}_{CKM}^{th}$ becomes a function of the two free parameters $(Z_u^{1/2}, Z_d^{1/2})$ instead of (δ_u, δ_d).

A simple approximate expression for the constraining condition (5.3), (5.11) in terms of $(Z_u^{1/2}, Z_d^{1/2})$, valid for $0 \leq Z_q \leq 10$, is obtained from Eq. (5.11), writing $\delta_q(Z_q)$ in the leading order of magnitude

$$\sqrt{\delta_d} - \sqrt{\delta_u} \simeq \frac{Z_d^{1/2}(\tilde{m}_s - \tilde{m}_d)}{\sqrt{(1 + \tilde{m}_s)(1 - \tilde{m}_d) + 2Z_d(\tilde{m}_s - \tilde{m}_d)}}$$
$$- \frac{Z_u^{1/2}(\tilde{m}_c - \tilde{m}_u)}{\sqrt{(1 + \tilde{m}_c)(1 - \tilde{m}_u) + 2Z_u(\tilde{m}_c - \tilde{m}_u)}} \simeq 0.040. \qquad (5.19)$$

When the condition (5.19) is satisfied, to each value of $Z_u^{1/2}$ corresponds one value of $Z_d^{1/2}$. But, since we have only one condition to fix the value of two parameters, $Z_u^{1/2}$ would still be a free parameter. Therefore, to avoid this continuous am-

biguity, we will further assume that the up and down mass matrices are generated following the same symmetry breaking pattern, that is,

$$Z_u^{1/2} = Z_d^{1/2} \equiv Z^{1/2}. \qquad (5.20)$$

Then, the value of Z which satisfies the constraining conditions (5.11) and (5.19) may be read directly from Fig. 1. We find $Z^* \simeq 2.5$.

A more precise numerical computation of the best value of Z was made using the exact numerical solution of Eq. (3.11), given in Eqs. (3.13)–(3.16), to compute the entries in $\mathbf{V}_{CKM}^{th}$ as functions of only two free parameters, Φ and $Z^{1/2}$. As previously, we kept the mass ratios fixed at the values given in Eqs. (5.1) and (5.2). Then, we made a new χ^2 fit of the exact expressions for the absolute values of the entries in the theoretical expressions for $|V_{CKM}^{th}|$ and the Jarlskog invariant J^{th}, to the experimentally determined values of $|V_{CKM}^{exp}|$ and J^{exp}. We found the following best values for Φ and Z,

$$\Phi = 89.3° \qquad (5.21)$$

and

$$2.40 \leq Z^* \leq 2.55, \qquad (5.22)$$

corresponding to a value of $\chi^2 \leq 0.33$.

When the best value of the *CP* violating phase $\Phi = 89.33°$ is changed by one degree, the computed values of all entries in the matrix of moduli $|V_{CKM}^{th}|$ change in the fourth decimal place, except $|V_{us}^{th}|$ and $|V_{cd}^{th}|$ which change in the third decimal place by an amount which is about one fourth of the uncertainty in the experimentally determined values of $|V_{us}^{exp}|$ and $|V_{cd}^{exp}|$ as reported in Caso *et al.* [26]. The value of χ^2 changes from 0.33 to 0.44 which is not statistically significant. Therefore, we will adopt as the best value of Φ the simple figure

$$\Phi^* = 90°. \qquad (5.23)$$

Once the best value of Z has been found, the question posed at the beginning of this section takes a new form: What is the symmetry breaking pattern corresponding to $Z^* \simeq 2.5$?

An answer would be readily found if $Z^{*1/2}$ could be written as a simple, non-trivial, linear combination of $Z_A^{1/2}$ and $Z_S^{1/2}$, which, are equal to $-\sqrt{8}$ and $1/\sqrt{8}$, respectively. From these numbers, we find that $Z^{*1/2}$ may indeed be written as

$$Z^{*1/2} = \frac{1}{2}[Z_s^{1/2} - Z_A^{1/2}] = \frac{1}{2}[1/\sqrt{8} + \sqrt{8}], \qquad (5.24)$$

then

$$Z^* = \frac{81}{32} = 2.53125. \qquad (5.25)$$

The corresponding values of $\delta_u(Z)$ and $\delta_d(Z)$ are

$$\delta_u(Z^*) = 0.000056, \quad \delta_d(Z^*) = 0.0023. \qquad (5.26)$$

Let us remark again that the numerical value of $Z^{*1/2}$ was extracted from a fit of $|V_{CKM}^{th}|$ to the experimentally determined absolute values of the elements of the CKM mixing matrix. The identification of $Z^{*1/2}$ with the expression (5.24) gives a clear and precise indication about the preferred pattern for the breaking of the $S_L(3) \otimes S_R(3)$ permutational flavor symmetry by the quark mass matrices.

VI. MASS TEXTURES FROM THE "BEST" SYMMETRY BREAKING SCHEME

Once the best value of $Z^{1/2}$ has been determined, we may turn the argument around, and propose it as a symmetry breaking ansatz in the form of the following assumption. The $S_L(3) \otimes S_R(3)$ flavor symmetry is broken down to $S_L(2) \otimes S_R(2)$ according to a mixed symmetry breaking pattern, which, in the hierarchical basis, is characterized by

$$Z^{*1/2} = \frac{1}{2}(Z_S^{1/2} - Z_A^{1/2}). \tag{6.1}$$

Then, the mass matrix with the modified Fritzsch texture takes the form

$$\mathbf{M}_{q,H}^* = m_{3q} \begin{pmatrix} 0 & \sqrt{\dfrac{\tilde{m}_{1q}\tilde{m}_{2q}}{1-\delta_q^*}}e^{-i\phi_q} & 0 \\ \sqrt{\dfrac{\tilde{m}_{1q}\tilde{m}_{2q}}{1-\delta_q^*}}e^{i\phi_q} & -\tilde{m}_{2q}+\tilde{m}_{1q}+\delta_q^* & \dfrac{9\sqrt{2}}{8}(-\tilde{m}_{2q}+\tilde{m}_{1q}+\delta_q^*) \\ 0 & \dfrac{9\sqrt{2}}{8}(-\tilde{m}_{2q}+\tilde{m}_{1q}+\delta_q^*) & 1-\delta_q^* \end{pmatrix}_H, \tag{6.2}$$

where δ_q^* is the solution of the cubic equation

$$113\delta_q^{*3} - [194(\tilde{m}_{2q}-\tilde{m}_{1q})+145]\delta_q^{*2}$$
$$+ [81(\tilde{m}_{2q}-\tilde{m}_{1q})^2 + 194(\tilde{m}_{2q}-\tilde{m}_{1q})$$
$$- 32\tilde{m}_{1q}\tilde{m}_{2q}+32]\delta_q^* - 81(\tilde{m}_{2q}-\tilde{m}_{1q})^2 = 0, \tag{6.3}$$

obtained from Eq. (3.12) when $\sqrt{81/32}$ is substituted for $Z^{*1/2}$.

The mass matrix $\mathbf{M}_{q,H}$ was built up adding three terms, $\mathbf{M}_{1q,H}$, $\mathbf{M}_{2q,H}$, and $\mathbf{M}_{3q,H}$. The term $\mathbf{M}_{3q,H}$ is a singlet irreducible representation of $S_L(3) \otimes S_R(3) \supset S_{diag}(3)$

$$\mathbf{M}_{3q,H} = (m_{3q}-m_{2q}+m_{1q})\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}_H. \tag{6.4}$$

The matrix $\mathbf{M}_{2q,H}$ breaks $S_L(3) \otimes S_R(3)$ down to $S_L(2) \otimes S_R(2)$, mixing the singlet and doublet representation of $S_{diag}(3)$ in a proportion precisely determined by the mixing parameter $Z^{*1/2} = \sqrt{81/32}$,

$$\mathbf{M}_{2q,H}^* = m_{3q}(-\tilde{m}_{2q}+\tilde{m}_{1q}+\delta_q^*)$$
$$\times \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & \sqrt{81/32} \\ 0 & \sqrt{81/32} & -1 \end{pmatrix}_H. \tag{6.5}$$

The mixing parameter $\sqrt{81/32}$ corresponds to what was called in Sec. II, a mixed symmetry breaking pattern, that is, it may be split in the sum of a term $\mathbf{M}_{2q,H}^A$ corresponding to a purely antisymmetric, plus a term $\mathbf{M}_{2q,H}^S$ corresponding to a purely symmetric breaking pattern. The coefficients in each term, $N_S = \frac{25}{18}$ and $N_A = -\frac{7}{18}$, are obtained solving the pair of coupled equations (3.27) and (3.28) when $Z^{*1/2} = \sqrt{81/32}$.

Hence,

$$\mathbf{M}_{2q,H}^* = m_{3q}(-\tilde{m}_{2q}+\tilde{m}_{1q}+\delta_q^*)$$
$$\times \left[\frac{-7}{18}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & -\sqrt{8} \\ 0 & -\sqrt{8} & -1 \end{pmatrix}_H \right.$$
$$\left. + \frac{25}{18}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & \dfrac{1}{\sqrt{8}} \\ 0 & \dfrac{1}{\sqrt{8}} & -1 \end{pmatrix}_H \right]. \tag{6.6}$$

The $S_L(2) \otimes S_R(2)$ symmetry of this term and its splitting in

the sum of a purely symmetric plus a purely antisymmetric breaking pattern term is evident in the weak representation

$$\mathbf{M}^*_{2q,W}=m_{3q}(-\tilde{m}_{2q}+\tilde{m}_{1q}+\delta^*_q)\left(\frac{1}{24}\right)$$

$$\times\begin{pmatrix} 14 & 14 & -25 \\ 14 & 14 & -25 \\ -25 & -25 & -28 \end{pmatrix}_W .\qquad(6.7)$$

Finally, the term $\mathbf{M}_{1q,H}$ breaks the $S_L(2)\otimes S_R(2)$ symmetry

$$M^*_{1q,H}=m_{3q}\sqrt{\frac{\tilde{m}_{1q}\tilde{m}_{2q}}{1-\delta^*_q}}\begin{pmatrix} 0 & e^{-i\phi_q} & 0 \\ e^{i\phi_q} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}_H .\qquad(6.8)$$

Since δ^*_q is a function of the mass ratios $(\tilde{m}_{1q},\tilde{m}_{2q})$ the phase ϕ_q is the only free parameter left in the mass matrix $\mathbf{M}^*_q$.

VII. THE MIXING MATRIX, V_{CKM}, FROM THE BEST SYMMETRY BREAKING SCHEME

We have seen that, once the symmetry breaking *Ansatz* fixes the value of the mixing parameter $Z^{1/2}$ at $\sqrt{81/32}$, the

entries in the mass matrices $\mathbf{M}_q$ are functions of the mass ratios $(\tilde{m}_{1q},\tilde{m}_{2q})$ and the phase ϕ_q which is a free parameter.

After factorizing the phases, as in Eq. (4.1), all entries in the real symmetric matrices $\bar{\mathbf{M}}_q$ are functions of the mass ratios $(\tilde{m}_{1q},\tilde{m}_{2q})$ only. Hence, the orthogonal matrices $\mathbf{O}_q$ which bring $\bar{\mathbf{M}}_q$ to diagonal form are also functions of $(\tilde{m}_{1q},\tilde{m}_{2q})$ only.

According to Eq. (4.4), $\mathbf{V}_{CKM}$ is given by $\mathbf{O}^{\mathbf{T}}_{\mathbf{u}}\mathbf{P}^{(\mathbf{u}-\mathbf{d})}\mathbf{O}_{\mathbf{d}}$, where $\mathbf{P}^{(\mathbf{u}-\mathbf{d})}$ is the diagonal matrix of the relative phases. Therefore, once the symmetry breaking *Ansatz* determines the value of $Z^{*1/2}=\sqrt{81/32}$, the theoretical expressions for the entries in the mixing matrix, $\mathbf{V}^{th}_{CKM}$, are written in terms of the four mass ratios $(\tilde{m}_u,\tilde{m}_c,\tilde{m}_d,\tilde{m}_s)$ and only one free parameter, namely, the CP violating phase Φ.

We made a new χ^2 fit of the absolute value of the entries in the mixing matrix, $|V^{th}_{CKM}|$, to the experimentally determined values $|V^{exp}_{CKM}|$. We kept the mass ratios fixed at the values given in Eqs. (5.1) and (5.2). We varied only the CP violating phase Φ. The best value of Φ was found to be $89.3°$ corresponding to a minimum value of χ^2 equal to 0.33. As explained at the end of Sec. V, we may round off to

$$\Phi^*=90°\qquad(7.1)$$

without spoiling the good quality of the fit.

The mixing matrix $\mathbf{V}^{th}_{CKM}$, computed with this value of Φ is

$$\mathbf{V}^{th}_{CKM}=\begin{pmatrix} 0.9750+i0.0188 & -0.2020+i0.0906 & 0.0003+i0.0037 \\ -0.0907+i0.2019 & 0.0188+i0.9742 & -0.0000+i0.0396 \\ -0.0000-i0.0084 & -0.0000-i0.0388 & 0.0000+i0.9992 \end{pmatrix}.\qquad(7.2)$$

The matrix of the moduli, computed from Eq. (7.2), is

$$|V^{th}_{CKM}|=\begin{pmatrix} 0.9752 & 0.2214 & 0.0037 \\ 0.2213 & 0.9744 & 0.0396 \\ 0.0084 & 0.0388 & 0.9992 \end{pmatrix},\qquad(7.3)$$

which is to be compared with the experimental value [26]

$$|V^{exp}_{CKM}|=\begin{pmatrix} 0.9745-0.9760 & 0.217-0.224 & 0.0018-0.0045 \\ 0.217-0.224 & 0.9737-0.9753 & 0.036-0.042 \\ 0.004-0.013 & 0.035-0.042 & 0.9991-0.9994 \end{pmatrix}.\qquad(7.4)$$

We see that the absolute values of the entries in the mixing matrix computed from the theoretical expressions for $\mathbf{V}_{CKM}^{th}$, with the values of the mass ratios given in Eqs. (5.1) and (5.2) [20] reproduce the central values of the experimentally determined entries in $|\mathbf{V}_{CKM}^{exp}|$, almost exactly, well within the bounds of experimental error.

We also computed the Jarlskog invariant J [22]. The value obtained from Eq. (4.14) is

$$J^{th} = 3.00 \times 10^{-5}, \tag{7.5}$$

in good agreement with current data on CP violation in the K^0-$\bar{K}^0$ mixing system [26].

The three inner angles of the unitarity triangle may now be readily computed from the expressions (5.15)–(5.17). We found the following values:

$$\alpha = 84°, \quad \beta = 24°, \quad \gamma = 72°. \tag{7.6}$$

These three angles will be determined from CP asymmetries in a variety of weak B decays at the forthcoming B factories.

An estimation of the range of values of these angles compatible with the experimental information on the absolute values of the matrix elements of $\mathbf{V}_{CKM}^{exp}$, is given by Mele [27] and Ali [28]. According to these authors, $79° \leq \alpha \leq 102°$, $21° \leq \beta \leq 28°$ and $55° \leq \gamma \leq 78°$. We see that the value of β obtained in this work coincides almost exactly with the central value of β according to Mele [27], while our γ is close to the upper limit given by Mele [27] and α is in the allowed range given by these authors.

VIII. SUMMARY AND CONCLUSIONS

In this work we derived theoretical expressions for the mixing matrix $\mathbf{V}_{CKM}^{th}$ from quark mass matrices $\mathbf{M}_q$ with a modified Fritzsch texture. The mass matrices were built up adding three terms $\mathbf{M}_{3q}$, $\mathbf{M}_{2q}$, and $\mathbf{M}_{1q}$ corresponding to stages of less symmetry in a simple scheme for breaking the flavor permutational symmetry.

The breaking pattern of the $S_L(3) \otimes S_R(3)$ symmetry down to $S_L(2) \otimes S_R(2)$ was characterized in terms of the parameter $Z^{1/2} = (M_{2q,H})_{23}/(M_{2q,H})_{22}$ which is a measure of the amount of mixing of singlet and doublet irreducible representations of $S_L(3) \otimes S_R(3)$. This breaking pattern was classified in terms of the symmetric $(Z_S^{1/2} = 1/\sqrt{8})$, and antisymmetric $(Z_A^{1/2} = -\sqrt{8})$ representations of an auxiliary group $\tilde{S}(2)$ of permutations of the two fields in the Yukawa term $\bar{\mathbf{q}}_{L,W} \mathbf{M}_{2q,W} \mathbf{q}_{R,W}$.

A careful comparison of the theoretical expression for the

absolute values of the elements of the CKM matrix with the experimentally determined values of $|V_{CKM}^{exp}|$, J^{exp} and the inner angles of the unitarity triangle α^{exp}, β^{exp}, and γ^{exp} gives a clear and precise indication on the existence of a preferred pattern for breaking the $S_L(3) \otimes S_R(3)$ flavor symmetry down to $S_L(2) \otimes S_R(2)$. The preferred or best symmetry breaking pattern is characterized by

$$Z^{*1/2} = \frac{1}{2}(Z_S^{1/2} - Z_A^{1/2}) = \sqrt{\frac{81}{32}}. \tag{8.1}$$

Once the numerical value of $Z^{*1/2}$ is fixed at $\sqrt{81/32}$, the mass matrices $\mathbf{M}_q$ are functions of the quark masses and only one phase. In consequence, the resulting best theoretical V_{CKM} matrix is parametrized in terms of the four quark mass ratios $(\tilde{m}_u, \tilde{m}_c, \tilde{m}_d, \tilde{m}_s)$ and only one CP violating phase Φ. The best value of Φ was found to be

$$\Phi = 90°. \tag{8.2}$$

The moduli of the matrix elements of the mixing matrix computed from the theoretical expression V_{CMK}^{th} are in excellent agreement with all the experimentally determined absolute values of the CKM matrix $|V_{CKM}^{exp}|$. For the Jarlskog invariant we found the value $J = 3.00 \times 10^{-5}$ and for the inner angles of the unitarity triangle we found the values $\alpha = 84$, $\beta = 24$, and $\gamma = 72$ also in very good agreement with current data on CP violation in the K^0-$\bar{K}^0$ mixing system [26] and the most recent data on oscillations in the B_s^0-$\bar{B}_s^0$ system [27] and [28].

In the standard electroweak model both the masses of the quarks as well as the weak mixing angles appear as free parameters. In this work, we have shown that, starting from the flavor permutational symmetry, a simple and explicit *Ansatz* about the pattern of symmetry breaking leads to a parametrization of the CKM mixing matrix in terms of four quark mass ratios $(m_u/m_t, m_c/m_t, m_d/m_b, m_s/m_b)$ and one CP violating phase in very good agreement with all the available experimental information on quark mixings and CP violation.

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