Four-point functions in the CFT-AdS correspondence

Hong Liu* and A. A. Tseytlin†
Theoretical Physics Group, Blackett Laboratory, Imperial College, London SW7 2BZ, United Kingdom
(Received 24 July 1998; published 3 March 1999)

We discuss the properties of four-point functions in the context of the correspondence between a classical supergravity theory in the bulk of the anti–de Sitter (AdS) space and quantum conformal field theory (CFT) at the boundary. The contribution to a four-point function from the exchange of a scalar field of arbitrary mass in AdS space is explicitly identified with that of the corresponding operator in the conformal partial-wave expansion of a four-point function on the CFT side. Integral representations are found for the massless vector and graviton exchanges. We also discuss some aspects of the four-point functions of tr F^2 and tr FF^* (''dilaton'' and ''axion'') operators in $\mathcal{N}=4$ supersymmetric SU(N) Yang-Mills theory as predicted by type-IIB supergravity in the five-dimensional AdS background. [S0556-2821(98)02324-8]

PACS number(s): 11.25.Hf, 04.65.+e, 11.25.Mj, 11.25.Pm

I. INTRODUCTION

There has been a recent revival of interest in the connection between large-N Yang-Mills theory [1] and string theory [2] following, in particular, the conjecture [3] that there is an exact correspondence between string or M theory on the $(d+1)$-dimensional anti–de Sitter space (AdS_{d+1}) and certain superconformal field theory (CFT_d) defined at the boundary of the AdS_{d+1} (see also [4]). According to the conjecture, quantum $\mathcal{N}=4$ supersymmetric Yang-Mills (SYM) theory with gauge group SU(N) in the large-N and large 't Hooft coupling limit can be described by the classical type-IIB supergravity on $\mathrm{AdS}_5\times \mathrm{S}^5$ space.

The formulation of the conjecture was made more explicit in [5,6], where it was proposed that the partition function of supergravity or string theory with fixed boundary values of the fields is to be identified with the generating functional of the composite operators in CFT. There is a one-to-one correspondence between certain local operators Φ_i of the boundary CFT and the bulk fields ϕ_i in AdS space [5,7,6,8]. The boundary CFT operator Φ_i and the associated bulk field ϕ_i carry the same unitary, irreducible, and highest weight representation of the conformal group SO(d,2), where the scale dimension λ_i of Φ_i is identified with the lowest energy value of ϕ_i and can be further related to the mass of ϕ_i. The correlation functions of the CFT operators are identified with the classical ''S-matrix elements'' of the bulk fields with their boundary values fixed. Two-and three-point functions follow simply from the quadratic terms and cubic vertices of the bulk theory, while the four-point functions, in general, contain the contact contributions as well as the exchanges of virtual particles.

Using this proposal, some ''model'' and ''realistic'' two- and three-point functions have been computed in [9,10,11,12,13,14,15,16]. In particular, a family of three-point functions of chiral primary operators in $\mathcal{N}=4$ SYM has been evaluated and shown to be equal to their free-field values, suggesting a nonrenormalization theorem in large-N limit [17].

In this paper we investigate the properties of the four-point functions in the context of the CFT-AdS correspondence. Four-point functions from contact interactions (quartic vertices) were considered before in [10]. The scalar exchange diagrams with some special values of mass were also discussed in [18]. The exchange diagrams in AdS space are, in general, very difficult to evaluate explicitly, as the propagators and integrals are quite complicated. Here we follow a different approach.

In CFT_d, the states generated by acting by a product of the conformal operators on the vacuum can be decomposed into a direct sum of irreducible representations of the conformal group

$$\Phi_1(x_1)\Phi_2(x_2)|0\rangle=\sum_k \int d^dx\ Q_k(x|x_1,x_2)|k,x\rangle, \tag{1.1}$$

where k sums over all the irreducible representations in Hilbert space and states $|k,x\rangle=\Phi_k(x)|0\rangle$ span the space of irreducible representation T_k. This conformal partial-wave expansion (CPWE) was obtained in the early 1970s by several authors [19,20] (see [21,22,23] for reviews). Using Eq. (1.1), the four-point functions can be written as

$$\langle 0|\Phi_1(x_1)\Phi_2(x_2)\Phi_3(x_3)\Phi_4(x_4)|0\rangle=\sum_k G_k, \tag{1.2}$$

where G_k is the contribution to the four-point function from the intermediate states generated by the operator Φ_k:

$$G_k=\int d^dx\ d^dy\ Q_k^*(x_1,x_2|x)\langle k,x|k,y\rangle Q_k(y|x_3,x_4). \tag{1.3}$$

Since the bulk propagator of the field ϕ_k in AdS_{d+1} associated with the conformal operator Φ_k by the AdS-CFT correspondence can be also written as a sum over normal modes in AdS_{d+1} which span the same irreducible representation T_k

*Email address: hong.liu@ic.ac.uk
†Also at the Lebedev Physics Institute, Moscow, Russia. Email address: tseytlin@ic.ac.uk

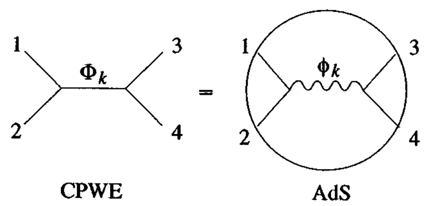

FIG. 1. Equivalence between the CPWE and scattering in AdS space.

of the isometry-conformal group SO$(d,2)$, one is tempted to conjecture that G_k represents the contribution of the ϕ_k exchange diagram in AdS$_{d+1}$. Diagrammatically, this equivalence can be expressed as in Fig. 1.

One of the aims of the present paper is to prove that this is indeed the case for the intermediate states corresponding to the scalar operators. It should be possible to generalize our method also to operators of higher spin. We shall attempt to consider the two cases which are of particular interest: massless vector and massless tensor (graviton) exchanges. They correspond to the conserved current vector and the energy momentum tensor operators in CFT. The explicit demonstration of the equivalence between the CPWE representation for the CFT correlator and the AdS amplitude here appears to be more difficult and will not be given in the present paper. The expressions for the propagators for the photon and graviton in AdS space are quite involved (useful expressions for them which are suitable for explicit calculations were not previously given in the literature; cf. [24,25]). There are also complications related to the presence of the gauge degrees of freedom and the fact that the current and the energy momentum tensor carry indecomposable representations of the conformal group. In this paper we shall use a noncovariant gauge fixing and will be able to write down the AdS amplitudes with the photon and graviton exchanges in a complete integral form. The detailed analysis of these amplitudes and establishing their relation to the CPWE will not be attempted here.

Having identified the exchange diagrams with the CPWE, a question that naturally arises is the interpretation of contact interactions (quartic or higher vertices) on the CFT side, as there is no obvious counterpart for them in the CPWE. One possibility is that since contact terms can always be formally written as special exchange diagrams, e.g.,

$$\int d^{d+1}x\ \phi_1\phi_2\phi_3\phi_4$$

$$=-\int d^{d+1}x\, d^{d+1}y\ \phi_1\phi_2 G(x,y)\partial^2(\phi_3\phi_4), \qquad (1.4)$$

where G is the massless field propagator and $-\partial^2 G=\delta(x-y)$, they might be contained in the CPWE. Another possibility could be that the boundary CFT is not closed. Let us consider, for example, $\mathcal{N}=4$ SYM theory in the large-N and large t Hooft coupling $\lambda=g^2N$ limit. Suppose that a four-point function at finite λ is expanded in the form of Eqs. (1.2) and (1.3). As we increase λ, certain correlators may approach zero as some inverse powers of λ and thus may not contribute in the $\lambda\to\infty$ limit. But if there is a very large number of such vertices, their total contribution to the sum (1.2) may not vanish. This would correspond to the presence of contact interactions in supergravity.[1] As there are many contact terms in type-IIB supergravity on AdS$_5\times$S^5, such a possibility deserves a detailed investigation.

The existence of the AdS-CFT correspondence puts by itself strong constraints on the theories on both sides. Since conformal field theories always contain the energy-momentum tensor which generates the conformal algebra, the theory in AdS space must be a gravitational theory. If the theory in AdS space is a field theory (supergravity), then on the CFT side the possible three-point functions and intermediate states contributing to Eq. (1.2) are highly constrained as only vertices which can be written as local invariants in AdS space are allowed. For example, consider a correlation function of four scalars. In general, symmetric tensor operators of spin greater than 2 can contribute to it as intermediate states in Eq. (1.2). However, there is no local covariant interaction vertex for two scalars and a higher spin tensor in supergravity theory (though it may be present in string theory); so it should vanish also on the CFT side. Assuming the equivalence between the CPWE in CFT and the scattering amplitude in AdS space, we see also that different channels for the CPWE in CFT should correspond to s-t-u channels in the scattering amplitudes in AdS space, which seems to imply that the scattering amplitudes in AdS space should have s-t-u crossing symmetry.[2]

In case of the "$\mathcal{N}=4$ SYM–type-IIB supergravity on AdS$_5\times$S^5" correspondence, the supergravity four-point functions in general are quite complicated.[3] In this paper we shall focus on the dilaton-axion sector, where the corresponding four-point functions are given by a relatively small number of diagrams. We will show that the main nontrivial contribution in this sector comes from a graviton exchange (the "mixed" scalar four-point function contains also a contact contribution). Using the graviton propagator found here in the noncovariant $h_{0\mu}=0$ gauge, we will be able to present the complete expressions for the scalar four-point functions

[1]These contact supergravity vertices may be thought of as originating from string field theory (with only cubic interactions between massless and massive modes) in the low-energy approximation in which all massive string modes are integrated out.

[2]One could think that this might be an indication that the theory on the AdS side should actually be a string-type theory. One does not expect to find crossing symmetry in the bulk supergravity amplitudes, but this is less clear when the bulk-to-boundary propagators are attached. Duality is, of course, restored in the bulk amplitudes once one replaces the supergravity amplitudes by the full string amplitudes, i.e., includes all α' corrections. At the same time, the boundary theory at large N and large g^2N corresponding just to supergravity with no α' corrections is also a CFT which should have a CPWE.

[3]Higher-order α' and nonperturbative string-theory contributions to four-point functions in $\mathcal{N}=4$ large-N SYM theory were discussed in [26].

in terms of the formal integrals. We shall discuss briefly certain properties of the integrals, leaving their evaluation and establishing the correspondence with the CPWE in CFT for the future.

The structure of the paper is as follows. In Sec. II we shall review some aspects of CFT in d dimensions, in particular, the conformal partial-wave expansion for the four-point functions. In Sec. III we shall discuss the scattering diagrams involving scalar exchanges in AdS space. We will show that they can be identified with the contributions to the CPWE coming from the corresponding operators on the CFT side. In Sec. IV we shall study the scattering amplitudes involving exchanges of massless vectors and gravitons. In Sec. V we shall consider the dilaton and axion four-point functions in $D=5$ supergravity corresponding to the correlators of the tr F^2 and tr FF^* operators in $\mathcal{N}=4$ SYM theory. Appendix A contains the notation and some technical details about the scalar propagator in AdS_{d+1}. In Appendix B we recall the expression [19] (see also [27]) for the scalar operator contribution to the scalar four-point function in CFT_d.

II. FOUR-POINT FUNCTIONS AND CONFORMAL PARTIAL-WAVE EXPANSION IN CFT

Let us first review certain aspects of the conformal field theory in d dimensions [21,22,23]. Denote the space of an irreducible representation T_σ of the conformal group[4] as M_σ, $\sigma=(\lambda,\vec{s})$, where λ is the conformal dimension and $\vec{s}$ is a set of quantum numbers labeling the spin degrees of freedom. We assume that Hilbert space can be represented as a direct sum of spaces M_{σ_i}, i.e.,

$$\mathcal{H}=M_{\sigma_1}+M_{\sigma_2}+\cdots+M_{\sigma_i}+\cdots,$$

$$\sigma_i=(\lambda_i,\vec{s}_i),\quad i=1,2,\ldots\,. \tag{2.1}$$

Conformal fields $\Phi_{\sigma_i}(x)$ are defined as the operators which generate spaces M_{σ_i}:

$$M_{\sigma_i}=\{|\sigma_i,x\rangle \quad \text{for all}\quad x\}=\{\Phi_{\sigma_i}(x)|0\rangle \quad \text{for all}\quad x\}.$$

States of the type $\Phi_{\sigma_i}(x_1)\Phi_{\sigma_j}(x_2)|0\rangle$ can be decomposed in terms of the basis in Eq. (2.1):

$$\Phi_{\sigma_i}(x_1)\Phi_{\sigma_j}(x_2)|0\rangle=\sum_k \int d^dx\; Q_{ijk}(x|x_1,x_2)|\sigma_k,x\rangle. \tag{2.2}$$

This implies the operator product expansion (OPE)

$$\Phi_{\sigma_i}(x_1)\Phi_{\sigma_j}(x_2)=\sum_k \int d^dx\; Q_{ijk}(x|x_1,x_2)\Phi_{\sigma_k}(x). \tag{2.3}$$

The standard OPE in the nearby points can be obtained from Eq. (2.3) by expanding the integrand in $y=x_1-x_2$:

$$\Phi_{\sigma_i}(y)\Phi_{\sigma_j}(0)|_{y\to 0}\sim\sum_{k,m} A_{\sigma_k,m}(y)\Phi_{\sigma_k,m}(0), \tag{2.4}$$

where $A_{\sigma_k,m}\sim y^{-(\lambda_i+\lambda_j-\lambda_k-m)}$ and $\Phi_{\sigma_k,m}$ are mth-order derivatives of the field Φ_{σ_k}.

When Φ_σ's are orthogonal to each other, Q's are just the amputated three-point functions:

$$Q_{ijk}(x|x_1,x_2)=\int d^dx'\, W^{-1}_{\sigma_k}(x-x')\times\langle 0|\Phi_{\sigma_k}(x')\Phi_{\sigma_i}(x_1)\Phi_{\sigma_j}(x_2)|0\rangle, \tag{2.5}$$

with $W_\sigma(x-x')=\langle 0|\Phi_\sigma(x)\Phi_\sigma(x')|0\rangle$ and its inverse W^{-1} defined by

$$\int d^dx\; W_\sigma(x_1-x)W^{-1}_\sigma(x-x_2)=I_+(x_1-x_2), \tag{2.6}$$

$$I_+(x)=\frac{1}{(2\pi)^d}\int d^dp\;\vartheta(p^0)\vartheta(p^2)e^{ipx}. \tag{2.7}$$

That I_+ (instead of the Dirac δ function) appears on the right side of Eq. (2.6) follows from the spectrality condition.

States involving higher-order products of Φ_σ's can be written in the basis (2.1) by repeatedly using Eq. (2.3). The problem of solving the theory thus becomes equivalent to finding the spectrum and the couplings for the infinite set of fields Φ_{σ_i}.

Applying Eq. (2.3) to the four-point functions, we find

$$\begin{aligned}W_{ijkl}&(x_1,x_2,x_3,x_4)\\&=\langle 0|\Phi_{\sigma_i}(x_1)\Phi_{\sigma_j}(x_2)\Phi_{\sigma_k}(x_3)\Phi_{\sigma_l}(x_4)|0\rangle\\&=\sum_m \int d^dx\; d^dy\; Q_{ijm}(x_1,x_2|x)\\&\quad\times W_{\sigma_m}(x-y)Q_{mkl}(y|x_3,x_4).\end{aligned} \tag{2.8}$$

We now switch to the Euclidean signature. The conformal partial-wave expansion of the four-point function in Euclidean region takes the form[5] $[\sigma=(\lambda,\vec{s})]$

$$G(x_1,x_2,x_3,x_4)=\sum_{\vec{s}}\int_C d\lambda\;\hat{G}_\sigma(x_1,x_2,x_3,x_4), \tag{2.9}$$

[4]For the Minkowskian signature, the representations under consideration are those of the infinite covering group of SO$(d,2)$, which are unitary and satisfy the spectrality condition $p_0>0$, $p^2>0$. For the Euclidean signature, they are the irreducible representations of SO$(d+1,1)$, which can be analytically continued from the Minkowskian counterparts.

[5]We denote the Euclidean correlators by G and omit some subscripts to simplify the notation.

where the integral is a contour integral in the complex plane of the conformal dimension λ (see below) and

$$\hat{G}_\sigma(x_1,x_2,x_3,x_4) = n(\sigma)\int d^d x\ G_{ij\tilde{\sigma}}(x_1,x_2,x)G_{kl\sigma}(x,x_3,x_4). \quad (2.10)$$

Here

$$G_{kl\sigma}(x,x_3,x_4)=\langle\Phi_\sigma(x)\Phi_k(x_3)\Phi_l(x_4)\rangle,$$

$$G_{ij\tilde{\sigma}}(x_1,x_2,x)=\langle\Phi_i(x_1)\Phi_j(x_2)\Phi_{\tilde{\sigma}}(x)\rangle$$

are the Euclidean three-point functions with $\tilde{\sigma}=(d-\lambda,\vec{s})$ and $n(\sigma)$ is some normalization constant. The field $\Phi_{\tilde{\sigma}}$ is called the conformal partner (or shadow operator) of Φ_σ, related to it by[6]

$$\Phi_{\tilde{\sigma}}(x)=\int d^d y\ \Delta_{\tilde{\sigma}}(x-y)\Phi_\sigma(y),$$

$$\Delta_{\tilde{\sigma}}(x-y)=\langle\Phi_{\tilde{\sigma}}(x)\Phi_{\tilde{\sigma}}(y)\rangle. \quad (2.11)$$

$\hat{G}_\sigma$ includes the contributions from both Φ_σ and its conformal partner $\Phi_{\tilde{\sigma}}$. The integration contour in Eq. (2.9) is chosen in order to select only the contribution from Φ_σ. One can decompose $\hat{G}_\sigma$ into the parts coming from Φ_σ and $\Phi_{\tilde{\sigma}}$:

$$\hat{G}_\sigma=G_\sigma+G_{\tilde{\sigma}}. \quad (2.12)$$

Since G_σ and $G_{\tilde{\sigma}}$ have different pole structure $(x_1-x_2\to 0)$,

$$G_{ij\sigma}\sim\frac{1}{|x_1-x_2|^{\lambda_i+\lambda_j-\lambda}},\quad G_{ij\tilde{\sigma}}\sim\frac{1}{|x_1-x_2|^{\lambda_i+\lambda_j+\lambda-d}},$$

the decomposition (2.12) is unique. Using Eq. (2.12) instead of the contour integral, we can rewrite Eq. (2.9) as

$$G(x_1,x_2,x_3,x_4)=\sum_{\sigma_i} G_{\sigma_i}(x_1,x_2,x_3,x_4). \quad (2.13)$$

In addition to Eqs. (2.8) and (2.9), there are two other ways to write partial-wave expansions: in terms of "*u* channel" and "*t* channel." The equivalence of the three channels is guaranteed by the associativity of the operator algebra (2.3). This is usually called the *crossing symmetry* of the four-point functions. As was already mentioned in the Introduction, the crossing symmetry of the CFT four-point functions should have interesting implications for the structure of the corresponding scattering amplitudes in AdS space.

[6]Note that there is no spectrality condition in Euclidean space so that $\Delta_{\tilde{\sigma}}$ and Δ_σ can be chosen to satisfy

$$\int d^d x\ \Delta_\sigma(x_1-x)\Delta_{\tilde{\sigma}}^{-1}(x-x_2)=\delta(x_1-x_2).$$

III. FOUR-POINT FUNCTIONS IN CFT-AdS CORRESPONDENCE: SCALAR EXCHANGE

In this section we consider the contribution to a four-point function of the exchange of a scalar field of an arbitrary mass. For definiteness, we shall study the "model" four-point functions of scalar fields and consider scattering in AdS space resulting from vertices of the type $\phi\phi_1\phi_2$ and $\phi\phi_3\phi_4$. The scattering amplitude is given by (see Appendix A for our notation)

$$S_\nu(x_1,x_2,x_3,x_4) = \int \frac{du_0 d^d u}{u_0^{d+1}}\frac{dv_0 d^d v}{v_0^{d+1}}\mathcal{K}_{\lambda_1}(u,x_1)\mathcal{K}_{\lambda_2}(u,x_2) \times G(u,v)\mathcal{K}_{\lambda_3}(v,x_3)\mathcal{K}_{\lambda_4}(v,x_4), \quad (3.1)$$

where $\mathcal{K}_\lambda(u,x)$ is the boundary propagator [6] corresponding to a conformal field with dimension λ,

$$\mathcal{K}_\lambda(u,x)=c_\lambda\left(\frac{u_0}{|u-x|^2}\right)^\lambda, \quad (3.2)$$

and $G(x,y)$ is the bulk propagator for a scalar field of mass m (see Appendix A) [10]:

$$G(x,y)=(x_0y_0)^{d/2}\int\frac{d^d k}{(2\pi)^d}e^{i\vec{k}\cdot(\vec{x}-\vec{y})}I_\nu(kx_0^<)K_\nu(kx_0^>). \quad (3.3)$$

Here I and K are the modified Bessel functions, the parameter ν is related to the mass by $\nu=\sqrt{m^2+\frac{1}{4}d^2}$, and $x_0^<(x_0^>)$ is the smaller (larger) number among x_0 and y_0. The amplitude (3.1) with Eq. (3.3) inserted is not manifestly conformally invariant. Its conformal invariance can be seen by using an alternative representation for $G(x,y)$ [24]:

$$G(x,y)=rz^{-\lambda}F\left(\lambda,\nu-\frac{1}{2};2\nu+1,z^{-1}\right), \quad (3.4)$$

where r is a normalization constant, F is a hypergeometric function, and $z=[(x_0+y_0)^2+(\vec{x}-\vec{y})^2]/4x_0y_0$. It is clear from Eq. (3.4) that $G(x,y)$ is invariant under the transformation $x\to x/|x|^2$, $y\to y/|y|^2$.

Let us first look at the "pseudopropagator," given by

$$\hat{G}(x,y)=(x_0y_0)^{d/2}\int\frac{d^d k}{(2\pi)^d}e^{i\vec{k}\cdot(\vec{x}-\vec{y})}K_\nu(kx_0)K_\nu(ky_0). \quad (3.5)$$

The value of Eq. (3.1) with Eq. (3.5) inserted instead of Eq. (3.3) will be denoted $\hat{S}$. After the substitution of boundary propagators and Eq. (3.5), it can be written as

$$\hat{S}_\nu(x_1,x_2,x_3,x_4)=\int\frac{d^d k}{(2\pi)^d}F_{12}^*(k;\vec{x}_1,\vec{x}_2)F_{34}(k;\vec{x}_3,\vec{x}_4), \quad (3.6)$$

where

$$F_{12}(k;\vec{x}_1,\vec{x}_2)=\int \frac{du_0 d^d u}{u_0^{d+1}}\, u_0^{d/2} e^{-i\vec{k}\cdot\vec{u}} \times K_\nu(ku_0)\mathcal{K}_{\lambda_1}(u,x_1)\mathcal{K}_{\lambda_2}(u,x_2). \quad (3.7)$$

The Fourier transformation for the boundary propagator with $\lambda=\nu+d/2$ is

$$\mathcal{K}_\lambda(u,x)=c_\lambda\left(\frac{u_0}{|u-x|^2}\right)^\lambda=\int \frac{d^d k}{(2\pi)^d}\, e^{i\vec{k}\cdot(\vec{u}-\vec{x})} f(k,u_0), \quad (3.8)$$

with

$$f(k,u_0)=\frac{1}{b_\lambda}\, u_0^{d/2} k^\nu K_\nu(ku_0), \quad b_\lambda=2^{\nu-1}\Gamma(\nu).$$

Thus

$$u_0^{d/2} K_\nu(ku_0)=b_\lambda k^{-\nu}\int d^d x\; e^{-i\vec{k}\cdot(\vec{x}-\vec{u})}\mathcal{K}_\lambda(u,x).$$

Plugging this into Eq. (3.7), we find

$$\begin{aligned} F_{12}(k;\vec{x}_1,\vec{x}_2) &= b_\lambda k^{-\nu}\int \frac{du_0 d^d u}{u_0^{d+1}} \int d^d x\; e^{-i\vec{k}\cdot\vec{x}} \\ &\quad\times \mathcal{K}_\lambda(u,x)\mathcal{K}_{\lambda_1}(u,x_1)\mathcal{K}_{\lambda_2}(u,x_2) \\ &= b_\lambda k^{-\nu}\int d^d x\; e^{-i\vec{k}\cdot\vec{x}} G_{\lambda\lambda_1\lambda_2}(x,x_1,x_2) \\ &= b_\lambda k^{-\nu} G_{\lambda\lambda_1\lambda_2}(k,x_1,x_2), \end{aligned}$$

where

$$G_{\lambda\lambda_1\lambda_2}(x,x_1,x_2)=\int \frac{du_0 d^d u}{u_0^{d+1}}\, \mathcal{K}_\lambda(u,x)\mathcal{K}_{\lambda_1}(u,x_1)\mathcal{K}_{\lambda_2}(u,x_2) \quad (3.9)$$

is the three-point function according to the CFT-AdS correspondence. Similarly, we find

$$F_{34}(k;\vec{x}_3,\vec{x}_4)=b_\lambda k^{-\nu} G_{\lambda\lambda_3\lambda_4}(k,x_3,x_4).$$

Then it follows from Eq. (3.6) that

$$\hat{S}_\nu=b_\lambda^2\int \frac{d^d k}{(2\pi)^d}\, G^*_{\lambda\lambda_1\lambda_2}(k,x_1,x_2) k^{-2\nu} G_{\lambda\lambda_3\lambda_4}(k,x_3,x_4). \quad (3.10)$$

From Eqs. (3.8) and (3.9) we can see that

$$G_{\tilde{\lambda}\lambda_1\lambda_2}(k,x_1,x_2)=\frac{b_\lambda}{b_{\tilde{\lambda}}}\, k^{-2\nu} G_{\lambda\lambda_1\lambda_2}(k,x_1,x_2). \quad (3.11)$$

Thus Eq. (3.10) can be written as

$$\hat{S}_\nu=b_\lambda b_{\tilde{\lambda}}\int \frac{d^d k}{(2\pi)^d}\, G^*_{\tilde{\lambda}\lambda_1\lambda_2}(k,x_1,x_2) G_{\lambda\lambda_3\lambda_4}(k,x_3,x_4),$$

and in coordinate space it becomes

$$\hat{S}_\nu=b_\lambda b_{\tilde{\lambda}}\int d^d x\, G_{\lambda_1\lambda_2\tilde{\lambda}}(x_1,x_2,x) G_{\lambda\lambda_3\lambda_4}(x,x_3,x_4). \quad (3.12)$$

We notice that the above expression (3.12) for $\hat{S}_\nu$ is precisely the same as the CFT expression (2.10) with $\sigma=(\lambda,0)$. Thus we have identified the amplitude $\hat{S}_\nu$ with the CFT correlator $\hat{G}_\sigma$ in Eq. (2.9).

Let us now look at the relation between S_ν, Eq. (3.1), and $\hat{S}_\nu$, Eq. (3.6). Using that

$$K_\nu=\frac{\pi}{2}\,\frac{1}{\sin\,\nu\pi}\,(I_{-\nu}-I_\nu)$$

and

$$\int_0^\infty dx_0\int_0^\infty dy_0=\int_0^\infty dx_0\int_0^{x_0} dy_0+\int_0^\infty dy_0\int_0^{y_0} dx_0,$$

it is easy to see that

$$\hat{S}_\nu=S_\nu+S_{-\nu}, \quad (3.13)$$

which can be understood as the sum of the contributions from the fields of dimensions λ and $d-\lambda$, respectively. Comparing Eq. (3.13) with Eq. (2.12) and assuming that S_ν has an analytic dependence on ν, we find that with $\hat{S}_\nu$ identified with $\hat{G}_\sigma$ in Eq. (2.10), S_ν and $S_{-\nu}$ are equal to G_σ and $G_{\tilde{\sigma}}$, demonstrating the required relation between the AdS amplitude S_ν and the CFT correlator.

It is easy to see that the above procedure applies without change to other types of three-point interactions (e.g., $\phi\partial\phi_1\partial\phi_2$) and to scattering of higher-spin fields involving scalar exchange. Thus we have established the correspondence between the exchange diagrams in AdS space and the conformal partial-wave expansion in CFT for the case of the *scalar* intermediate states.

We mention here that the contribution of a scalar operator to the CPWE [Eqs. (2.8), (2.9)] of a four-point function was evaluated a while ago in [19]. The expression can be written in a closed form in terms of double hypergeometric functions (see Appendix B). From the identification of Eqs. (3.1) and (2.8), we see that it can also be interpreted as the scattering amplitude (3.1) in AdS space.

IV. PHOTON AND GRAVITON PROPAGATORS IN AdS$_{d+1}$

In this section we shall consider the scattering amplitudes in AdS space involving massless vector and graviton exchanges. The photon and graviton propagators in covariant gauges were discussed before in [25]. The expressions found were complicated and do not seem to be useful in explicit calculations. Here we shall choose the noncovariant gauges, $A_0=0$ for the vector and $h_{0\mu}=0$ for the graviton. It turns out that the resulting AdS propagators are quite simple and have structure similar to that of their flat space counterparts.

A. Massless vector

Let us start with the case of a vector field in AdS space described by the action

$$I=\int d\alpha\left(\frac{1}{4}F_{\mu\nu}F^{\mu\nu}+A_\mu\mathcal{T}^\mu\right),\quad \int d\alpha=\int d^{d+1}x\sqrt{g_0}. \tag{4.1}$$

We fix the Coulomb gauge $A_0=0$. In this gauge the equation for A_i $(i=1,\dots,d)$ becomes

$$(\partial_0^2+\partial_j^2)A_i-\partial_j\partial_iA_j-\frac{d-3}{x_0}\partial_0A_i=-\frac{1}{x_0^2}\mathcal{T}_i. \tag{4.2}$$

The equation for A_0 gives the constraint

$$\partial_0\partial_iA_i=\frac{1}{x_0^2}\mathcal{T}_0. \tag{4.3}$$

We decompose A_i as

$$A_i=A_i^\perp+\partial_i\xi,$$

where $A_i^\perp$ is the transverse part and $\xi=(1/\partial^2)\partial_iA_i$, with $\partial^2=\partial_i\partial_i$. Then Eqs. (4.2) and (4.3) reduce to

$$\partial_0\xi=\frac{1}{x_0^2}\frac{1}{\partial^2}\mathcal{T}_0,\quad (\partial_0^2+\partial^2)A_i^\perp-\frac{d-3}{x_0}\partial_0A_i^\perp=-\frac{1}{x_0^2}\mathcal{T}_i^\perp,$$

where $\mathcal{T}_i^\perp$ is the transverse part of $\mathcal{T}_i$, i.e., $\partial_i\mathcal{T}_i^\perp=0$. Thus the equation for $x_0A_i^\perp$ reduces to that for a free scalar of $m^2=-(d-1)$ in AdS_{d+1} space with a source $x_0\mathcal{T}_i^\perp$. It can be easily solved to obtain

$$x_0A_i^\perp(x)=\int d\alpha\ G(x,y)y_0\mathcal{T}_i^\perp(y), \tag{4.4}$$

where $G(x,y)$ is the propagator for a free scalar with $m^2=-(d-1)$.

Then

$$I=\frac{1}{2}\int d\alpha\ A_i\mathcal{T}^i=\frac{1}{2}\int d\alpha(A_i^\perp\mathcal{T}^i+\partial_i\xi\mathcal{T}^i). \tag{4.5}$$

Note that the conservation of the current gives

$$D_\mu\mathcal{T}^\mu=0\Rightarrow\partial_i\mathcal{T}^i=-\frac{1}{\sqrt{g_0}}\partial_0(\sqrt{g_0}\mathcal{T}^0).$$

After a partial integration, use of the current conservation, and the substitution of Eq. (4.4) into Eq. (4.5), we get

$$I=\frac{1}{2}\int\frac{dx_0d\vec{x}}{x_0^{d+1}}\frac{dy_0d\vec{y}}{y_0^{d+1}}x_0y_0\mathcal{T}_i^\perp(x)G(x,y)\mathcal{T}_i^\perp(y)$$
$$-\frac{1}{2}\int\frac{dx_0d\vec{x}}{x_0^{d+1}}\mathcal{T}_0\frac{1}{\partial^2}\mathcal{T}_0. \tag{4.6}$$

This expression determines the photon propagator.[7]

B. Graviton

We consider the metric as a sum of the AdS metric and a graviton perturbation $g_{\mu\nu}=(1/x_0^2)(\delta_{\mu\nu}+h_{\mu\nu})$ and consider the action (see also [13])

$$I=\int d\alpha\left(R-2\Lambda+\frac{1}{2}h_{\mu\nu}T_\nu^\mu\right).$$

In the following we shall assume that there is no index raising for h, but for the energy-momentum tensor $T_{\mu\nu}$ the indices will be raised by $g_0^{\mu\nu}=x_0^2\delta^{\mu\nu}$. We shall consider the gauge $h_{0\mu}=0$. In this gauge, the linearized Einstein equations for h_{ij} become (see also [28])

$$\partial_0^2(h_{ij}-\delta_{ij}h)-\frac{d-1}{x_0}\partial_0(h_{ij}-\delta_{ij}h)$$
$$+\partial_k^2\tilde{h}_{ij}-\partial_l\partial_i\tilde{h}_{jl}-\partial_l\partial_j\tilde{h}_{il}+\delta_{ij}\partial_l\partial_m\tilde{h}_{lm}=-2T_{ij}, \tag{4.7}$$

where $h=\delta_{ij}h_{ij}$ and $\tilde{h}_{ij}=h_{ij}-\frac{1}{2}\delta_{ij}h$. There are also two constraints following from the 00 and $0i$ components of the Einstein equations:

$$\partial_0(\partial_jh_{ij}-\partial_ih)=2T_{0i}, \tag{4.8}$$

$$-\partial_i\partial_jh_{ij}+\partial_i^2h-\frac{d-1}{x_0}\partial_0h=2T_{00}. \tag{4.9}$$

We decompose h_{ij} as

$$h_{ij}=\overline{h_{ij}^\perp}+\partial_iB_j^\perp+\partial_jB_i^\perp+\partial_i\partial_j\eta+\frac{1}{d-1}\left(\delta_{ij}-\frac{\partial_i\partial_j}{\partial^2}\right)h',$$

with $\partial^2=\partial_i\partial_i$ and

$$B_i^\perp=\frac{1}{\partial^2}\partial_kh_{ik}-\frac{\partial_i}{(\partial^2)^2}\partial_k\partial_lh_{kl},$$

$$\eta=\frac{1}{(\partial^2)^2}\partial_k\partial_lh_{kl},\quad h'=h-\frac{1}{\partial^2}\partial_k\partial_lh_{kl}.$$

It is easy to check that

[7] In Eq. (4.6) there is also a boundary term $-\frac{1}{2}\int_{\partial M}d^dx\ x_0^{1-d}\xi\mathcal{T}_0$. In the noncovariant gauge we are choosing, this term is responsible for the Ward identity of three-point functions, but we do not expect it to contribute to higher-point functions.

$$\delta_{ij}\overline{h_{ij}^{\perp}}=0, \quad \partial_i\overline{h_{ij}^{\perp}}=0, \quad \partial_i B_i^{\perp}=0.$$

Then Eqs. (4.7) and (4.8) become

$$(\partial_0^2+\partial_k^2)\overline{h_{ij}^{\perp}}-\frac{d-1}{x_0}\,\partial_0\overline{h_{ij}^{\perp}}=-2t_{ij}, \quad t_{ij}\equiv P_{ijkl}T_{kl}, \tag{4.10}$$

$$\partial_0 B_i^{\perp}=\frac{1}{\partial^2}\,(2T_{0i}+\partial_0\partial_i h'), \quad \partial_0 h'=-\frac{2}{\partial^2}\,\partial_j T_{0j}, \tag{4.11}$$

$$\partial_0 h=-\frac{2x_0}{d-1}\,T_{00}+\frac{x_0}{d-1}\,\partial^2 h', \tag{4.12}$$

where P_{ijkl} is the transverse traceless projector in flat space.

Thus the equations for the transverse traceless part $\overline{h_{ij}^{\perp}}=P_{ijkl}h_{kl}$ reduce to that of a free massless scalar, so that we get $[d\beta=d^{d+1}y\sqrt{g_0(y)}]$

$$h_{ij}^{\perp}(x)=2\int d\beta\; y_0^2 G(x,y)t_{ij}(y). \tag{4.13}$$

Using the conservation of the energy-momentum tensor and Eqs. (4.11) and (4.12), we find that the quadratic (graviton propagator) part of the Einstein action takes the form

$$\begin{aligned} I=\frac{1}{2}\int d\alpha\; h_{ij}T_j^i&=\int d\alpha\; d\beta(x_0y_0)^2 t_{ij}(x)G(x,y)t_{ij}(y) \\ &-2\int d\alpha\; x_0^2 T_{0i}\,\frac{1}{\partial^2}\,T_{0i}-\frac{2}{d-1}\int d\alpha\; x_0^2\partial_i T_{0i}\,\frac{1}{\partial^2}\,T_{00} \\ &-\frac{d-2}{d-1}\int d\alpha\; x_0^2\partial_i T_{0i}\left(\frac{1}{\partial^2}\right)^2\partial_i T_{0i}, \end{aligned} \tag{4.14}$$

where we have ignored boundary terms resulting from partial integration.

V. SCALAR FOUR-POINT FUNCTIONS IN AdS_5 SUPERGRAVITY/$\mathcal{N}=4$ SYM THEORY

In this section we shall consider the four-point functions involving the scalar operators $\mathcal{O}=\mathrm{tr}\,F^2$ and $\mathcal{O}'=\mathrm{tr}\,FF^*$ in $\mathcal{N}=4$ SYM theory in the large-N and large 't Hooft coupling limit as predicted by the type-IIB supergravity in $\mathrm{AdS}_5\times\mathrm{S}^5$. Here $\mathcal{O}$ and $\mathcal{O}'$ correspond to the massless fields ϕ and C in AdS_5 coming from the dilaton and axion of type-IIB supergravity on $\mathrm{AdS}_5\times\mathrm{S}^5$. The four-point functions of interest are $\langle\mathcal{O}(x_1)\mathcal{O}(x_2)\mathcal{O}(x_3)\mathcal{O}(x_4)\rangle$, $\langle\mathcal{O}'(x_1)\mathcal{O}'(x_2)\mathcal{O}(x_3)\mathcal{O}(x_4)\rangle$, and $\langle\mathcal{O}'(x_1)\mathcal{O}'(x_2)\mathcal{O}'(x_3)\mathcal{O}'(x_4)\rangle$. They correspond in AdS_5 to the tree-level scattering amplitudes of four dilatons, two dilatons and two axions, and four axions, respectively. Thus we need to look for vertices in the Lagrangian of type-IIB supergravity on $\mathrm{AdS}_5\times\mathrm{S}^5$ involving two dilatons (axions) and one other field or quartic vertices involving only dilatons and axions. For this purpose it is sufficient to concentrate on the graviton-dilaton-axion sector of the supergravity[8]

$$\hat{I}=\frac{1}{2\kappa_{10}^2}\int_{\mathrm{AdS}_5}d^5x\int_{\mathrm{S}^5}d^5y\sqrt{-\hat{g}}\times\left[\hat{R}-\frac{1}{2}\,(\partial\hat{\phi})^2-\frac{1}{2}\,e^{2\hat{\phi}}(\partial\hat{C})^2\right]. \tag{5.1}$$

Since the fields ϕ and C are associated with the zeroth spherical harmonics of $\hat{\phi}$ and $\hat{C}$ on S^5, it is clear that the desired vertices cannot involve fields from higher spherical harmonics. Thus the fields in Eq. (5.1) can be assumed to have only x dependence, i.e.,

$$\hat{\phi}(x,y)=\phi(x), \quad \hat{C}(x,y)=C(x),$$
$$\hat{h}_{\hat{\mu}\hat{\nu}}(x,y)=\{h_{\mu\nu}(x),h_{\mu\alpha}(x),h_{\alpha\beta}(x)\}.$$

After dimensional reduction to AdS_5 and a Weyl scaling,[9] the relevant part of Eq. (5.1) becomes

$$I=\frac{1}{2\kappa^2}\int_{\mathrm{AdS}_5}d^5x\sqrt{-g'}\left[R-\frac{1}{2}\,(\partial\phi)^2-\frac{1}{2}\,e^{2\phi}(\partial C)^2\right], \tag{5.2}$$

where $R=R(h')$ is the 5D Ricci scalar. The $\phi-C$ sector of the type-IIB supergravity on $\mathrm{AdS}_5\times\mathrm{S}^5$ is thus very simple. The only cubic vertex involving two ϕ is $\partial\phi\;\partial\phi\;h'$. In particular, there is no $h_\alpha^\alpha\phi\phi$ vertex, where h_α^α is the trace of the internal (S^5) part of the metric (massive fixed scalar) which in $\mathcal{N}=4$ SYM theory corresponds to the operator of the structure $\mathcal{O}_8=\mathrm{tr}\,F^4-\frac{1}{4}\,\mathrm{tr}(F^2)^2$. Since in the free Maxwell theory there is a nonvanishing three-point function $\langle\mathcal{O}\mathcal{O}\mathcal{O}_8\rangle$ [29], this three-point function must have nontrivial dependence on the t Hooft coupling (cf. [17]).

Following the same reasoning, we can see that the diagrams contributing to the four-graviton scattering may come only from the R term in Eq. (5.2). In particular, there is no cubic vertex involving two gravitons and other fields. This is consistent with the expectation [30] that the OPE of the energy momentum tensor in $\mathcal{N}=4$ SYM theory closes on itself.[10]

Starting with Eq. (5.2), we can write down the scattering diagrams in AdS_5 contributing to the four-point functions under consideration (see Figs. 2 and 3). Let us first look at the diagrams which do not involve graviton exchange. The scattering amplitude with the dilaton or axion exchange, e.g., $\phi(1)C(2)\phi(3)C(4)$ in Fig. 3, can be written as

[8]Our notation will be as follows. All fields in $D=10$ will be written with carets and their indices. $\mu,\nu,\ldots$ will refer to the indices of AdS_5, while $\alpha,\beta,\ldots$ will refer to those of S^5.

[9]The resulting graviton $h'_{\mu\nu}$ differs from $h_{\mu\nu}$ by a Weyl scaling, i.e., $h'_{\mu\nu}=h_{\mu\nu}+\frac{1}{3}g_{0\mu\nu}h_\alpha^\alpha$, where $g_{0\mu\nu}$ is the background metric in AdS_5.

[10]This is not so in most four-dimensional theories. For a discussion of the case of $\mathcal{N}=1$ supersymmetric theories, see [31].

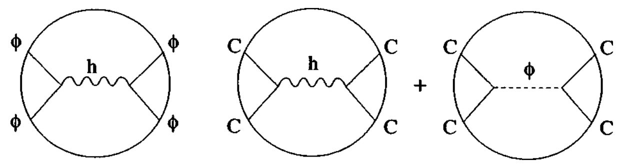

FIG. 2. Scattering diagrams for $\phi\phi\phi\phi$ and $CCCC$. Only s-channel diagrams are displayed here.

$$A_1=\int \frac{du_0 d^d u}{u_0^{d+1}}\frac{dv_0 d^d v}{v_0^{d+1}}(u_0 v_0)^2 \partial_\mu \mathcal{K}_d(u,x_1)\partial_\mu \mathcal{K}_d(u,x_2) \times G(u,v)\partial_\nu \mathcal{K}_d(v,x_3)\partial_\nu \mathcal{K}_d(v,x_4). \tag{5.3}$$

The contact $\phi C \phi C$ interaction gives

$$A_2=\int \frac{du_0 d^d u}{u_0^{d+1}} u_0^2 \mathcal{K}_d(u,x_1)\partial_\nu \mathcal{K}_d(u,x_2) \times \mathcal{K}_d(u,x_3)\partial_\nu \mathcal{K}_d(u,x_4). \tag{5.4}$$

Here $\mathcal{K}_d(d=4)$ and $G(u,v)$ [see Eqs. (3.2) and (3.3)] are the boundary and bulk propagators for the massless scalars. Since these propagators satisfy

$$\frac{1}{\sqrt{g_0}}\partial_\mu(\sqrt{g_0}g_0^{\mu\nu}\partial_\nu)\mathcal{K}_d=0,$$

$$\frac{1}{\sqrt{g_0}}\partial_\mu(\sqrt{g_0}g_0^{\mu\nu}\partial_\nu)G(u,v)=-\frac{1}{\sqrt{g_0}}\delta(u-v),$$

we find by a partial integration[11] that $A_1(1,2,3,4)$ reduces to $-\frac{1}{2}A_2(1,2,3,4)$.

Direct evaluation of A_2 is quite tedious. It is easy to show, however, that the symmetric in $x_1,\ldots,x_4$ part of Eq. (5.4), $(A_2)_{\rm sym}$ [and thus $(A_1)_{\rm sym}$] vanishes. One way to see that is to make a field redefinition $\phi=-\log(1+\psi)$. Then the free action for ϕ becomes

$$I=\int d^{d+1}x\sqrt{g_0}(1+\psi)^{-2}(\partial\psi)^2. \tag{5.5}$$

The new action generates cubic and quartic dilaton vertices, and it can be seen that the resulting four-dilaton scattering amplitude is proportional to $(A_2)_{\rm sym}$. Since the field redefinition changes the correlators only by unimportant contact terms, we conclude that $(A_2)_{\rm sym}$ should contain only contact terms, i.e., should vanish for separated points.[12]

The fact that $(A_2)_{\rm sym}=-2(A_1)_{\rm sym}=0$ implies, in particular, that as in the case of the $\phi\phi\phi\phi$ amplitude, the only nontrivial contribution to the $CCCC$ amplitude comes from the sum of the s,t,u graviton exchanges, so that the $\phi\phi\phi\phi$ and $CCCC$ amplitudes are actually equal.[13] The structure of the $\phi C\phi C$ amplitude is more complicated as in addition to the graviton exchange it contains also the sum of the axion exchange and the contact contributions given by a combination of A_2 amplitudes with different orders of end points (see also [32]).

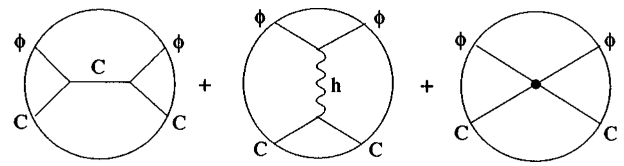

FIG. 3. Scattering diagrams for $\phi\phi CC$.

The formal integral expression for the graviton contribution to the four-point functions can be written down by substituting into Eq. (4.14) the appropriate expression for the energy-momentum tensor corresponding to the massless scalar fields:

$$T_{\mu\nu}=\partial_\mu \mathcal{K}_d(u,x_1)\partial_\nu \mathcal{K}_d(u,x_2) -\frac{1}{2}\delta_{\mu\nu}\partial_\lambda \mathcal{K}_d(u,x_1)\partial_\lambda \mathcal{K}_d(u,x_2).$$

Here $\mathcal{K}_d$ is again the boundary propagator (with $d=4$). The resulting expression is quite long and will not be explicitly presented here. We leave its detailed analysis for future work.

To conclude, let us make some speculative remarks on the consequences of the possible crossing symmetry of CFT four-point functions for the structure of scattering amplitudes in AdS space. From the correspondence between the scattering amplitudes in AdS space and correlators in CFT given by the CPWE's, we would expect that the amplitudes in Figs. 2 and 3 should possess s-t-u symmetry. That would imply, for example, that the s, t, and u channel graviton exchanges in the $\phi\phi\phi\phi$ amplitude are all equal. Then in the $CCCC$ amplitude (which does not contain contact contribution) the s,t,u dilaton exchanges would also need to be equal. Since their sum $(A_1)_{\rm sym}$ vanishes, each of them should also be vanishing, at least up to some singular contributions (cf.

[11]Boundary terms resulting from partial integration give only unimportant contact terms (terms containing a Dirac delta function and its derivatives).

[12]In the original version of this paper, we were assuming that this argument demonstrates the triviality of A_2, but it actually applies only to $(A_2)_{\rm sym}$ as the four legs here correspond to the *same* field [while $\psi^2(\partial\psi)^2$ reduces after integration by parts to $-\frac{1}{3}\psi^3\partial^2\psi$, which vanishes on shell, there is no similar rearrangement for $\phi^2(\partial C)^2$]. We noticed this mistake after reading the recent paper in [32], with the conclusions of which we now agree.

[13]This may be viewed as a consequence of the SL$(2,R)$ symmetry of the action (5.2).

[32]) that would need to be subtracted as part of a definition of the CFT-AdS correspondence.[14] To be able to draw any conclusions from the assumption of crossing symmetry in the case of the $\phi C \phi C$ amplitude (Fig. 3), one needs first to interpret the contact bulk diagram contributing to it. For example, we may replace it by a massless scalar exchange in the t channel using that, as shown above, $A_2(1,2,3,4) = -2A_2(1,3,2,4)$.[15] Given that in the $\phi C \phi C$ case the graviton exchange contributes only to the t channel, one would be led to the conclusion that the scalar and graviton exchange amplitudes should be proportional to each other (up to possible subtractions mentioned above). That would, in turn, imply that all these massless scalar four-point amplitudes should be vanishing. Given these somewhat surprising conclusions, it remains to be seen if crossing symmetry can actually be realized in a four-dimensional CFT.

Note added in proof. After this paper was completed, it became clear that the assumption made in the paragraph below Eq. (3.13) about analytic dependence of S_ν on ν is not valid. Thus the argument presented in Sec. III only leads to the conclusion that $S_\nu + S_{-\nu} = G_\sigma + G_{\tilde{\sigma}}$. For the same reason, the identity presented in Fig. 1 in Sec. I is only correct up to terms that are symmetric under $\nu \leftrightarrow -\nu$.

ACKNOWLEDGMENTS

We acknowledge the support of PPARC and of European Commission TMR program grant ERBFMRX-CT96-0045. We are grateful to S. Ferrara, D. Freedman, I. Klebanov, R. Metsaev, and A. Ritz for stimulating discussions.

APPENDIX A: SCALAR PROPAGATOR IN AdS$_{d+1}$

As in [6], we consider the anti–de Sitter space of dimension $D = d+1$ with Euclidean signature and the (half-space) metric

$$ds^2 = g_{0\mu\nu} dx^\mu dx^\nu = \frac{1}{x_0^2}\,(dx_0^2 + dx_i^2). \tag{A1}$$

The AdS$_{d+1}$ bulk indices will be denoted by $\mu,\nu,\ldots$ and will take values $0,1,\ldots,d$. We shall use the notation $x = (x_0,\vec{x})$, $\vec{x} = (x_i)$, $i = 1,\ldots,d$.

Considering the Euclidean action for a massive scalar,

$$I = \frac{1}{2}\int d^{d+1}x \sqrt{g_0}[(\partial_\mu\phi)^2 + m^2\phi^2],$$

we are to solve

$$(D^2 - m^2)G(x,y) = -\frac{1}{\sqrt{g_0}}\,\delta(x-y).$$

More explicitly, in the coordinate system (A1), this equation becomes

$$x_0^n \partial_\mu [x_0^{-n+2}\partial_\mu G(x,y)] - m^2 G(x,y) = -\delta(\vec{x}-\vec{y})\,\delta(x_0-y_0) y_0^n.$$

Let $G(x,y) = x_0^{d/2} H(x,y)$. Then

$$(\Delta_\nu + \partial_i^2) H(x,y) = -y_0^{(d-2)/2}\delta(\vec{x}-\vec{y})\,\delta(x_0-y_0), \tag{A2}$$

where $\nu^2 = m^2 + \frac{1}{4}d^2$ and the operator Δ_ν, defined by

$$\Delta_\nu = \partial_0^2 + \frac{1}{x_0}\,\partial_0 - \frac{\nu^2}{x_0^2},$$

has Bessel functions as its eigenvalues:

$$\Delta_\nu J_\nu(wx_0) = -w^2 J_\nu(wx_0).$$

The delta functions can be written in terms of orthonormal functions:

$$\delta(x_0-y_0) = y_0\int_0^\infty dw\; wJ_\nu(wx_0)J_\nu(wy_0),$$

$$\int \frac{d^dk}{(2\pi)^d}\, e^{i\vec{k}\cdot(\vec{x}-\vec{y})} = \delta(\vec{x}-\vec{y}).$$

We expand the Green function in a similar fashion:

$$H(x,y) = \int \frac{d^dk}{(2\pi)^d}\, e^{i\vec{k}\cdot\vec{x}} \int_0^\infty dw\; wJ_\nu(wx_0)\tilde{H}(w,k;y).$$

Then substitution into Eq. (A2) leads to

$$\tilde{H}(w,k;y) = y_0^{d/2}\,\frac{J_\nu(wy_0)}{w^2+k^2}\, e^{-i\vec{k}\cdot\vec{y}}.$$

The scalar Green function in the anti–de Sitter space (A1) is thus given by

$$G(x,y) = (x_0y_0)^{d/2}\int \frac{d^dk}{(2\pi)^d}\int_0^\infty dw\; w\,\frac{1}{w^2+k^2}$$

$$e^{i\vec{k}\cdot(\vec{x}-\vec{y})} J_\nu(wx_0)J_\nu(wy_0)$$

$$= (x_0y_0)^{d/2}\int \frac{d^dk}{(2\pi)^d}\, e^{i\vec{k}\cdot(\vec{x}-\vec{y})} \times I_\nu(kx_0^<)K_\nu(kx_0^>), \tag{A3}$$

where $x_0^<$ ($x_0^>$) is the smaller (larger) number among x_0 and y_0.

[14] Note that in Euclidean space the massless scalar exchange diagrams are generated by a contact-type (δ-function) vertex $\langle \mathcal{O}(x)\mathcal{O}'(y)\mathcal{O}'(z)\rangle \sim [\delta(x-y)+\delta(x-z)][1/(|y-z|^8)] - \delta(y-z)[1/(|x-z|^8)]$ represented in the bulk by the $\phi\partial C\partial C$ interaction (its noncontact part vanishes since upon integration by parts it reduces to $\frac{1}{2}C^2\partial^2\phi - \phi C\partial^2 C$). However, the corresponding Minkowskian (Wightman) three-point function is zero and thus should not contribute to the CPWE of the four-point Wightman function in CFT.

[15] The massless scalar here may be identified with dilaton, as there is a nonvanishing contact-type Euclidean three-point function in SYM theory: $\langle \mathcal{O}(x)\mathcal{O}(y)\mathcal{O}(z)\rangle \sim \delta(x-y)[1/(|y-z|^8)] + \delta(x-z)[1/(|y-x|^8)] + \delta(y-z)[1/(|x-z|^8)]$. This is also suggested by the field redefinition argument in Eq. (5.5).

APPENDIX B: SCALAR CONTRIBUTION TO FOUR-POINT FUNCTIONS

The contribution of a scalar operator Φ of dimension λ to the CPWE of a four-point function of scalar operators $\langle 0|\Phi_1(x_1)\Phi_2(x_2)\Phi_3(x_3)\Phi_4(x_4)|0\rangle$ was worked out a while ago in [19]. The same expression should also correspond to the scattering amplitude in AdS space, Eq. (3.1). Here we shall quote the result of [19]. We will use the following definitions: $\Sigma_{ij}=\lambda_i+\lambda_j$, $\Delta_{ij}=\lambda_i-\lambda_j$, $x_{ij}=|x_i-x_j|$, and

$$\rho=\frac{x_{14}^2x_{23}^2}{x_{12}^2x_{34}^2},\qquad \eta=\frac{x_{13}^2x_{24}^2}{x_{12}^2x_{34}^2},$$

where λ_i, $i=1,2,3,4$, are the conformal dimensions of Φ_i. Then

$$\langle 0|\Phi_1(x_1)\Phi_2(x_2)\Phi_3(x_3)\Phi_4(x_4)|0\rangle=x_{12}^{\Sigma_{12}-\Delta_{12}}x_{13}^{-\Delta_{12}-\Delta_{34}}x_{14}^{\Delta_{34}-\Delta_{12}}x_{34}^{\Delta_{12}-\Sigma_{34}}f_0(\rho,\eta),$$

with

$$f_0(\rho,\eta)=c_1\eta^{-(\lambda-\Delta_{12})/2}\Bigg[F_4\left(\frac{1}{2}(\lambda+\Delta_{34}),\ \frac{1}{2}(\lambda-\Delta_{12}),\lambda+1-\frac{1}{2}d,\ \frac{1}{2}(\Delta_{34}-\Delta_{12})+1;\ \frac{1}{\eta},\ \frac{\rho}{\eta}\right)$$
$$+c_2\left(\frac{\rho}{\eta}\right)^{-(\lambda-\Delta_{12})/2}F_4\left(\frac{1}{2}(\lambda-\Delta_{34}),\ \frac{1}{2}(\lambda-\Delta_{12}),\lambda+1-\frac{1}{2}d,1-\frac{1}{2}(\Delta_{34}+\Delta_{12});\ \frac{1}{\rho},\ \frac{\eta}{\rho}\right)\Bigg].$$

When $\Delta_{12}=\Delta_{34}=0$, this simplifies into

$$f_0(\rho,\eta)=c_1'\eta^{-\lambda/2}\Bigg[F_4\left(\frac{1}{2}\lambda,\ \frac{1}{2}\lambda,\lambda+1-\frac{1}{2}d,1;\ \frac{1}{\eta},\ \frac{\rho}{\eta}\right)+c_2'\left(\frac{\rho}{\eta}\right)^{-\lambda/2}F_4\left(\frac{1}{2}\lambda,\ \frac{1}{2}\lambda,\lambda+1-\frac{1}{2}d,1;\ \frac{1}{\rho},\ \frac{\eta}{\rho}\right)\Bigg].$$

In the above, c_1,c_2,c_1',c_2' are some numerical constants and F_4 is a double hypergeometric function.

[1] G. 't Hooft, Nucl. Phys. **B72**, 461 (1974).
[2] A. M. Polyakov, Nucl. Phys. B (Proc. Suppl.) **68**, 1 (1998); Les Houches Summer School, 1992, hep-th/9304146, p. 783.
[3] J. Maldacena, ''The large *N* limit of superconformal field theories and supergravity,'' hep-th/9711200.
[4] I. R. Klebanov, Nucl. Phys. **B496**, 231 (1997); S. S. Gubser, I. R. Klebanov, and A. A. Tseytlin, *ibid.* **B499**, 41 (1997); S. S. Gubser and I. R. Klebanov, Phys. Lett. B **413**, 41 (1997).
[5] S. S. Gubser, I. R. Klebanov, and A. M. Polyakov, Phys. Lett. B **428**, 105 (1998).
[6] E. Witten, ''Anti de Sitter space and holography,'' hep-th/9802150.
[7] G. Horowitz and H. Ooguri, Phys. Rev. Lett. **80**, 4116 (1998).
[8] S. Ferrara, C. Fronsdal, and A. Zaffaroni, ''On $N=8$ Supergravity on AdS_5 and $N=4$ Superconformal Yang-Mills theory,'' hep-th/9802203.
[9] I. Ya. Aref'eva and I. V. Volovich, ''On large *N* conformal theories, field theories in Anti de Sitter space and singletons,'' hep-th/9803028; Phys. Lett. B **433**, 49 (1998).
[10] W. Mück and K. S. Viswanathan, Phys. Rev. D **58**, 041901 (1998).
[11] M. Henningson and K. Sfetsos, Phys. Lett. B **431**, 63 (1998).
[12] D. Z. Freedman, S. D. Mathur, A. Matusis, and L. Rastelli, ''Correlation functions in the CFT_d/AdS_{d+1} correspondence,'' hep-th/9804058.
[13] H. Liu and A. A. Tseytlin, ''$D=4$ Super-Yang-Mills, $D=5$ Gauged Supergravity, and $D=4$ Conformal Supergravity,'' hep-th/9804083.
[14] G. Chalmers, H. Nastase, K. Schalm, and R. Siebelink, ''*R*-Current Correlators in $\mathcal{N}=4$ SYM from AdS,'' hep-th/9805105.
[15] W. Mück and K. S. Viswanathan, Phys. Rev. D **58**, 106006 (1998).
[16] A. M. Ghezelbash, K. Kaviani, S. Parvizi, and A. H. Fatollahi, ''Interacting Spinors-Scalars and the AdS/CFT Correspondence,'' hep-th/9805162.
[17] S. Lee, S. Minwalla, M. Rangamani, and N. Seiberg, ''Three-Point Functions of Chiral Operators in $D=4$, $\mathcal{N}=4$ SYM at Large *N*,'' hep-th/9806074.
[18] D. Z. Freedman, S. D. Mathur, A. Matusis, and L. Rastelli, presented at Strings 98, Santa Barbara, 1998.
[19] S. Ferrara, R. Gatto, A. F. Grillo, and G. Parisi, Nucl. Phys. **B49**, 77 (1972); Nuovo Cimento **26**, 226 (1975).
[20] S. Ferrara, R. Gatto, A. F. Grillo, and G. Parisi, Lett. Nuovo Cimento **4**, 115 (1972); **5**, 147 (1972); A. M. Polyakov, Sov. Phys. JETP **39**, 10 (1974); M. Ya. Palchik, Phys. Lett. **66B**, 259 (1977); G. Mack, Commun. Math. Phys. **53**, 155 (1977).
[21] E. S. Fradkin and M. Ya. Palchik, *Conformal Quantum Field Theory in D dimensions* (Kluwer, Dordrecht, 1996).
[22] E. S. Fradkin and M. Ya. Palchik, Phys. Rep., Phys. Lett. **44C**, 249 (1978).
[23] I. T. Todorov, M. C. Mintchev, and V. B. Petkova, *Conformal Invariance in Quantum Field Theory* (Scuola Normale Superiore, Pisa, 1978).
[24] B. Allen and T. Jacobson, Commun. Math. Phys. **103**, 669 (1986).

[25] B. Allen, Phys. Rev. D **34**, 3670 (1986); B. Allen and M. Turyn, Nucl. Phys. **B292**, 813 (1987).
[26] T. Banks and M. B. Green, J. High Energy Phys. **05**, 002 (1998); M. Bianchi, M. B. Green, S. Kovacs, and G. Rossi, ''Instantons in supersymmetric Yang-Mills and *D*-instantons in IIB superstring theory,'' hep-th/9807033.
[27] S. Ferrara and A. Zaffaroni, ''Bulk gauge fields in AdS supergravity and supersingletons,'' hep-th/9807090.
[28] G. E. Arutyunov and S. Frolov, ''On the origin of supergravity boundary terms in the AdS/CFT correspondence,'' hep-th/9806216.
[29] S. S. Gubser, A. Hashimoto, I. R. Klebanov, and M. Krasnitz, Nucl. Phys. **B526**, 393 (1998); S. S. Gubser and A. Hashimoto, ''Exact absorption probabilities for the D3-brane,'' hep-th/9805140.
[30] P. S. Howe and P. C. West, Phys. Lett. B **389**, 273 (1996).
[31] D. Anselmi, M. Grisaru, and A. Johansen, Nucl. Phys. **B491**, 221 (1997).
[32] D. Z. Freedman, S. D. Mathur, A. Matusis, and L. Rastelli, ''Comments on 4-point functions in the CFT/AdS correspondence,'' hep-th/9808006.