Compactification for a three-brane universe

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A fully realistic and systematic effective field theory model of a 3-brane universe is constructed. It consists of a six-dimensional gravitating spacetime, containing several, approximately parallel $(3+1)$-dimensional defects, or ''3-branes.'' The standard model particles are confined to live on one of the 3-branes while different four-dimensional field theories may inhabit the others, in literally a case of ''parallel universes.'' The effective field theory is valid up to the six-dimensional Planck scale, where it must be replaced by a more fundamental theory of gravity and 3-brane structure. Each 3-brane induces a conical geometry in the two dimensions transverse to it. Collectively, the curvature induced by the 3-branes can compactify the extra dimensions into a space of spherical topology. It is possible to take the six-dimensional Planck scale to be not much larger than the weak scale, and the compact space not much smaller than a millimeter, thereby realizing the recent proposal by Arkani-Hamed, Dimopoulos and Dvali for eliminating the gauge hierearchy problem. In this case, an extra force is required to stabilize the compact space against collapse. This is provided by a six-dimensional (compact) U(1) gauge field with a magnetic flux quantum trapped in the compact space. The nature of the cosmological constant problem in this scenario is discussed. [S0556-2821(99)00408-7]

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I. INTRODUCTION

It is usually assumed that the fundamental dynamical scale underlying gravity is the Planck scale,

$$M_{Pl}\sim 10^{18}\ \text{GeV}, \tag{1}$$

set by the observed value of Newton's constant. If so, one is faced with the problem of understanding the mechanism which stabilizes the very large hierarchy between this scale and the electroweak scale, $v=246\ \text{GeV}$. Recently however, it has been proposed [1] that the dynamical scale of gravity, M, is not much larger than the weak scale, thereby eliminating the usual hierarchy problem. This is accomplished by taking general relativity to be fundamentally *six*-dimensional, with two large compact dimensions, and identifying M with the six-dimensional Planck mass. Using the standard relation [1] (also see Sec. VI within)

$$M_{Pl}^2\sim \mathcal{A}M^4, \tag{2}$$

where $\mathcal{A}$ is the area of the compact two-dimensional space, one finds that if M is not much larger than the weak scale, then the typical (linear) dimension of the compact space is not much smaller than a millimeter. That is, the compactification mass scale is not much larger than $10^{-4}\ \text{eV}$. Reference [1] proposed that the reason we do not experimentally observe finely spaced Kaluza-Klein excitations of the standard model (SM) particles is because the entire SM is confined to a $(3+1)$-dimensional defect, which we will refer to as a ''3-brane,'' which is point-like in the two-dimensional compact space and extended in the non-compact directions. On the other hand, gravity is not confined in this manner and light Kaluza-Klein excitations of the graviton are present. The net result is that gravity is effectively described by four-dimensional general relativity at distances larger than a millimeter, but at shorter distances the Kaluza-Klein excitations propagate and gravity reveals its six-dimensional nature. To date, gravity has only been tested down to a distance of a centimeter with no sign of extra dimensions, but if we are lucky the six-dimensional transition may appear in upcoming sub-millimeter tests of gravity [1,2].

There is an ongoing effort to theoretically realize a phenomenologically acceptable version of the above scenario, either within quantum field theory or within string theory [1,3–5]. Related ideas involving 3-brane universes and/or relatively low compactification mass scales appear in Refs. [6–15] The purpose of the present paper is to construct a realistic model of a 3-brane universe using the effective field theory methods developed in Ref. [16], focussing on the compactification mechanism. This approach is analogous to the chiral Lagrangian approach to the soft pion sector of the strong interactions. Just as the chiral lagrangian describes the most general structure of the low-energy interactions among (pseudo) Nambu-Goldstone bosons, without explicitly describing the mechanism that gave rise to the associated spontaneous symmetry-breaking, the effective field theory we use here will describe the general structure of low-energy interactions among the 3-brane fluctuations, six-dimensional gravity and the SM fields, without explictly describing the mechanism that gave rise to the 3-brane and the fields living on it. Just as the chiral Lagrangian description is valid up to energies at which the detailed QCD mechanism for chiral symmetry-breaking becomes important, the effective theory we will use is valid up to energies of order M, at which point the internal structure of the 3-brane and the physics of strongly-coupled gravity become important.

The basic model proposed in this paper consists of a six-dimensional gravitating spacetime, containing several, approximately parallel 3-branes. They act as sources for the curvature needed to compactify the extra dimensions into a

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space of spherical topology. Each of the 3-branes may be inhabited by a separate four-dimensional quantum field theory, one of which is the familiar SM. These parallel, four-dimensional sub-universes interact weakly with each other via the bulk six-dimensional gravity, so that they can be considered as hidden sectors relative to each other. This is qualitatively similar to the ideas put forth in Refs. [10].

This paper is organized so as to progressively build up to a fully realistic model. Section II briefly reviews the necessary effective field theory formalism detailed in Ref. [16]. Section III describes the effective field theory that results from integrating out the massive SM physics. Section IV deals with the case of a single 3-brane in six fully non-compact dimensions. The classical equations of motion reveal that the 3-brane induces a conical geometry in the two transverse dimensions. In Sec. V, the cones from several 3-branes are patched together to fully compactify the extra dimensions. In Sec. VI, the effective field theory below the compactification mass scale is derived. It is pointed out that when the VEV of the effective field corresponding to the size of the compact space is large, this field can mediate effects in conflict with experimental post-Newtonian gravitational tests. This problem is resolved in Sec. VII by introducing a six-dimensional Abelian gauge field with a non-trivial magnetic flux through the compact space. Section VIII discusses the nature of the cosmological constant problem in the present scenario. Section IX contains the final discussion.

II. THE STANDARD MODEL ON A 3-BRANE

This section summarizes some of the key aspects of the 3-brane effective field theory formalism described in Ref. [16].

Our starting point will be the action governing the SM fields on a 3-brane, which in turn is coupled to six-dimensional ''bulk'' gravity [16]:

$$S=S_{3\text{-}brane}+S_{bulk}, \tag{3}$$

$$\begin{aligned}S_{3\text{-}brane}=\int d^4x\sqrt{-g}\Big\{&-f_0^4-g^{\mu\nu}D_\mu H^* D_\nu H-V(H,H^*)\\&-\frac{1}{4}g^{\mu\rho}g^{\nu\sigma}F_{\mu\nu}F_{\rho\sigma}+\overline{\psi}_L i e^\mu_\alpha \sigma^\alpha D_\mu \psi_L+yH\psi_L\psi_L\\&+\text{H.c.}+\cdots\Big\},\end{aligned} \tag{4}$$

$$S_{bulk}=\int d^6X\sqrt{-G}\Big\{-\Lambda_0-2M_0^4R-\frac{1}{4}\mathcal{F}^{MN}\mathcal{F}_{MN}+\cdots\Big\}. \tag{5}$$

The SM scalar, chiral spinor and vector fields are denoted H,ψ_L,A_μ respectively, the last of these being used to form the covariant derivatives and the gauge field strength, $F_{\mu\nu}$. (Gauge and flavor indices have not been explicitly written.) The SM fields are functions of intrinsic coordinates on the 3-brane, $x^{\mu=0,\ldots,3}$. The gravitational field is the six-dimensional metric, G_{MN}, used to construct the six-dimensional curvature scalar, R, and is a function of coordinates for the bulk spacetime, $X^{M=0,\ldots,5}$. A six-dimensional compact $U(1)$ gauge field is also included, with field strength $\mathcal{F}_{MN}(X)$. It will not play an important role until Sec. VII. The 3-brane embedding in the bulk spacetime is given by fields, $Y^M(x)$. The SM fields ''feel'' a four-dimensional metric on the 3-brane induced by this embedding, given by

$$g_{\mu\nu}(x)=G_{MN}(Y(x))\partial_\mu Y^M\partial_\nu Y^N. \tag{6}$$

(The case of chiral fermions is more subtle, involving an induced vierbein, e^α_μ. It is given careful treatment in Ref. [16], but we will not need the details here.) The dimensionful constants, f_0^4, Λ_0 and M_0, are the ''bare'' 3-brane tension, six-dimensional cosmological constant and six-dimensional Planck mass respectively. In this paper we will consider the case where $f_0\sim\mathcal{O}(M_0)$.

The terms in Eqs. (4) and (5) are the lowest dimension operators which are invariant under both general X-coordinate transformations and x-coordinate transformations, the ellipses containing higher-dimension invariants suppressed by powers of M_0. The resulting theory, written in terms of canonical fields, is necessarily non-renormalizable and must be treated by the methods of effective field theory, the effective theory being valid up to energies of order M_0. This scale constrains both the allowed energy-momenta in physical processes, and also the size of metric fluctuations away from six-dimensional Minkowski space and 3-brane fluctuations away from a flat four-dimensional hypersurface. Physics at higher energies can only be understood within a more fundamental theory, describing the internal structure of the 3-brane and strongly-coupled gravity. The fact that SM experiments are not sensitive to such exotic physics indicates that M_0 (and hence f_0) are at least larger than the weak scale [appearing in the SM potential, $V(H,H^*)$].

Finally, consider the embedding fields, $Y^M(x)$. Because of the coordinate invariances, not all of the Y^M are physical. A convenient gauge-fixing (in the effective theory's domain of validity) is provided by choosing

$$Y^\mu(x)=x^\mu,\quad Y^{m=4,5}(x)\ \text{arbitrary}. \tag{7}$$

The two physical fields, Y^m, acquire explicit kinetic terms and interactions upon expanding Eq. (4) for small fluctuations, using Eq. (6). They appear derivatively coupled because they are the Nambu-Goldstone modes corresponding to spontaneous breaking of transverse translations by the 3-brane. We will generally use lower-case Roman letters, $m,n,\ldots=4,5$, to denote these transverse directions.

III. INFRARED DYNAMICS OF GRAVITY AND THE 3-BRANE

Let us now imagine integrating out the physics of the effective theory described above, down to the far infrared. In particular, all the massive SM particles are completely integrated out. We are left with an effective theory valid at very low energies, consisting of six-dimensional gravity and the Abelian gauge field, the 3-brane embedding fields, and massless SM particles. Now at these energies the massless SM particles are essentially decoupled from each other and from the gravitational and (derivatively-coupled) embedding fields. Therefore, since we are more interested in the dynamics of the 3-brane itself, we can drop reference to the massless SM fields since they have a negligible effect. Similarly we can ignore the six-dimensional gauge field (which has no sources in this paper). Then the general form of this infrared effective theory is given by Eq. (3), where now

$$S_{3\text{-}brane}=\int d^4x\sqrt{-g}\{-f^4+\cdots\}, \tag{8}$$

$$S_{bulk}=\int d^6X\sqrt{-G}\{-\Lambda-2M^4R+\cdots\}. \tag{9}$$

The higher dimension terms in the ellipses are irrelevant in the far infrared. The dimensionful constants, f^4, Λ and M are the bottom-line renormalized 3-brane tension, six-dimensional cosmological constant and six-dimensional Planck constant respectively. The fact that we took $f_0 \sim \mathcal{O}(M_0)$ implies that, barring fine cancellations, $f \sim \mathcal{O}(M)$. For now we shall assume that $\Lambda=0$, but will take it to be non-zero but small in Secs. VI and VII.

IV. A FLAT 3-BRANE SOLUTION

The dynamics of the 3-brane and six-dimensional gravity is weakly coupled in the far infrared, and well-approximated by the classical equations of motion. We will look for a static solution of the following form:

$$Y^m=\overline{Y^m}=\text{const},\quad Y^\mu(x)=x^\mu,$$

$$ds^2\equiv G_{MN}\,dX^M\,dX^N=\eta_{\mu\nu}\,dx^\mu\,dx^\nu+\mathcal{G}_{mn}(X^4,X^5)dX^m\,dX^n. \tag{10}$$

That is, the bulk spacetime has the form, $\text{Mink}_4\times\mathcal{M}_2$, where Mink_4 is four-dimensional Minkowski space and $\mathcal{M}_2$ is a two-dimensional manifold. In this section we will take $\mathcal{M}_2$ to have non-compact planar topology. The 3-brane is embedded along the Minkowski directions and at some point, $\overline{Y^m}$, in $\mathcal{M}_2$. Note that, by Eq. (6), with such an embedding the SM fields would feel an induced Minkowski metric $g_{\mu\nu}=\eta_{\mu\nu}$. Also notice that our ansatz for Y satisfies the gauge condition, Eq. (7).

When obtaining the classical equations of motion by functionally differentiating the action, it is legitimate to derive the Y^m equations of motion by first setting G_{MN} to its ansatz form, and then to derive the metric equations of motion by first setting Y to its ansatz form. The Y^m equations of motion are then

$$\partial_\mu[\sqrt{-g}g^{\mu\nu}\mathcal{G}_{mn}(Y(x))\partial_\nu Y^n]=0, \tag{11}$$

where

$$g_{\mu\nu}=\eta_{\mu\nu}+\mathcal{G}_{mn}(Y)\partial_\mu Y^m\partial_\nu Y^n. \tag{12}$$

Clearly, $Y^n=\overline{Y^n}=\text{const}$ provides a solution to these equations.

The metric equations of motion (six-dimensional Einstein equations) are

$$\sqrt{-G}\left(R_{MN}-\frac{1}{2}RG_{MN}\right)(X)=\frac{f^4}{4M^4}G_{M\mu}(X)\,\eta^{\mu\nu}G_{\nu N}(X)\,\delta^2(X^m-\overline{Y^m}). \tag{13}$$

Now let us try the metric ansatz. The only non-trivial components of the curvature tensor can be R_{mn}, and Einstein's equations split into two:

$$\sqrt{\mathcal{G}}R^{(2)}=-\frac{f^4}{2M^4}\,\delta^2(X^m-\overline{Y^m}) \tag{14}$$

$$R_{mn}-\frac{1}{2}R^{(2)}\mathcal{G}_{mn}=0, \tag{15}$$

where $R^{(2)}$ denotes the two-dimensional curvature scalar constructed from $\mathcal{G}_{mn}$. Equation (15) holds identically for any two-dimensional metric, $\mathcal{G}_{mn}$. Equation (14) is closely analogous to Einstein's equations in $(2+1)$ dimensions in the presence of a static particle source [17], and has a very simple solution: $\mathcal{G}_{mn}$ corresponds to a conical geometry on $\mathcal{M}_2$, with the tip of the cone at $\overline{Y^m}$. The deficit angle of the cone is given by

$$\delta=\frac{f^4}{4M^4}. \tag{16}$$

Although we will not need it here, an explicit form for this metric can be given in radial coordinates centered at $\overline{Y^m}$, that is $X^4-\overline{Y^4}\equiv\rho, X^5-\overline{Y^5}\equiv\phi$:

$$\mathcal{G}_{\rho\rho}=1, \mathcal{G}_{\rho\phi}=\mathcal{G}_{\phi\rho}=0,\quad \mathcal{G}_{\phi\phi}=\left(1-\frac{f^4}{8\pi M^4}\right)^2\rho^2. \tag{17}$$

Away from the 3-brane, the bulk spacetime has Minkowskian geometry.

V. COMPACTIFICATION WITH SEVERAL 3-BRANES

Of course, the above solution does not provide a realistic background spacetime, because gravitational fluctuations can propagate freely in the six non-compact dimensions. For example, this leads to a $1/r^4$ Newtonian force instead of the experimental $1/r^2$ law [1]. To cure this problem we will consider $\mathcal{M}_2$ to be compact, with spherical topology. However, we must reconcile this with the fact that the static 3-brane considered in the previous section yields a locally flat $\mathcal{M}_2$ with a conical singularity at the 3-brane location. The simplest way to proceed is to consider the case of several 3-branes, labelled by an index j. Each of these 3-branes may be inhabited by different four-dimensional field theories, one of which is the SM. At low enough energies the details of these field theories are irrelevant, the 3-branes are characterized just by their renormalized tensions, f_j^4. We will look for a solution to the classical equations of motion using the same ansatz for the metric as in the previous section and with each 3-brane again extended in the Minkowski directions and occupying a fixed point, $\overline{Y}_j^m$, in $\mathcal{M}_2$. This configuration is a case of ''parallel universes'' linked only by the higher-dimensional gravity.

As in the previous section, it is straightforward to see that the ansatz satisfies the Y_j equations of motion. The non-trivial Einstein equation generalizes to

$$\sqrt{\mathcal{G}}R^{(2)}=-\sum_j \frac{f_j^4}{2M^4}\delta^2(X^m-\overline{Y}_j^m). \tag{18}$$

The solution is now analogous to the case of several static point masses in $(2+1)$-dimensional gravity on a space of spherical topology [17]. The geometry is flat everywhere in $\mathcal{M}_2$ except at the locations of the 3-branes, where there are conical singularities with deficit angles:

$$\delta_j=\frac{f_j^4}{4M^4}. \tag{19}$$

However this static solution is not generally possible because of the constraint provided by the Gauss-Bonnet theorem for spherical topology:

$$\int_{\mathcal{M}_2} dX^4\, dX^5 \sqrt{\mathcal{G}}R^{(2)}=-8\pi. \tag{20}$$

By Eq. (18) this implies that the static solution is only possible if the 3-brane tensions satisfy the sum rule

$$\sum_j \frac{f_j^4}{4M^4}=4\pi. \tag{21}$$

That is, according to Eq. (18), the conical singularities are the only source of curvature for $\mathcal{M}_2$, and the deficit angles must add up to 4π in order to yield a surface of spherical topology. A simple example of such an $\mathcal{M}_2$ is the surface of a tetrahedron, where the four vertices correspond to the positions of four 3-branes.

As will be seen in Sec. VI and discussed further in Sec. VIII, the fine-tuning of 3-brane tensions to satisfy Eq. (21) is just the re-incarnation of the cosmological constant problem in the present context. Granting this fine-tuning, we have a minimal mechanism for compactification of the higher dimensions, namely the curvature induced by the 3-branes themselves.

VI. EFFECTIVE FIELD THEORY BELOW THE COMPACTIFICATION SCALE

Let us now derive the effective field theory for the massless degrees of freedom after compactification in the manner described in the previous section. As usual when higher dimensions are compactified, the higher dimensional fields, in the present case just the six-dimensional metric (neglecting the Abelian gauge field for now), give rise to a Kaluza-Klein tower of four-dimensional states, most of which acquire masses of order the compactification mass scale. The massless states correspond to ''zero-modes'' of the compactified configuration. The 3-brane is an added source of massless fields, namely the Y^m fields and the massless SM fields. We will neglect the massless SM fields here since their inclusion is rather trivial. Obviously the fields on the 3-brane do not give rise to any massive Kaluza-Klein states.

Let us first consider the six-dimensional metric tensor, whose components can be decomposed in four-dimensional Minkowski space as a tensor $G_{\mu\nu}(X)$, vector fields, $G_{\mu m}(X)$, and scalars, $G_{mn}(X)$. We found a global Minkowski four-dimensional spacetime factor as part of our classical solution in Sec. V. We can deform this continuously so that this spacetime factor is a curved manifold described by a four-dimensional metric,

$$G_{\mu\nu}=\overline{g}_{\mu\nu}(x), \tag{22}$$

in coordinates where $X^\mu\equiv x^\mu$. Non-trivial $X^{4,5}$-dependence in $G_{\mu\nu}$ corresponds to Kaluza-Klein excitations with masses of order the compactification scale. In Kaluza-Klein theory, massless components of the $G_{\mu m}$ correspond to continuous isometries of the compactified space. In the present case however, there are no continuous isometries (for example, consider the case where $\mathcal{M}_2$ is a tetrahedron). Therefore there are no massless $G_{\mu m}$ states, and below the compactification scale we effectively have

$$G_{\mu m}=0. \tag{23}$$

The fate of the $G_{mn}(X)$ is tied up with the 3-brane fields, $Y^m(x)$. The identification of zero-modes is made somewhat ambiguous by general coordinate invariance. In one convenient choice of language, we can observe that the metric on $\mathcal{M}_2$, given by $G_{mn}=\mathcal{G}_{mn}$, is the Euclidean metric plus a number of conical singularities at the 3-brane positions, the deficit angles fixed by the 3-brane tensions according to Eq. (19). Therefore the zero-modes are the 3-brane separations, $|Y_j-Y_k|$, measured with the Euclidean metric, which then determine $\mathcal{G}_{mn}$. Note that by Eqs. (6), (7), the induced metric on the jth 3-brane is given by

$$g_{\mu\nu}(x)=\overline{g}_{\mu\nu}(x)+\mathcal{G}_{mn}(Y_j^4(x),Y_j^5(x))\partial_\mu Y^m\partial_\nu Y^n. \tag{24}$$

This formula requires careful interpretation because $\mathcal{G}$ is being evaluated at a position where it has a conical singularity. We do not expect a 3-brane to ''feel'' the curvature singularity for which it is itself the source, anymore than we expect this for a point particle. Indeed, given any physical ultraviolet regularization of our original effective theory at scale M, such curvature singularities would be smoothed out over distances of order $1/M$. In the 3-brane effective Lagrangian, we are using $\mathcal{G}_{mn}$ to measure distances involved in 3-brane fluctuations. In the effective theory's domain of validity the typical distances are much larger than $1/M$, so the curvature singularity is unimportant. In the present situation, throwing out the curvature singularity leaves us with

$$g_{\mu\nu}(x)=\overline{g}_{\mu\nu}(x)+\delta_{mn}\partial_\mu Y_j^m\partial_\nu Y_j^n, \tag{25}$$

where δ_{mn} denotes the two-dimensional Euclidean metric.

The effective four-dimensional theory below the compactification scale is then given by substituting the massless metric components into Eqs. (8), (9),

$$S=-\int d^4x\sqrt{-\overline{g}}\Bigg\{\sum_j\left[f_j^4+\frac{f_j^4}{2}\delta_{mn}\overline{g}^{\mu\nu}\partial_\mu Y_j^m\partial_\nu Y_j^n\right] +\int dX^4\,dX^5\sqrt{\mathcal{G}}[2M^4R^{(4)}+2M^4R^{(2)}]+\cdots\Bigg\}, \tag{26}$$

where $R^{(4)}$ is the four-dimensional curvature scalar due to $\overline{g}_{\mu\nu}$, and $R^{(2)}$ is the two-dimensional curvature scalar due to $\mathcal{G}_{mn}$. The kinetic terms for the Y_j arose by expanding $\sqrt{-g}$ in Eq. (8) about $\overline{g}_{\mu\nu}$, using Eq. (25). By the Gauss-Bonnet theorem, Eq. (20), and the fact that $R^{(4)}$ constructed from $\overline{g}_{\mu\nu}$ is independent of $X^{4,5}$, we get

$$S=-\int d^4x\,\sqrt{-\overline{g}}\Bigg\{\sum_j f_j^4-16\pi M^4 +\sum_j\frac{f_j^4}{2}\delta_{mn}\overline{g}^{\mu\nu}\partial_\mu Y_j^m\partial_\nu Y_j^n+2\mathcal{A}(Y)M^4R^{(4)}+\cdots\Bigg\}, \tag{27}$$

where

$$\mathcal{A}(Y)\equiv\int dX^4dX^5\sqrt{\mathcal{G}} \tag{28}$$

is the area of $\mathcal{M}_2$ determined by the 3-brane separations. It is now clear that our previous sum rule requirement on the 3-brane deficit angles, Eq. (21), is precisely the tuning of parameters necessary to cancel the effective four-dimensional cosmological constant.

From Eq. (27) we see that the effective four-dimensional Planck constant is given by

$$M_{Pl}^2=\mathcal{A}(Y)M^4. \tag{29}$$

The fact that M_{Pl} depends on massless fields leads to a conflict with post-Newtonian experimental tests of general relativity. See Ref. [18] for a review. These tests are sensitive to the tensorial nature of the macroscopic gravitational force, and imply that the scalar admixture of the gravitational force can be at most a fraction of a percent. The reason that one must perform tests sensitive to relativistic (''post-Newtonian'') effects in order to distinguish scalar from tensor exchange is because in the Newtonian limit both exchanges give rise to a $1/r^2$ force. At first sight this issue does not appear to pose a problem for us since the derivative couplings of the (canonically normalized [16]) Y scalars to SM states (our standard probes of gravity), as given by Eqs. (4) and (6), are negligible at distances larger than a centimeter where gravity is tested.[1] However the problem is that a Y-dependent four-dimensional Planck mass in Eq. (27) corresponds to an order one mixing of the (normalized) Y scalars with the metric tensor, $\overline{g}_{\mu\nu}$, which leads to an unacceptable scalar admixture to gravity of order one.[2]

To escape from this phenomenological problem we require a potential energy term for the compactified area, $\mathcal{A}(Y)$, which stabilizes it and gives the corresponding combination of Y-scalars a finite Yukawa range below a centimeter. The simplest way to introduce a potential term is to begin with a small positive six-dimensional cosmological constant, Λ. At the level of the effective theory below the compactification scale this clearly leads to an extra term,

$$\delta S_\Lambda=-\int d^4x\sqrt{-\overline{g}}\mathcal{A}(Y)\Lambda. \tag{30}$$

This term favors the reduction of $\mathcal{A}$. We now require a force that opposes this reduction in order to obtain stability. It was proposed in Refs. [19] that this can be provided by intrinsically quantum mechanical matching corrections at the compactification scale. Although this is not the method of stabilization preferred in this paper, it is useful to briefly consider it first in order to understand its merits and problems.

The important point is that there is a tower of Kaluza-Klein states of the graviton with masses set by the compactification scale. Quantum loops of these states will then contribute to the effective potential. Dimensional analysis suggest the rough form,

$$\delta S_{quantum}=-\int d^4x\sqrt{-\overline{g}}\frac{k}{\mathcal{A}^2}. \tag{31}$$

Indeed these are just the types of corrections, with $k>0$, that are induced in more standard Kaluza-Klein compactifications. For example, see Ref. [20]. We see that combining our original effective action with δS_Λ and $\delta S_{quantum}$ gives a stabilizing effective potential for $\mathcal{A}(Y)$,

$$\mathcal{V}_{eff}=\mathcal{A}(Y)\Lambda+\frac{k}{\mathcal{A}(Y)^2}+\sum_j f_j^4-16\pi M^4. \tag{32}$$

At the minimum of this potential,

$$\mathcal{A}\sim\mathcal{O}(\Lambda^{-1/3}). \tag{33}$$

[1]Recall that we are considering the case where $f_j\sim\mathcal{O}(M)$ are larger than the weak scale. [That is, by Eqs. (18) and (20), we are considering the case of several order one deficit angles adding up to 4π.] The scales f_j suppress the derivative couplings of the canonically normalized Y scalars.

[2]One can also perform a Weyl transformation to eliminate the field dependence from the Einstein action. It then resurfaces in direct gravitational strength couplings of the scalars to the SM.

Thus $\Lambda^{1/6}$ is the compactification scale and is a free parameter of the effective theory. The minimum value of $\mathcal{V}_{eff}$ is now the effective four-dimensional cosmological constant. It can be set to zero by fine-tuning the SM 3-brane tension. (See Sec. VIII.)

The curvature at the minimum of the effective potential sets the mass of the combination of Y-scalars determining $\mathcal{A}$. We can estimate this as follows. The area $\mathcal{A}$ scales quadratically with the Y's, so that near the minimum of $\mathcal{V}_{eff}$,

$$\mathcal{V}_{eff} \sim \Lambda(\delta Y)^2. \tag{34}$$

From Eq. (27) we see that the canonically normalized scalar fields are $f^2 Y$, so that Eq. (34) corresponds to a mass of order $\Lambda^{1/2}/f^2$. By Eqs. (29), (33) and the fact that $f \sim \mathcal{O}(M)$, we see that this mass is just $1/\mathcal{A}M_{Pl}$. To be phenomenologically acceptable we must then have [18]

$$\mathcal{A}M_{Pl} < 1 \text{ cm}. \tag{35}$$

This corresponds to a compactification length scale smaller than 10^{-16} cm and M larger than 10^{10} GeV.

The mechanism considered above for acceptably stabilizing $\mathcal{A}$ is minimal and attractive, but for millimeter scale compactifications it is unacceptable and we must turn to something else.

VII. STABILITY FROM TRAPPED MAGNETIC FLUX

In this section we make use of the six-dimensional compact $U(1)$ gauge field and six-dimensional cosmological constant to stabilize the compact space. The mechanism is essentially a limiting case of that of Ref. [21] where the $U(1)$ was the remnant of a spontaneously broken six-dimensional $SU(2)$ gauge theory. The intuitive idea is simple. Compact $U(1)$ gauge fields can have non-zero magnetic flux through closed two-dimensional surfaces such as $\mathcal{M}_2$. This flux is a quantized topological invariant of the $U(1)$ fiber bundle structure and is therefore fixed. This forces the flux density to increase as $\mathcal{A}$ decreases, leading to a higher magnetic energy density. This provides the stabilizing potential we seek. As $\mathcal{A}$ increases this potential reduces, but the potential due to a small six-dimensional cosmological constant increases, as seen in Eq. (30). The size of the compact space is determined by the balance between these two effects.

Explicitly, the magnetic flux through $\mathcal{M}_2$ is given by

$$\Phi \equiv \int_{\mathcal{M}_2} dX^4 dX^5 \epsilon^{mn} \mathcal{F}_{mn}(X) = \frac{2\pi N}{e}, \tag{36}$$

where $\epsilon^{45} = -\epsilon^{54} = 1$, N must be an integer, and e is the elementary charge that defines the abelian gauge group as a compact $U(1)$. Note that in six-dimensions, e has units of mass^{-1}. Let us briefly recall the reason for the flux quantization. Naively, the flux must vanish by Stokes' theorem. However this can be evaded by adding a compensating point-like vortex of flux, $-\Phi$ (corresponding to the "Dirac string" of three dimensional space) somwhere on $\mathcal{M}_2$. The presence of the vortex can only be physically detected by the Aharnov-Bohm phase it induces in test charges, namely $-e\Phi$. Thus the vortex is unphysical (akin to an unphysical coordinate singularity) precisely when the flux is quantized as in Eq. (36). In this section we will consider the case $N = 1$.

The equations of motion for the gauge field following from Eq. (5), with the six-dimensional metric given by $G_{\mu\nu} = \bar{g}_{\mu\nu}(x)$, $G_{\mu m} = 0$, $G_{mn} = \mathcal{G}_{mn}(X^4, X^5)$, are

$$\partial_\mu[\sqrt{-\bar{g}}\bar{g}^{\mu\rho}\bar{g}^{\nu\sigma}\mathcal{F}_{\rho\sigma}] = 0, \tag{37}$$

$$\partial_\mu[\sqrt{-\bar{g}}\bar{g}^{\mu\rho}\mathcal{F}_{\rho l}] = 0, \tag{38}$$

$$\partial_m[\sqrt{\mathcal{G}}\mathcal{G}^{mk}\mathcal{F}_{k\nu}] = 0, \tag{39}$$

$$\partial_m[\sqrt{\mathcal{G}}\mathcal{G}^{mk}\mathcal{G}^{nl}\mathcal{F}_{kl}] = 0. \tag{40}$$

We will seek a solution where the gauge field has vanishing μ-components and the m-components are x-independent. Therefore Eqs. (37)–(39) are automatically satisfied, leaving only Eq. (40).

Since $\mathcal{F}_{mn}$ is an antisymmetric tensor it can conveniently be written in terms of a scalar field, B, on $\mathcal{M}_2$,

$$\mathcal{F}_{mn} \equiv \frac{\epsilon_{mn}}{\sqrt{\mathcal{G}}} B, \tag{41}$$

where the indices of the ϵ-tensor have been lowered with the $\mathcal{G}_{mn}$ metric. Equation (40) can then simplify to

$$\partial_m B = 0. \tag{42}$$

Taking into account Eq. (36) (for $N=1$) gives the solution,

$$B = \frac{2\pi}{e\mathcal{A}}. \tag{43}$$

Substituting Eqs. (41) and (43) into Eq. (5) we can read off the contribution that the magnetic energy of the trapped flux makes to the effective potential of the effective theory below the compactification scale. The result is

$$\mathcal{V}_{eff} = \sum_j f_j^4 - 16\pi M^4 + \int dX^4 dX^5 \sqrt{\mathcal{G}}\left[\Lambda + \frac{1}{2}B^2\right]$$
$$= \Lambda\mathcal{A}(Y) + \frac{2\pi^2}{e^2\mathcal{A}(Y)} + \sum_j f_j^4 - 16\pi M^4, \tag{44}$$

where we have omitted the quantum corrections discussed in the last section since they are negligible for large compactifications. Minimizing this effective potential for $\mathcal{A}$ we find that

$$\mathcal{A} = \frac{\sqrt{2}\pi}{e\sqrt{\Lambda}}. \tag{45}$$

The effective potential at this minimum will correspond to an effective four-dimensional cosmological constant which can be made to vanish by tuning the SM 3-brane tension. As in the previous section we can estimate the mass of the canonically normalized combination of Y-scalars corresponding to $\mathcal{A}(Y)$ fluctuations, and find that it is still of order $\Lambda^{1/2}/f^2$. What is new is that we have the extra parameter e to play with, so that we can consistently arrange for both the Compton wavelength of these fluctuations and the compactification length scale to be (roughly) of order a millimeter. This phenomenologically interesting but safe choice is accomplished by taking

$$e \sim \mathcal{O}\left(\frac{1}{\sqrt{\mathcal{A}}M^2}\right) \sim \mathcal{O}\left(\frac{1}{M_{Pl}}\right),$$

$$\Lambda \sim \mathcal{O}\left(\frac{M^4}{\mathcal{A}}\right). \quad (46)$$

In this section we have treated the six-dimensional cosmological constant and magnetic energy as perturbations to the basic picture developed in Secs. V and VI. Strictly, this analysis is valid in the regime where the $\mathcal{M}_2$ curvature due to these new sources is small compared to the zeroth order sources, namely the 3-branes themselves. A six-dimensional cosmological constant is a source for a constant curvature of order Λ/M^4, which integrates to a total of order $\mathcal{A}\Lambda/M^4$. This should be smaller than the 3-brane deficit angles, which were order one, but not necessarily much (parametrically) smaller. Thus at the order of magnitude level this is consistent with Eq. (46). The magnetic energy balances the cosmological constant at the minimum of the effective potential and is therefore also consistent with the approximation we have made. In fact an exact solution of the classical equations of motion including the magnetic energy is possible, and agrees with what we have found.

VIII. WHICH COSMOLOGICAL CONSTANT PROBLEM?

We now consider the nature of the cosmological constant problem in the present model. For a general review see [22]. In fact there are two cosmological constants that we should consider, the effective four-dimensional constant below the compactification scale and the six-dimensional constant, Λ. They pose quite different problems.

Let us begin with the four-dimensional effective cosmological constant. It is given by the value of the effective potential at its minimum,

$$\mathcal{V}_{eff}^{min} = 2\Lambda\mathcal{A}^{min} + \sum f_j^4 - 16\pi M^4. \quad (47)$$

Equation (4) shows that, in the absence of exact supersymmetry, the SM vacuum energy will renormalize the corresponding 3-brane tension by an amount roughly set by the weak scale v,

$$f^4 = f_0^4 + \mathcal{O}(v^4). \quad (48)$$

Consequently, the *natural* size of $|\mathcal{V}_{eff}^{min}|$ is at least $\mathcal{O}(v^4)$. The extreme fine tuning of f_0 in order to get $|\mathcal{V}_{eff}^{min}|$ to be less than the experimental bound of $10^{-56}v^4$, is precisely the usual cosmological constant problem. Note that in the absence of the stabilization mechanism, the requirement that $\mathcal{V}_{eff}$ vanish reduces to the sum rule for the 3-brane deficit angles, Eq. (21).

It is amusing to note that the above fine-tuning problem disappears in the case where the extra dimensions are *not* compactified, as in Sec. IV. There we found a solution to the effective classical equations of motion, where the induced metric on the 3-brane seen by the SM particles is exactly four-dimensional Minkowksi space, without any need to fine tune the 3-brane tension. A change in this tension only led to a change in the deficit angle of the conical singularity in the extra dimensions. Of course this is merely trading one major problem for another since without compactifying $\mathcal{M}_2$, gravity remains six-dimensional at all distances, in obvious conflict with experiment. (For example, Newton's $1/r^2$ law is replaced by a $1/r^4$ law.)

Let us now turn to the six-dimensional constant, Λ. The crucial observation is that *the SM vacuum energy does not renormalize* Λ. This is because the SM fields are confined to a 3-brane whereas Λ represents a gravitational interaction throughout the six-dimensional bulk spacetime. It can only be renormalized by quantum loops of six-dimensional fields. In fact using dimensional regularization there is no renormalization of Λ by quantum loops of six-dimensional gravitons and gauge fields, since they do not have a mass scale that can appear in divergent cosmological constant terms. So any choice of Λ seems technically natural. This argument requires qualification however. Six-dimensional general relativity breaks down as an effective field theory at scale M and therefore there must be new physics by this scale which replaces it. Therefore naturalness would require this dynamical scale to set the size of Λ, which is much larger than we can tolerate [see Eq. (46)]. We have two choices in our effective field theory, simply accept that Λ is also fine-tuned to be as small as in Eq. (46), or consider the bulk dynamics to be supersymmetric so that a small Λ is technically natural. In the latter case, we should take the view that *all* of the fundamental dynamics are exactly supersymmetric, but that the SM sector appears non-supersymmetric because of spontaneous supersymmetry breaking dynamics on (or by) the SM 3-brane. This supersymmetry breaking only feeds into the gravitational sector below the compactification mass scale.

To summarize, there are two potential cosmological constant problems, associated with the two cosmological constants in six and four dimensions. While the six-dimensional cosmological constant can be be kept naturally small if the bulk dynamics is supersymmetric, this is not an option for the effective four-dimensional cosmological constant because we know experimentally that supersymmetry is too badly broken in at least the SM sector. Ideas along the lines put forth in Ref. [23] may be required to resolve this tough naturalness problem.

IX. DISCUSSION

In this paper, a 3-brane effective field theory has been constructed which is consistent with all experimental standard model and gravitational tests. The size of the compactified extra dimensions is effectively a free parameter of the model. If it is almost a millimeter, as proposed in Ref. [1], upcoming tests of short-distance gravity [2] will see the transition to six-dimensional gravity, while future particle accelerators will be sensitive to the physics of strongly coupled gravity at the six-dimensional Planck scale, whether this is provided by strings or something else. The present model should also have interesting cosmological implications although we have not pursued these here.

In the present paper, the compactification scale is effectively put in by hand among the parameters of our starting effective theory. Although it is technically natural to have the compactification mass scale be much smaller than the weak scale and six-dimensional Planck scale, thereby realizing the proposal of Ref. [1] for solving the gauge hierarchy problem, it would be more attractive if there were a single dynamical scale, roughly of order the six-dimensional Planck scale M, with the compactification scale emerging dynamically in terms of this scale and some dimensionless parameters, g, say in the form $e^{-1/g^2}M$. This would be a true elimination of the hiererchy problem. It is worth exploring if a model of this type can be constructed.

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