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Electromagnetic duality and light-front coordinates

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We review the light-front Hamiltonian approach for the Abelian gauge theory in 3+1 dimensions, and then study electromagnetic duality in this framework. [S0556-2821(99)02608-9]

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I. INTRODUCTION

In recent years we have witnessed a resurrection of interest in light-front Hamiltonian physics in two areas. The first one, the new nonperturbative approach to QCD [1], is related to the original application of light-front coordinates [2], i.e. hadron spectroscopy, and the other comes from string theory [3]. The popular M-theory [4] is formulated in light-front coordinates [5]. With the rise of the second application, some rather academic questions became of interest. For example, dualities in string theories are one of the most powerful tools, yet not much is known about how they work on the light front. In this paper, we attempt to study one of the simplest cases of known dualities — the electromagnetic duality in the Abelian gauge field theory in 3+1 dimensions.

Susskind has conjectured [6] that since light-front coordinates are non-local in the longitudinal direction, it might be possible to formulate a light-front theory with both electric and magnetic sources without having to introduce any additional non-localities corresponding to Dirac strings [7]. He observed that the role of electric and magnetic fields reverses in the light-front Hamiltonian (which contains only the physical, transverse fields) when the original fields are replaced by (transverse) fields perpendicular to them. He concluded that the above described transformation of fields is the electromagnetic duality on the light front. Then he suggested that magnetic sources be added to the Hamiltonian by symmetry.

Here we investigate this idea. The paper is organized as follows: In Sec. II we summarize the formalisms of free Abelian fields, show how classical electric sources can be added to the theory, and establish the connection between the components of the $F^{\mu\nu}$ tensor and electric and magnetic fields. For completeness, we list the surface terms even though they do not enter the calculation presented here, and we list some manipulations with the $(\partial^+)^{-1}$ operator. Further, we wish to mention that there are, in general, problems regarding other than + components of light-front currents, even though this does not affect our calculation since we restrict ourselves to external classical currents. Section II is rather formal; a reader familiar with the light front may want to skip most of it. Section III is devoted to Susskind's idea. The last section contains our conclusions.

II. LIGHT-FRONT FIELD THEORY: FORMALITIES

In a light-front quantum field theory, fields are quantized at an equal light-front time $x^+=t+z$ [8]. The remaining coordinates, $x^-=t-z$ and $x^\perp\equiv(x^1,x^2)$, are spatial.[1]

Let $p=(p^-,\ p^+,\ p^1,\ p^2)$ be the four-momentum of a free particle with mass m in light-front coordinates. Then

$$p_\mu x^\mu=\frac{1}{2}p^+x^-+\frac{1}{2}p^-x^+-p^\perp\cdot x^\perp. \tag{1}$$

p^+ is the *longitudinal* momentum, p^1 and p^2 are the *transverse* momenta, and p^- is *the light-front energy*:

$$p^-=\frac{p^{\perp 2}+m^2}{p^+}. \tag{2}$$

The light-front energy is well defined apart from peculiar modes which have zero longitudinal momentum (so-called *zero modes*). p^+, which is equal to p^0+p^3, satisfies $p^+\geqslant 0$. This means that in the vacuum all particles must have precisely zero longitudinal momentum. From the expression for the light-front energy one can see that the energy diverges as $p^+\to 0$ for massive particles. For massless particles, the light-front energy can be finite even at $p^+=0$, but the vacuum can be made trivial by imposing a small longitudinal momentum cutoff, e.g. requiring that all longitudinal momenta satisfy $p_i^+>\epsilon$. Another frequently used method of regularization is a discretized light-cone quantization (DLCQ) [11] which removes both ultraviolet and infrared divergences. The physics of $p^+=0$ cannot be recovered by renormalization with respect to high energy states, and has to be added (at present) by hand using counterterms which can be functions of $x^\perp$. The so-called "constraint zero mode" [12] is a specific counterterm consequent of DLCQ, and it does not require a nontrivial vacuum structure.

A. Abelian gauge fields

Let us start with pure electromagnetism. The Lagrangian density is

$$\mathcal{L}=-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}, \tag{3}$$

where

$$F^{\mu\nu}=\partial^\mu A^\nu-\partial^\nu A^\mu. \tag{4}$$

[1] There are two slightly different conventions regarding the +,− components. The other one differs from the one used here by a factor of $\sqrt{2}$, $a^\pm=(a^0\pm a^3)/\sqrt{2}$, so that the +− component of the metric tensor is $g^{+-}=1$. In the convention $a^\pm=(a^0\pm a^3)$ used here, $g^{+-}=2$.

In a light-front formulation, indices μ,ν run through $+,-$, and $\perp=(1,2)$.

In the *light-front gauge*, $A^+=0$, and the Lagrangian density reduces to

$$\mathcal{L}=\frac{1}{8}(\partial^+A^-)^2+\frac{1}{2}\partial^+A^i\partial^-A^i-\frac{1}{2}\partial^+A^i\partial^iA^- -\frac{1}{4}(\partial^iA^j-\partial^jA^i)^2. \tag{5}$$

The Lagrangian density does not contain a time derivative of A^-, and so it is immediately obvious that this component of A^μ is not dynamical. A^- can be eliminated using the equations of motion

$$(\partial^+)^2A^-=2\partial^+\partial^iA^i. \tag{6}$$

Apart from zero modes (i.e. $p^+=0$ states introduced above), ∂^+ can be inverted, and A^- is then given by

$$A^-=\frac{2}{\partial^+}\partial^iA^i. \tag{7}$$

To proceed further, some manipulations with $(\partial^+)^{-1}$ are needed. Up to a constant in x^- which can depend on remaining coordinates, the operator $(\partial^+)^{-1}$ is defined as follows:

$$\frac{1}{\partial^+}f(x^-)\equiv\int\frac{dy^-}{4}\epsilon(x^--y^-)f(y^-),$$

where $\epsilon(x)=1$, if $x>0$, and $\epsilon(-x)=-\epsilon(x)$, and $f(x^-)$ is an arbitrary function. Using the properties of $\epsilon(x)$ it is straightforward to find

$$\int d^3x\left(\frac{1}{\partial^+}f(x)\right)^2=-\int d^3x f(x)\left(\frac{1}{\partial^+}\right)^2f(x),$$

$$\int d^3x f(x)\left(\frac{1}{\partial^+}g(x)\right)=-\int d^3x\left(\frac{1}{\partial^+}f(x)\right)g(x).$$

Substituting for A^-, and using the properties of the operator $(\partial^+)^{-1}$ shown above, gives the Lagrangian in terms of physical degrees of freedom:

$$\mathcal{L}=\frac{1}{2}\partial^+A^i\partial^-A^i-\frac{1}{2}(\partial^iA^i)^2-\frac{1}{4}(\partial^iA^j-\partial^jA^i)^2 +\text{surface terms}, \tag{8}$$

where the surface terms

$$-\left\{\partial^+\left(A^j\partial^j\frac{1}{\partial^+}\partial^iA^i\right)-\partial^j(A^j\partial^iA^i)\right\}$$

are traditionally dropped. The light-front Hamiltonian density is found to be

$$\mathcal{H}=\tfrac{1}{2}(\partial^iA^i)^2+\tfrac{1}{4}(\partial^iA^j-\partial^jA^i)^2, \tag{9}$$

not dependent on the longitudinal derivative of the fields.

The field strength tensor $F^{\mu\nu}$ in terms of A^i is

$$F^{+-}=2\partial^iA^i,\quad F^{+i}=\partial^+A^i,$$

$$F^{-i}=\partial^-A^i-\partial^i\frac{2}{\partial^+}\partial^jA^j,\quad F^{ij}=\partial^iA^j-\partial^jA^i. \tag{10}$$

Let us also introduce the dual tensor $\tilde{F}^{\mu\nu}=1/2\epsilon^{\mu\nu\lambda\rho}F_{\lambda\rho}$, where $\epsilon^{\mu\nu\lambda\rho}$ is totally antisymmetric, $\epsilon^{+-12}\equiv2$:

$$\tilde{F}^{+-}=\epsilon_{ij}F^{ij},\quad \tilde{F}^{+i}=\epsilon_{ij}F^{+j},$$

$$\tilde{F}^{ij}=-\frac{1}{2}\epsilon_{ij}F^{+-},\quad \tilde{F}^{-i}=-\epsilon_{ij}F^{-j}, \tag{11}$$

where $\epsilon_{12}=1$ is antisymmetric. Let us note that F and $\tilde{F}$ are related by electromagnetic duality $\vec{B}\to\vec{E}$, $\vec{E}\to-\vec{B}$. The connection between electric and magnetic fields $\vec{E}$ and $\vec{B}$ and the tensor $F^{\mu\nu}$ given in light-front coordinates is shown in the next section.

Finally, we briefly describe what happens in the presence of classical electric sources. As in the free case, A^- is not dynamical and can be eliminated using the equations of motion. The interaction term in the Lagrangian, $-j_\mu A^\mu$, produces an additional term in the expression for A^-, viz.

$$A^-=A^-_{\text{free}}-\frac{2}{(\partial^+)^2}j^+, \tag{12}$$

where A^-_{free} is given in Eq. (7). Consequently, $F^{\mu\nu}$ is modified also. In particular, the $+$ component of the current is absorbed into F^{+-} and F^{-i}:

$$F^{+-}=F^{+-}_{\text{free}}-\frac{2}{\partial^+}j^+,$$

$$F^{-i}=F^{-i}_{\text{free}}-\partial^i\frac{2}{(\partial^+)^2}j^+, \tag{13}$$

where $F^{\mu\nu}_{\text{free}}$ is given Eq. (10). The remaining two components of $F^{\mu\nu}$ are unchanged.

The Lagrangian equations of motion are

$$\partial_\mu F^{\mu i}=j^i, \tag{14}$$

$\partial_\mu F^{\mu+}=j^+$ is satisfied identically, and using equations of motion it can be shown that

$$\partial_\mu F^{\mu-}=-\frac{2}{\partial^+}\left[\frac{1}{2}\partial^-j^+-\partial^ij^i\right],$$

which implies a continuity equation for j^μ.

The Hamiltonian density in the presence of classical sources is

$$\mathcal{H}=\frac{1}{2}\left(\partial^iA^i-\frac{1}{\partial^+}j^+\right)^2+\frac{1}{4}(\partial^iA^j-\partial^jA^i)^2-j^\perp A^\perp, \tag{15}$$

and the fields A^i can be quantized as if they were free.

B. Connection between electric and magnetic fields $\vec{E}$ and $\vec{B}$ and the tensor $F^{\mu\nu}$ in light-front coordinates

The connection between $\vec{B}$, $\vec{E}$ and $F^{\mu\nu}$ can be established using the definition of the potential:

$$\vec{E} = -\vec{\nabla}A^0 - \frac{\partial}{\partial t}\vec{A},$$

$$\vec{B} = \vec{\nabla}\times\vec{A}. \tag{16}$$

Substituting

$$A^0 = \tfrac{1}{2}(A^+ + A^-),$$

$$A^3 = \tfrac{1}{2}(A^+ - A^-),$$

$$\frac{\partial}{\partial t} = \partial^0 = \tfrac{1}{2}(\partial^+ + \partial^-),$$

$$\frac{\partial}{\partial z} = -\partial^3 = -\tfrac{1}{2}(\partial^+ - \partial^-), \tag{17}$$

we obtain

$$E^i = -\tfrac{1}{2}(F^{+i} + F^{-i}), \quad E^z = \tfrac{1}{2}F^{+-},$$

$$B^i = \epsilon_{ij}\tfrac{1}{2}(F^{+j} - F^{-j}), \quad B^z = -\tfrac{1}{2}\epsilon_{ij}F^{ij}, \tag{18}$$

or

$$F^{+-} = 2E^z, \quad \tilde{F}^{+-} = -2B^z,$$

$$F^{+i} = -(E^i + \epsilon_{ij}B^j), \quad \tilde{F}^{+i} = (B^i - \epsilon_{ij}E^j),$$

$$F^{ij} = -\epsilon_{ij}B^z, \quad \tilde{F}^{ij} = -\epsilon_{ij}E^z,$$

$$F^{-i} = -(E^i - \epsilon_{ij}B^j), \quad \tilde{F}^{-i} = (B^i + \epsilon_{ij}E^j), \tag{19}$$

where i,j are transverse indices $[i,j=(1,2)]$. These definitions ensure that the Lagrangian equations of motion give the correct set of Maxwell's equations. In Ref. [10] the magnetic and electric fields are defined differently, in particular, in analogy with equal time, $E^\mu = 1/2F^{+\mu}$ and $B^- = F^{12}$, but this is misleading, because these definitions do not lead to Maxwell's equations. Moreover, $\vec{E}$ and $\vec{B}$ are not four-vectors, and E^0 and B^0 are not defined; so there is no natural way to form the minus and plus components.

III. ELECTROMAGNETIC DUALITY

In this section we investigate the question of whether it is possible to formulate electromagnetic duality as a transformation of the potential $A^\perp$ itself rather than the field strength tensor and its dual. Given the Hamiltonian in the transverse degrees of freedom, a natural starting point is the transformation

$$A^j \to \tilde{A}^i \equiv -\epsilon_{ij}A^j. \tag{20}$$

Indeed, under this transformation the first and second terms in the free Hamiltonian "interchange":

$$\mathcal{H} = \tfrac{1}{4}(\partial^i\tilde{A}^j - \partial^j\tilde{A}^i)^2 + \tfrac{1}{2}(\partial^i\tilde{A}^i)^2. \tag{21}$$

By comparison with the Hamiltonian including electric sources [see Eq. (15)], it appears that one can by symmetry add magnetic sources, as well as the electric sources. In complete analogy one could then expect that the $+$ component of the magnetic current $\tilde{j}^\mu$ was absorbed into the definition of the field strength tensor and/or the dual tensor [6]. Taking advantage of the kinematical boost invariance (for a review see [9]), it would be sufficient to consider the simple case of a magnetic current with only the $+$ component (the so-called *good component*) being non-zero, viz.

$$\mathcal{H} = \frac{1}{2}\left(\partial^i A^i - \frac{1}{\partial^+}j^+\right)^2 + \frac{1}{2}\left(\partial^i\tilde{A}^i - \frac{1}{\partial^+}\tilde{j}^+\right)^2 - j^\perp A^\perp.$$

It is straightforward to show that if one proceeds as described, the Hamiltonian leads to the desired equations of motion, including the continuity equation for the $-$ component of $\tilde{j}$.[2]

The catch is that the Hamiltonian itself is *not* equivalent to the complete set of Maxwell equations. It is, rather, the Hamiltonian *and* the gauge conditions [13]. In particular, *only* with the gauge conditions $A^+ = 0$ and $A^- = 2(\partial^+)^{-1}\partial^i A^i$ are all components of the field strength tensor defined unambiguously.

Let us look at what happens with the field strength tensor and its dual under the transformation (20). In order for the transformation (20) to be the operation of electromagnetic duality, it has to lead to

$$-\tilde{F}^{\mu\nu}(A^\perp) = F^{\mu\nu}(\tilde{A}^\perp),$$

$$F^{\mu\nu}(A^\perp) = \tilde{F}^{\mu\nu}(\tilde{A}^\perp). \tag{22}$$

However,

$$-\tilde{F}^{+-} = 2\partial^i\tilde{A}^i,$$

$$-\tilde{F}^{+i} = \partial^+\tilde{A}^i,$$

$$-\tilde{F}^{-i} = \partial^-\tilde{A}^i - \partial^i(2/\partial^+)\,\partial^k\tilde{A}^k - (2/\partial^+)\,\Box\tilde{A}^i,$$

$$-\tilde{F}^{ij} = \partial^i\tilde{A}^j - \partial^j\tilde{A}^i \tag{23}$$

shows that the transformation (20) is not quite electromagnetic duality: It works for all components except F^{-i}. The $\tilde{F}^{-i}$ contains an additional term $-2(\partial^+)^{-1}\Box\tilde{A}^i$.

For free fields, the additional term is zero, and Eq. (20) is therefore electromagnetic duality. Is it possible to remove the additional term in general, realizing the electromagnetic du-

[2] It is not really a mystery—as a result of the definition of the dual tensor as $\tilde{F}^{\mu\nu} \equiv 1/2\epsilon^{\mu\nu\lambda\rho}F_{\lambda\rho}$, it follows that $\partial_\mu\tilde{F}^{\mu\nu} \equiv 0$. However, absorbing the $\tilde{j}^+$ appropriately into the definition of $\tilde{F}$ produces a non-zero right-hand side. Note that in our case this trick does not work for $\tilde{j}^i \neq 0$.

ality as a generalization of the original Susskind's suggestion, e.g. Eq. (20) plus a gauge transformation?

After fixing the gauge, there is still a residual gauge freedom. In order not to disturb the gauge conditions used to derive the Hamiltonian, the residual gauge function Λ has to satisfy

$$\partial^+\Lambda=0, \quad \partial^-\Lambda=(2/\partial^+)(\partial^i)^2\Lambda. \tag{24}$$

Ignoring for a moment the question of zero modes, this implies that

$$(2/\partial^+)\Box\Lambda=0$$

and thus cannot cancel the unwanted term $-2(\partial^+)^{-1}\Box\tilde{A}^i$.

We now return to the question of zero modes. Since they correspond to a constant in x^-, they cannot cancel the $-2(\partial^+)^{-1}\Box\tilde{A}^i$ term which, in general, does depend on x^-.

IV. CONCLUSION AND SUMMARY

We reviewed the formalism of Abelian gauge theory in light-front coordinates. We argued that while the potential A^μ can be described in light-front coordinates, there is no light-front analogue to electric and magnetic fields $\vec{E},\vec{B}$ in the sense that if one defines electric and magnetic fields as components of the light-front field strength tensor, the definitions do not lead to Maxwell equations, and electromagnetic duality is not realized as $\vec{B}\rightarrow\vec{E}$, $\vec{E}\rightarrow-\vec{B}$.

We then studied electromagnetic duality on the level of fields A^μ (in the light-front gauge $A^+=0$). Our study was motivated by the fact that the light-front Hamiltonian is in this case expressed in terms of transverse fields only, and that by under a specific transformation of transverse fields the electric and magnetic terms in the Hamiltonian interchange.

However, the electromagnetic duality in light-front coordinates cannot be realized by a transformation of transverse fields only. Neither can it be written as a transformation of transverse fields plus a gauge transformation, not even when the gauge transformation has a zero mode. Altering A^- in addition to $A^\perp$ is not likely to fix the problem either, because it is only one of the two components of the dual tensor involving A^- (i.e. $\tilde{F}^{-i}$) that does not transform as desired; fixing $\tilde{F}^{-i}$ would spoil the transformation of $\tilde{F}^{-+}$.

To include magnetic monopoles one would have to allow for additional non-localities (in the gauge function), most likely equivalent to Dirac strings in an equal-time theory [7].

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