Comment on ''Brans-Dicke wormholes in the Jordan and Einstein frames''

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In the Brans-Dicke (BD) theory, wormhole existence is determined in the Jordan representation. The Einstein representation correctly describes the geodesic motion of a test black hole that responds to the tensor mass m_T of a bounded gravitational system. The central body's active mass m as calculated in the Jordan frame determines the change in time of the area of a bundle of light rays and the motion of a test particle in Keplerian orbit. In the Jordan representation, strong-field BD wormhole solutions can exist and the energy density and the radial and lateral tensions are negative at the wormhole throat. However, the Einstein representation minimally coupled scalar field energy condition eliminates wormhole solutions within the BD weak-field approximation, $C=-1/(\omega+2)$. [S0556-2821(98)01924-9]

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A recently published article by Nandi *et al.* [1] claims that in the Jordan frame wormhole existence can occur only for a very limited range of the Brans-Dicke (BD) parameter ω. The test for wormholes includes an examination of the areal radius [2,3] of a test object for a minimum at distances greater than the event horizon of a gravitational system. The space-time location of an event is the same whether calculated in the Einstein or in the Jordan frame. However, corresponding surface areas are different in the two representations because the area of an $r=\text{const}$ surface in the Einstein representation is (φ/φ_0) times the area of this same surface in the Jordan representation. The Einstein representation correctly describes the geodesic motion of a test black hole; the Jordan representation correctly describes the geodesic motion of a test particle. The central body's active mass m as calculated in the Jordan frame, determines how the area of a bundle of light rays changes in time; also a test particle in Keplerian orbit measures m [4,5]. In the Einstein representation, geodesic motion is limited to test-black-holes which determine the Keplerian tensor mass m_T of a bounded gravitational system.

I concur that an examination of the Einstein representation of the BD theory shows that [1]

$$\omega>-3/2 \tag{1}$$

since the stress energy for a massless minimally coupled scalar field satisfies all energy conditions [6]. Furthermore, from the BD theory condition (when $B>0$) [7]

$$\lambda=[1+C+(1+\tfrac{1}{2}\omega)C^2]^{1/2}>0 \tag{2}$$

and the wormhole existence condition, Eq. (3), that the areal radius $R=(g_{22})^{1/2}$ has a minimum as a function of the proper length $l=\int(g_{11})^{1/2}dr$,

$$[(C+1)/\lambda]^2>1,\ \ (C+1)>\lambda>0, \tag{3}$$

one finds [1]

$$C(1-\tfrac{1}{2}\omega C)>0,\ \ C>-1. \tag{4}$$

At Eq. (9) below, I remark that Eq. (4) is also valid for the case $\lambda<0, B<0$. However, I disagree with the conclusions reached in Ref. [1] regarding the existence of wormholes as determined in the Jordan representation. First, the conclusion that within the weak-field approximation wormholes can exist when ω is limited to

$$-3/2<\omega<-4/3 \tag{5}$$

is a computational error. The weak field approximation [7,8]

$$C=-1/(\omega+2) \tag{6}$$

is used. Besides the fact that the use of this approximation is inappropriate for strong field wormhole phenomena, the authors of Ref. [1] found that Eq. (6) together with Eq. (4) implies $\omega<-\frac{4}{3}$. This condition together with Eq. (1) yielded Eq. (5). However, Eqs. (3) and (6) imply that

$$C+1=\frac{\omega+1}{\omega+2}>0. \tag{7}$$

That is, ω cannot be chosen between -2 and -1 for wormhole existence in the weak field approximation. Also Eqs. (5) and (6) imply $-2<C<-\frac{3}{2}$ which is inconsistent with Eq. (3). Thus, Eq. (5) is incorrect. Instead, Eqs. (3), (6), and (7) imply $\omega<-2$ and $C>0$. This conclusion, together with the Einstein representation minimally coupled scalar field energy condition [1] that $\omega>-3/2$, eliminates the weak field approximation from wormhole investigations. At Eq. (9) below, it is shown that when $\lambda<0, B<0$ the same conclusions follow.

Secondly, the authors (in Ref. [1] as well as in their previous publication [9]), use the notation $r_0^{\pm}$ and $R_0^{\pm}$ for the wormhole throat location. Under the conditions of Eqs. (2) and (3) $r_0^+>B$ while $r_0^-<B$. This follows from Ref. [1] [Eqs. (13) and (14)]:

$$r_0^+r_0^-=B^2;$$

thus

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$$0<\nu-(\nu^2-1)^{1/2}\equiv(r_0^-/B)_{B>0}<1<(r_0^+/B)_{B>0}$$
$$\equiv\nu+(\nu^2-1)^{1/2},\quad \nu=(C+1)/\lambda>1, \tag{8}$$

and one must exclude r_0^- from consideration in this case $(B,\lambda>0)$. In Ref. [9] it was observed that the BD solutions are invariant under the simultaneous symmetry transformations $\lambda\to-\lambda$ and $B\to-B$, and that the post-Newtonian limit determines the relation $B=\lambda m/2$, where $m>0$ is the central body's active mass [4,5]. For the case $\lambda<0, B<0$, Eq. (3) becomes $\nu=(C+1)/\lambda<-1, (C+1)>-\lambda>0$, $C>-1$, and Eq. (4) still holds. Furthermore, in this case the wormhole throat location is specified by $r_0^-/B<0$, and r_0^+ is excluded. Note that this case is just a notational change and yields no new results:

$$-(r_0^-/B)_{B<0}=-[\nu-(\nu^2-1)^{1/2}]_{\nu<-1}$$
$$=[\nu+(\nu^2-1)^{1/2}]_{\nu>1}$$
$$=(r_0^+/B)_{B>0}>1,$$
$$(r_0^+)_{B>0}=(r_0^-)_{B<0}>|B|. \tag{9}$$

Equation (7) and the conclusions in the paragraph following Eq. (7) still applying under this symmetry transformation.

Thirdly, the limits on ρ_0 and R_0^+ as $r_0^+\to B$ as presented in Ref. [1] [between Eqs. (18) and (19)] are incorrect. Since

$$r_0^+/B=\nu+(\nu^2-1)^{1/2},\quad \nu=(C+1)/\lambda>1, \tag{10}$$

one cannot let $r_0^+\to B$ unless $\nu\to1$ and $C\to0$. Note that R_0^+ is a monotonic function of ν:

$$4B<R_0^+<\infty; \tag{11}$$

its minimum $4B$ occurs at $\nu=1$ and its maximum at $\nu=\infty$. The values at the wormhole throat $(r=r_0^+=r_0)$ of the areal radius $[R(r_0^+)=R_0]$, the energy density (ρ), and the radial (p_r) and lateral tensions (P) are given by [10]

$$R_0=2B(\nu^2-1)^{1/2}\left[\frac{\nu+1}{\nu-1}\right]^{\nu/2}$$
$$=r_0^{-1}(r_0+B)^{(r_0+B)^2/(2r_0B)}(r_0-B)^{(r_0-B)^2/(2r_0B)},$$
$$\rho_0=(p_r)_0=[2(\omega C)^{-1}-1]P_0=-[c^4\varphi(r_0)]/[8\pi R_0^2], \tag{12}$$

where

$$\nu\pm1=(r_0\pm B)^2/(2r_0B),$$
$$\varphi(r_0)/\varphi_0=\left[\frac{\nu+1}{\nu-1}\right]^{C/(2\lambda)}=\left[\frac{r_0+B}{r_0-B}\right]^{C/\lambda}, \tag{13}$$

and

$$\varphi(\nu\to1,r_0\to B)=\varphi_0,\quad R_0(\nu\to1,r_0\to B)=4B,$$
$$\rho_0(\nu\to1,r_0\to B)=-[c^4\varphi_0]/[8\pi(4B)^2]. \tag{14}$$

In Ref. [1] the limit $R_0\to0^+$ as $r_0\to B^+$ is presented. As shown in Eq. (13) this is incorrect. However, when $1>(C+1)/\lambda=\nu$, there is no wormhole throat and $R(r)\to0^+$ as $r\to B^+$.

Fourthly, as far as the traversability conditions, I find no singularities at the wormhole throat. In particular the $\langle0101\rangle$ component of the Riemann tensor, entering the expression for the radial tidal acceleration, is finite at the wormhole throat:

$$\mathbb{R}^0_{101}(r_0)=C/[(\nu^2-1)(\lambda r_0)^2]=[(1-\omega C/2)r_0^2]^{-1}. \tag{15}$$

At the wormhole throat $(R=R_0)$, the redshift function [2,4,10], Φ vs R has an infinite slope. However, in contrast to the statement in Ref. [1], I find no jump discontinuity in $\Phi(R)$ at $R=R_0$. Also at the throat the radial component of proper acceleration that an observer must maintain to remain at rest is given by $a^r=c^2r_0/[\lambda(\nu^2-1)^{1/2}R_0^2]$ which is positive indicating that an outward-directed radial acceleration is necessary to keep an observer from being pulled into the wormhole [2,6]. I have found that for $\omega>-\frac{3}{2}$, that $C>0$ and that both the energy density $\rho(r)$ and the radial tension $p_r(r)$ are negative and finite for all $r>B$. The lateral tension is positive (focusing) at large r and negative (defocusing) within the wormhole throat, changing sign outside the wormhole throat location at $r/B=x_1+(x_1^2-1)^{1/2}>r_0^+/B$, where $x_1=\nu+\frac{1}{2}\omega C/\lambda$ and C and $\omega>0$ [10]. The negative energy density is a special case of defocusing which implies exotic matter at the throat [6]. The negative radial tension indicates stretching and the negative surface tension is interpreted as positive (outward) pressure preventing collapse of the wormhole throat [2,6].

Finally, deep within the throat at $r=B$ a naked singularity occurs and all the measures of traversability diverge. This singularity can be avoided by placing appropriate material at $r>B$. In conclusion, in the BD theory the Einstein representation minimally coupled scalar field energy condition requires that $\omega>-\frac{3}{2}$ which disallows wormholes existence within the weak field approximation $C=-1/(\omega+2)$. The Jordan representation allows wormhole solutions to exist for strong fields [10].

Note added. In Nandi's accompanying Reply the two cases $(C+1<0, \lambda<0, B>0)$ and $(C+1<0, \lambda>0, B<0)$ are introduced to argue for wormhole existence when $C=-1/(\omega+2)$ and $-\frac{3}{2}<\omega<-\frac{4}{3}$. However, the Keplerian mass m is positive definite; hence B and λ cannot have opposite signs. Furthermore in both these cases the redshift function [2,4,10] $\Phi>0$, and hence $-g_{00}\to\infty$ as $r\to|B|$ rather than the Brans class I behavior $\Phi<0$, and $-g_{00}\to0$ as $r\to|B|$.

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