Auxiliary field method as a powerful tool for nonperturbative study

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The auxiliary field method, defined through introducing an auxiliary (also called the Hubbard-Stratonovich or mean) field, and utilizing a loop expansion, gives an excellent result for a wide range of coupling constants. The analysis is done for the anharmonic oscillator and the double-well cases in zero (a simple integral) and one (quantum mechanics) dimension. It is shown that the result becomes increasingly accurate by taking a higher loop into account in the weak coupling region; however, such is not the case in the strong coupling region. The two-loop approximation is shown to be still insufficient for the double-well case in quantum mechanics. [S0556-2821(99)02706-X]

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I. INTRODUCTION

In most actual situations, path integral expressions have a non-Gaussian form so that some approximation is always needed to evaluate them. Apart from perturbative treatments, such as a weak (strong) coupling expansion, which can only describe a small (large) coupling region, other approaches have been used to handle a wider coupling range such as the well-known variational method [1] which has been applied successfully to the polaron problem [2]. This method, combined with an optimization technique, has been discussed in Ref. [3]. A numerical estimation is also possible once expressed in the path integral form; for instance, computer simulations produce fruitful results such as in lattice QCD [4], but current technology does not yet permit imposing certain symmetries onto the lattice; chiral symmetry is a well-known example [5] of such a symmetry. An advantage of path integration, contrary to the operator formalism, is that we can easily switch from one variable to another by means of a simple change of variables, which opens up new possibilities. The auxiliary field is considered as one of these variables, and was introduced into the model by Gross and Neveu [6].

The Gross-Neveu model is a two-dimensional four-fermion model inspired by the work of Nambu and Jona-Lasinio [7]:

$$\mathcal{L}=\overline{\psi}i\partial\!\!\!/\psi+\frac{g^2}{2}(\overline{\psi}\psi)^2, \tag{1.1}$$

where ψ is an N component. Gross and Neveu [6] proposed an equivalent Lagrangian:

$$\mathcal{L}'=\overline{\psi}i\partial\!\!\!/\psi-\frac{\sigma^2}{2}-g\overline{\psi}\psi\sigma. \tag{1.2}$$

Here σ has no kinetic term and on elimination by using the equation of motion one recovers the original Lagrangian, Eq. (1.1). In this sense, σ is an auxiliary field.

One could paraphrase the above in terms of path integrals [8] as follows: the partition function (in an imaginary temperature) for the Lagrangian, Eq. (1.1), reads

$$Z\equiv\int\mathcal{D}\psi\mathcal{D}\overline{\psi}\exp\left[i\int d^2x\left(\overline{\psi}i\partial\!\!\!/\psi+\frac{g^2}{2}(\overline{\psi}\psi)^2\right)\right]. \tag{1.3}$$

Introducing the auxiliary field σ in terms of the Gaussian integrals, such that

$$1=\int\mathcal{D}\sigma\exp\left[-i\int d^2x\frac{1}{2}(\sigma+g\overline{\psi}\psi)^2\right], \tag{1.4}$$

and inserting into Eq. (1.3) we find

$$\begin{aligned}Z&=\int d\sigma\mathcal{D}\psi\mathcal{D}\overline{\psi}\exp\left(i\int d^2x\mathcal{L}'\right)\\&=\int d\sigma\mathcal{D}\psi\mathcal{D}\overline{\psi}\exp\left[i\int d^2x\left(\overline{\psi}i\partial\!\!\!/\psi-\frac{\sigma^2}{2}-g\overline{\psi}\psi\sigma\right)\right].\end{aligned} \tag{1.5}$$

Similar techniques are also utilized nowadays for a boson quartic interaction [9] instead of the four-fermion interaction. The nomenclature for σ field is, therefore, various: the mean field [10] and the Hubbard-Stratonovich field [11] in solid state physics.

In practice, we define the partition function (which can be obtained through $t\mapsto it$)

$$\begin{aligned}Z(T)\equiv&\int d\sigma\mathcal{D}\psi\mathcal{D}\overline{\psi}\\&\times\exp\left[-\int_0^T dt\int dx\left(\overline{\psi}\partial\!\!\!/\psi+\frac{\sigma^2}{2}+g\overline{\psi}\psi\sigma\right)\right],\end{aligned} \tag{1.6}$$

where the antiperiodic boundary condition for the fermi field, $\psi(T,x)=-\psi(0,x)$, should be understood. We then integrate out the fermion field to find

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$$Z(T)=\int d\sigma \exp\left[-\int d^2x\frac{\sigma^2}{2}+N\ln\det(\partial\!\!\!/+g\sigma)\right]$$
$$\equiv\int d\sigma e^{-S[\sigma]},$$
$$S[\sigma]\equiv\int d^2x\frac{\sigma^2}{2}-N\ln\det(\partial\!\!\!/+g\sigma). \tag{1.7}$$

Since we look for a vacuum with $T\mapsto\infty$, we should find a constant solution σ_0 in the equation of motion

$$\left.\frac{\delta S}{\delta\sigma(x)}\right|_{\sigma_0}=0, \tag{1.8}$$

which gives the gap equation

$$\sigma_0=2Ng^2\int\frac{d^2k}{(2\pi)^2}\frac{\sigma_0}{k^2+(g\sigma_0)^2}. \tag{1.9}$$

If σ_0 is nonzero, then dynamically symmetry breaking occurs. Indeed, the formula is legitimated if the number of fermion species becomes infinite, $N\to\infty$. However, this is not the case for most of the physical situations: N is finite or even 1. We ask the question as to how accurate is it when $N=1$, which is one of the motivations of this work.

The second purpose of this paper is as follows: performing the WKB approximation in the double-well potential we have to pursue instanton calculations [12], which are cumbersome as well as tedious. A simpler approach would be to use the auxiliary field method. We clarify these issues with examples of the quartic coupling of bosonic field.

The paper is organized as follows. In Sec. II a simple model calculation is performed for the integral expression. Here we realize the importance of the loop expansion with respect to the auxiliary field and find a more accurate result by taking a higher loop correction into account when the coupling g is small. However, when g becomes larger, higher loops do not always improve the situation. We then proceed to the quantum mechanical model in Sec. III, where we compare our results with those obtained numerically, and find that the two-loop correction gives a 4% error for $10^{-3}<g^2<10^3$ except for $O(10^{-1})<g^2<O(1)$ in the double-well case. The final section is devoted to a discussion.

II. SIMPLE (ZERO-DIMENSIONAL) MODEL

The starting point is

$$I\equiv\int_{-\infty}^{\infty}\frac{dx}{\sqrt{2\pi}}\exp\left[-\frac{\omega^2}{2}x^2-\frac{g^2}{8}x^4\right]. \tag{2.1}$$

The integral is expressed as

$$I=\sqrt{\frac{\pi|\omega^2|}{4g^2}}e^{\omega^4/4g^2}\left\{\mathrm{I}_{-1/4}\left(\frac{\omega^4}{4g^2}\right)-\epsilon(\omega^2)\mathrm{I}_{1/4}\left(\frac{\omega^4}{4g^2}\right)\right\}, \tag{2.2}$$

where

$$\epsilon(x)=\begin{cases}+1, & x>0,\\ -1, & x<0,\end{cases}$$

and $\mathrm{I}_\alpha(x)$ is the modified Bessel function. There are two cases depending on the sign of ω^2.

Case (i): $\omega^2>0$ (zero-dimensional anharmonic oscillator). In this case,

$$I=\sqrt{\frac{\omega^2}{2\pi g^2}}e^{\omega^4/4g^2}\mathrm{K}_{1/4}\left(\frac{\omega^4}{4g^2}\right)\overset{g^2\to0}{\sim}1+O(g^2), \tag{2.3}$$

where $\mathrm{K}_\alpha(x)$ is the modified Bessel function.

Case (ii): $\omega^2<0$ (zero-dimensional double well). In this case, in the limit $g^2\to0$,

$$I\overset{g^2\to0}{\sim}\sqrt{2}e^{\omega^4/2g^2}. \tag{2.4}$$

Here it should be noted that $g^2=0$ is an essential singularity; that is to say, the double-well case is non Borel summable.

Now introduce an auxiliary field such that

$$1=\int_{-\infty}^{\infty}\frac{dy}{\sqrt{2\pi}}\exp\left[-\frac{1}{2}\left(y+ig\frac{x^2}{2}\right)^2\right] \tag{2.5}$$

so as to cancel the x^4 term when inserted into Eq. (2.1), yielding

$$I=\int_{-\infty}^{\infty}\frac{dx}{\sqrt{2\pi}}\frac{dy}{\sqrt{2\pi}}\exp\left[-\frac{1}{2}(\omega^2+igy)x^2-\frac{y^2}{2}\right]$$
$$=\int_{-\infty}^{\infty}\frac{dy}{\sqrt{2\pi}}(\omega^2+igy)^{-1/2}e^{-y^2/2}$$
$$=\int_{-\infty}^{\infty}\frac{dy}{\sqrt{2\pi}}\exp\left[-\frac{y^2}{2}-\frac{1}{2}\ln(\omega^2+igy)\right]. \tag{2.6}$$

We rewrite the final expression as

$$I=\left.\int_{-\infty}^{\infty}\frac{dy}{\sqrt{2\pi}}\exp\left[-\frac{S(y)}{a}\right]\right|_{a=1},$$
$$S(y)\equiv\frac{1}{2}\ln(\omega^2+igy)+\frac{y^2}{2}, \tag{2.7}$$

where we have introduced a parameter a, which must be set to unity at the end. We call a the loop-expansion parameter. Next, denote the solution of $S'(y)=0$ as y_0;

$$S'(y)=y+\frac{ig}{2(\omega^2+igy)}=0, \tag{2.8}$$

which can be expressed as

$$\Omega^2-\omega^2=\frac{g^2}{2\Omega^2}, \tag{2.9}$$

where Ω^2 should obey

$$\Omega^2 \equiv \omega^2 + igy_0 > 0, \tag{2.10}$$

since the Gaussian integral of x, Eq. (2.6), must exist. Then use the saddle point method around y_0 to give

$$I = \int_{-\infty}^{\infty} \frac{dy}{\sqrt{2\pi}} \exp\left[-\frac{S(y_0)}{a} - \frac{S^{(2)}(y_0)}{2a}(y-y_0)^2 - \frac{S^{(3)}(y_0)}{3!a}(y-y_0)^3 - \frac{S^{(4)}(y_0)}{4!a}(y-y_0)^4 - \cdots \right]\Bigg|_{a=1}. \tag{2.11}$$

Making a change of variable, $y - y_0 \mapsto y/\sqrt{a}$, we obtain

$$\begin{aligned} I &= e^{-S_0/a} \int_{-\infty}^{\infty} \sqrt{\frac{a}{2\pi}} dy \\ &\quad \times \exp\left[-\frac{S_0^{(2)}}{2} y^2 - \sqrt{a}\frac{S_0^{(3)}}{3!} y^3 - a\frac{S_0^{(4)}}{4!} y^4 - \cdots \right]\Bigg|_{a=1} \\ &\cong e^{-S_0/a} \sqrt{\frac{a}{2\pi}} \int_{-\infty}^{\infty} dy\, e^{-S_0^{(2)} y^2/2} \\ &\quad \times \left(1 - a\left\{ \frac{S_0^{(4)}}{4!} y^4 - \frac{S_0^{(3)2}}{2(3!)^2} y^6 \right\} + O(a^2) \right)\Bigg|_{a=1}, \end{aligned} \tag{2.12}$$

where we have written

$$S^{(n)}(y_0) \equiv S_0^{(n)}. \tag{2.13}$$

The a^{L-1} term is called the L-loop term ($L=0$ is the tree term). From Eq. (2.9),

$$\Omega^2 = \frac{\omega^2 + \sqrt{\omega^4 + 2g^2}}{2}, \tag{2.14}$$

then

$$S_0 = -\frac{(\Omega^2 - \omega^2)^2}{2g^2} = -\frac{g^2}{8\Omega^4}, \quad S_0^{(2)} = 1 + \frac{g^2}{2\Omega^4},$$

$$S_0^{(3)} = -i\frac{g^3}{\Omega^6}, \quad S_0^{(4)} = -3\frac{g^4}{\Omega^8}, \quad S_0^{(6)} = \frac{5!g^6}{2\Omega^{12}}. \tag{2.15}$$

Using these and performing elementary integrations, we obtain

$$\begin{aligned} I &= \exp\left(\frac{g^2}{8\Omega^4}\right) \sqrt{\frac{\Omega^2}{\Omega^4 + \frac{g^2}{2}}} \left(1 + \frac{3g^4}{8\left(\Omega^4 + \frac{g^2}{2}\right)^2} \right. \\ &\quad - \frac{35g^6}{24\left(\Omega^4 + \frac{g^2}{2}\right)^3} + \frac{329g^8}{128\left(\Omega^4 + \frac{g^2}{2}\right)^4} \\ &\quad \left. - \frac{105g^{10}}{64\left(\Omega^4 + \frac{g^2}{2}\right)^5} + O(4\text{-loop}) \right). \end{aligned} \tag{2.16}$$

Stated as above, we assign the following definitions to the contributions to the integral I:

$$I_{\text{tree}} \equiv \exp\left(\frac{g^2}{8\Omega^4}\right),$$

$$I_{\text{1-loop}} \equiv I_{\text{tree}} \sqrt{\frac{\Omega^2}{\Omega^4 + \frac{g^2}{2}}},$$

$$I_{\text{2-loop}} \equiv I_{\text{1-loop}} \left(1 + \frac{3g^4}{8\left(\Omega^4 + \frac{g^2}{2}\right)^2} - \frac{5g^6}{24\left(\Omega^4 + \frac{g^2}{2}\right)^3} \right),$$

$$\begin{aligned} I_{\text{3-loop}} &\equiv \exp\left(\frac{g^2}{8\Omega^4}\right) \sqrt{\frac{\Omega^2}{\Omega^4 + \frac{g^2}{2}}} \\ &\quad \times \left(1 + \frac{3g^4}{8\left(\Omega^4 + \frac{g^2}{2}\right)^2} - \frac{35g^6}{24\left(\Omega^4 + \frac{g^2}{2}\right)^3} \right. \\ &\quad \left. + \frac{329g^8}{128\left(\Omega^4 + \frac{g^2}{2}\right)^4} - \frac{105g^{10}}{64\left(\Omega^4 + \frac{g^2}{2}\right)^5} \right). \end{aligned} \tag{2.17}$$

Let us analyze the individual cases.

Case (i): zero-dimensional anharmonic oscillator. Put $\omega^2 \mapsto 1$ so that Eq. (2.14) reads

$$\Omega^2 = \frac{\sqrt{1+2g^2}+1}{2}. \tag{2.18}$$

We plot the ratio of $I_{L\text{-loop}}(L=0,1,2,3)$ to the exact value in Fig. 1, (a) for $g^2 \leqslant 1$ and (b) $g^2 > 1$, respectively. Details are shown in Table I.

Case (ii): zero-dimensional double well. Put $\omega^2 \mapsto -1$ so that Eq. (2.14) reads

$$\Omega^2 = \frac{\sqrt{1+2g^2}-1}{2}. \tag{2.19}$$

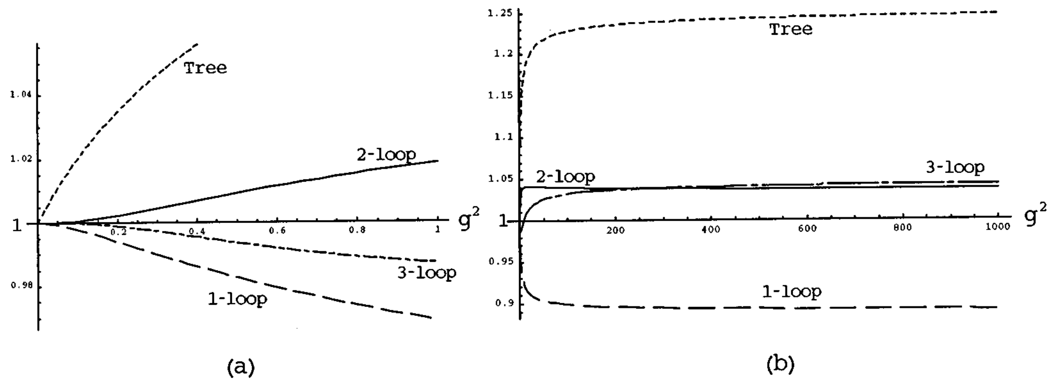

FIG. 1. Zero-dimensional anharmonic oscillator case. (a), $g^2\leqslant 1$. (b) $g^2>1$. Dotted lines represent $I_{\rm tree}/$exact, dashed lines $I_{\text{1-loop}}/$exact, solid lines $I_{\text{2-loop}}/$exact, and dash-dotted lines $I_{3\text{-}loop}/$exact, respectively.

We plot the same ratio as in case (i) in Figs. 2(a) and 2(b). However, in Fig. 2(a) we have omitted the tree graph because of the large deviation. Details are again shown in Table II.

From Figs. 1 and 2, it should be noted that higher loop corrections always improve the result when $g^2\leqslant 1$, but not in the strong coupling region as is seen from the graphs Figs. 1(b) and 2(b). Details for the numerical values are listed in Tables I and II. This fact implies that the loop expansion is merely an asymptotic expansion. In the anharmonic oscillator case, the result is especially satisfactory: the two-loop result gives a 4% error for $10^{-3}<g^2<10^3$. In the double-well case, the three-loop expansion spoils the agreement in the region $g^2\geqslant 1$, but gives a better result at $g^2<1$. However, it is remarkable that the error, in the two-loop expansion, still remains within $\sim 8\%$ for a large coupling region, $10^{-3}<g^2<10^3$.

Finally, the essential role of the loop expansion should be emphasized: if we stop at the g^4 term in the two- or three-loop expression, Eq. (2.17), the result deviates far away from the true value. Therefore we must abandon the coupling constant expansion in the auxiliary field method.

III. QUANTUM MECHANICAL MODEL

Encouraged by the foregoing results, in this section we analyze the quantum mechanical model defined by the Hamiltonian

$$H=\frac{p^2}{2}+\frac{\omega^2}{2}x^2+\frac{g^2}{8}x^4. \tag{3.1}$$

Here again, depending on the sign of ω^2, we consider two cases: (i) $\omega^2>0$, the anharmonic oscillator, and (ii) $\omega^2<0$, the double well. The partition function is given by

$$Z(T)=\mathrm{Tr}\,e^{-TH}=\int \mathcal{D}x \times\exp\left[-\int_0^T dt\left(\frac{\dot{x}^2}{2}+\frac{\omega^2}{2}x^2+\frac{g^2}{8}x^4\right)\right]\Bigg|_{x(T)=x(0)}, \tag{3.2}$$

TABLE I. Zero-dimensional anharmonic oscillator.

g^2	Exact	Tree Tree/Exact	One loop (One loop)/exact	Two loop (Two loop)/exact	Three loop (Three loop)/exact
10^{-3}	0.9996	0.9999 1.00	0.9996 1	0.9996 1	0.9996 1
10^{-2}	0.9963	0.9988 1.00	0.9963 1	0.9963 1	0.9963 1
10^{-1}	0.9685	0.9881 1.02	0.9664 1.00	0.9690 1.00	0.9683 1.00
1	0.8386	0.9149 1.1	0.8125 0.97	0.8541 1.02	0.8277 0.99
10	0.5954	0.7027 1.18	0.5484 0.92	0.6195 1.04	0.5976 1.00
10^2	0.3672	0.4510 1.23	0.3300 0.90	0.3817 1.04	0.3790 1.03
10^3	0.2131	0.2656 1.25	0.1899 0.89	0.2210 1.037	0.2222 1.043

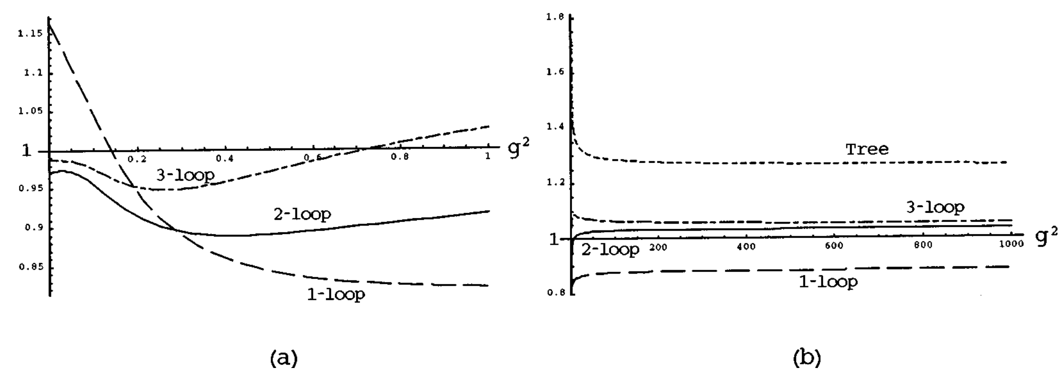

FIG. 2. Zero-dimensional double-well case. (a) $g^2 \leqslant 1$. (b) $g^2 > 1$. Dotted line represents $I_{\text{tree}}/\text{exact}$, which is omitted in (a), because of a large deviation. Dashed lines represent $I_{\text{1-loop}}/\text{exact}$, solid lines $I_{\text{2-loop}}/\text{exact}$, and dash-dotted lines $I_{\text{3-loop}}/\text{exact}$, respectively.

where $x(T)=x(0)$ designates the periodic boundary condition. Here (and hereafter) we put $\hbar \mapsto 1$, and use the continuous representation

$$\mathcal{D}x \equiv \lim_{N\to\infty} \prod_{j=1}^{N} \frac{dx_j}{\sqrt{2\pi\Delta t}}, \quad \Delta t \equiv \frac{T}{N}. \tag{3.3}$$

Introducing an auxiliary field in terms of the Gaussian identity

$$1 = \int \mathcal{D}y \exp\left[-\int_0^T dt \frac{1}{2}\left(y + \frac{igx^2}{2} \right)^2 \right], \tag{3.4}$$

so as to cancel the quartic term, we obtain

$$Z(T) = \int \mathcal{D}x\mathcal{D}y \exp\left[-\int_0^T dt \left(\frac{\dot{x}^2}{2} + (\omega^2 + igy)\frac{x^2}{2} + \frac{y^2}{2} \right) \right], \tag{3.5}$$

which, after integration with respect to x, becomes

$$Z(T) = \int \mathcal{D}y \exp\left[-\int_0^T dt \frac{y^2}{2} + \frac{1}{2} \ln\det\left(-\frac{d^2}{dt^2} + \omega^2 + igy \right) \right]$$
$$\equiv \int \mathcal{D}y \exp\left(-\frac{S[y]}{a} \right)\bigg|_{a=1}, \tag{3.6}$$

where

$$S[y] \equiv \int_0^T dt \frac{y^2}{2} + \frac{1}{2} \ln\det\left(-\frac{d^2}{dt^2} + \omega^2 + igy \right), \tag{3.7}$$

and, again, the loop-expansion parameter a has been introduced.

Denoting by $y_0(t)$ the solution of the equation of motion, $S'[y] = \delta S[y]/\delta y(t) = 0$, we obtain the gap equation

$$y_0(t) + \frac{ig}{2} G(t,t) = 0, \tag{3.8}$$

which can be rewritten as

$$\Omega(t)^2 - \omega^2 = \frac{g^2}{2} G(t,t), \tag{3.9}$$

TABLE II. Zero-dimensional double well.

g^2	Exact	Tree Tree/exact	One loop (One loop)/exact	Two loop (Two loop)/exact	Three loop (Three loop)/exact
10^{-3}	1.986×10^{217}	1.035×10^{219} 52.1	2.313×10^{217} 1.17	1.930×10^{217} 0.97	1.961×10^{217} 0.98
10^{21}	7.360×10^{21}	1.210×10^{23} 16.44	8.495×10^{21} 1.15	7.162×10^{21} 0.97	7.270×10^{21} 0.98
10^{-1}	2.208×10^{2}	1.107×10^{3} 5.01	2.311×10^{2} 1.06	2.112×10^{2} 0.96	2.156×10^{2} 0.98
1	2.350	4.202 1.79	1.932 0.82	2.155 0.92	2.411 1.03
10	0.8074	1.103 1.37	0.6897 0.85	0.8137 1.01	0.8768 1.09
10^2	0.4040	0.5196 1.29	0.3542 0.88	0.4159 1.03	0.4295 1.06
10^3	0.2196	0.2777 1.26	0.1942 0.88	0.2270 1.03	0.2312 1.05

where

$$\Omega(t)^2 \equiv \omega^2 + igy_0(t). \tag{3.10}$$

Here the Green's function $G(t,t')$ obeys the inhomogeneous equation

$$\left(-\frac{d^2}{dt^2}+\Omega(t)^2\right)G(t,t')=\delta(t-t'). \tag{3.11}$$

Again it should be noted that

$$\Omega(t)^2>0, \tag{3.12}$$

due to the existence of the Gaussian integration of x in Eq. (3.5).

Now expand $S[y]$ around y_0 [or $\Omega(t)^2$] such that

$$S[y]=S_0+\frac{1}{2}(y-y_0)^2 S_0^{(2)}+\frac{1}{3!}(y-y_0)^3 S_0^{(3)}+\cdots, \tag{3.13}$$

where we have used the abbreviations

$$S_0^{(n)} \equiv \left.\frac{\delta^n S}{\delta y(t_1)\,\delta y(t_2)\cdots\delta y(t_n)}\right|_{y=y_0} \tag{3.14}$$

and the notation

$$(y-y_0)^n S_0^{(n)} \equiv \int dt_1 dt_2\cdots dt_n (y-y_0) \times(t_1)(y-y_0)(t_2)\cdots(y-y_0) \times(t_n)\left.\frac{\delta^n S}{\delta y(t_1)\,\delta y(t_2)\cdots\delta y(t_n)}\right|_{y=y_0}. \tag{3.15}$$

Shifting and scaling the integration variables as before, we obtain

$$Z(T)=e^{-S_0/a}\int \mathcal{D}y\,\exp\left[-\frac{1}{2}\Delta^{-1}y^2 -\sqrt{a}\,\frac{y^3}{3!}S_0^{(3)}-a\frac{y^4}{4!}S_0^{(4)}-\cdots\right]\Bigg|_{a=1}, \tag{3.16}$$

where we have written $S_0^{(2)}\mapsto\Delta^{-1}$ which reads, explicitly,

$$\Delta^{-1}(t_1,t_2)\equiv\left.\frac{\delta^2 S}{\delta y(t_1)\,\delta y(t_2)}\right|_{y=y_0}=\delta(t_1-t_2) +\frac{g^2}{2}G(t_1,t_2)G(t_2,t_1). \tag{3.17}$$

Moreover,

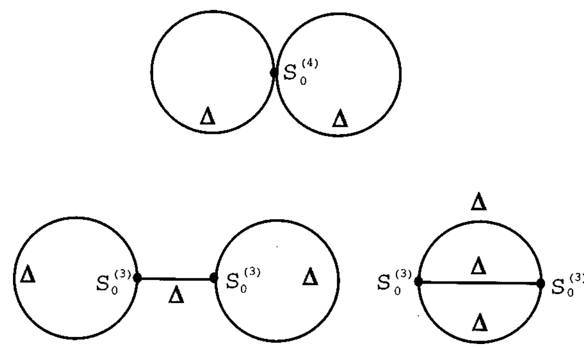

FIG. 3. Formal 2-loop graphs. Δ denotes the propagator of the auxiliary field.

$$S_0^{(3)}=-\frac{ig^3}{2}\{G(t_1,t_2)G(t_2,t_3)G(t_3,t_1) +G(t_1,t_3)G(t_3,t_2)G(t_2,t_1)\},$$

$$S_0^{(4)}=-g^4\{G(t_1,t_2)G(t_2,t_3)G(t_3,t_4)G(t_4,t_1) +G(t_1,t_2)G(t_2,t_4)G(t_4,t_3)G(t_3,t_1) +G(t_1,t_3)G(t_3,t_2)G(t_2,t_4)G(t_4,t_1)\}. \tag{3.18}$$

From these we have

$$Z(T)_{\text{tree}}\equiv\exp(-S_0),$$

$$Z(T)_{\text{1-loop}}\equiv\exp\left(-S_0-\frac{1}{2}\ln\det\Delta\right),$$

$$Z(T)_{\text{2-loop}}\equiv\exp\left(-S_0-\frac{1}{2}\ln\det\Delta\right)\times[1+(\text{2-loop graphs})], \tag{3.19}$$

where the two-loop graphs are formally given by Fig. 3.

The rest of the task is to fix the form of the Green's function [solution to Eq. (3.11)], and find the solution $y_0(t)$ of the gap equation (3.9). In this paper we confine ourselves to a time-independent solution, denoted by an overbar,

$$y_0(t)\mapsto\overline{y_0}:\text{const}, \quad \Omega(t)^2\mapsto\overline{\Omega}^2:\text{const}. \tag{3.20}$$

The Green's function, the solution to Eq. (3.11), can be obtained explicitly:

$$\overline{G}(t,t';\overline{\Omega})\equiv\frac{1}{T}\sum_{r=-\infty}^{\infty}\frac{e^{i2\pi r(t-t')/T}}{\left(\frac{2\pi r}{T}\right)^2+\overline{\Omega}^2} =\frac{1}{2\overline{\Omega}\sinh\frac{\overline{\Omega}T}{2}}\left\{\theta(t-t')\cosh\overline{\Omega}\left(\frac{T}{2}-t+t'\right) +\theta(t'-t)\cosh\overline{\Omega}\left(\frac{T}{2}+t-t'\right)\right\}, \tag{3.21}$$

where we have taken the periodic boundary condition into account.

Now $\overline{\Omega}$ is the solution of the gap equation:

$$ig\overline{y_0}=\overline{\Omega}^2-\omega^2=\frac{g^2}{2}\overline{G}(t,t;\overline{\Omega})=\frac{g^2}{4\overline{\Omega}}\coth\left(\frac{\overline{\Omega}T}{2}\right). \tag{3.22}$$

When $T\to$ large, $\overline{\Omega}$ can be expressed as

$$\overline{\Omega}=\Omega_0+\Omega_1 e^{-\Omega_0 T}+\Omega_2 e^{-2\Omega_0 T}+\cdots, \tag{3.23}$$

where Ω_0 is the solution of the third degree equation:

$$\Omega_0^3-\omega^2\Omega_0=\frac{g^2}{4}. \tag{3.24}$$

As an aside, we remark that in order to calculate the energy of the first excited state, Ω_1 must be known, which is easily obtained to be

$$\Omega_1=\frac{g^2}{2(3\Omega_0^2+\omega^2)}. \tag{3.25}$$

In this paper, however, only the ground state energy is considered. Other barred quantities are found straightforwardly, in particular

$$\overline{\Delta}(t,t')=\left(\left.\frac{\delta^2 S}{\delta y(t)\,\delta y(t')}\right|_{y=\overline{y_0}}\right)^{-1}$$
$$=\delta(t-t')-\frac{g^2}{2\overline{\Omega}}\overline{G}(t,t';\hat{\Omega}), \tag{3.26}$$

where

$$\hat{\Omega}^2\equiv 4\overline{\Omega}^2+\frac{g^2}{2\overline{\Omega}}. \tag{3.27}$$

The tree part in Eq. (3.19) becomes

$$\overline{Z}(T)_{\text{tree}}\equiv\exp(-\overline{S_0}), \tag{3.28}$$

with

$$\overline{S_0}=T\frac{\overline{y_0}^2}{2}+\frac{1}{2}\sum_{r=-\infty}^{\infty}\ln\left\{\left(\frac{2\pi r}{T}\right)^2+\overline{\Omega}^2\right\}$$
$$=-\frac{T}{2g^2}(\overline{\Omega}^2-\omega^2)^2+\ln\sinh\left(\frac{\overline{\Omega}T}{2}\right)$$
$$+(\overline{\Omega}-\text{independent part}), \tag{3.29}$$

where the gap equation (3.22) has been utilized in the first term in the final expression. In the one-loop part of Eq. (3.19), we need to know $\ln\det\overline{\Delta}(t,t')$ which is

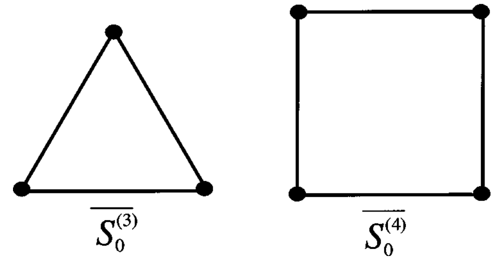

FIG. 4. Vertices for the constant classical solution. The solid lines represent $\overline{G}(t,t';\overline{\Omega})$ and the dots represent the vertices which should be attached by the auxiliary field propagator, $\overline{\Delta}(t,t')$, given by the double line. (See Fig. 5.)

$$\frac{1}{2}\ln\det\overline{\Delta}(t,t')$$
$$=\frac{1}{2}\sum_{r=-\infty}^{\infty}\ln\left\{1+\frac{g^2T^2}{8\pi^2\overline{\Omega}}\frac{1}{r^2+(\overline{\Omega}T/\pi)^2}\right\}$$
$$=\frac{1}{2}\sum_{r=-\infty}^{\infty}\left\{\ln\left[r^2+\left(\frac{\hat{\Omega}T}{2\pi}\right)^2\right]-\ln\left[r^2+\left(\frac{\overline{\Omega}T}{\pi}\right)^2\right]\right\}$$
$$=\ln\left(\frac{\sinh(\hat{\Omega}T/2)}{\sinh(\overline{\Omega}T)}\right). \tag{3.30}$$

As for the two-loop part, $\overline{S_0^{(3)}}$ and $\overline{S_0^{(4)}}$ are now expressed as in Fig. 4. Accordingly, the two-loop part is shown in Fig. 5.

From the graphs, we note that one needs the three- and four-loop calculations in the two loops of the auxiliary field, since our vertices $\overline{S_0^{(3)}}$ and $\overline{S_0^{(4)}}$ are nonlocal. As a result of this complexity, we confine ourselves to the case that $T\mapsto\infty$, that is, to the ground state. Write the Fourier transformations of $\overline{G}$ and $\overline{\Delta}$ such that

$$\overline{G}(t,t';\Omega_0)=\int_{-\infty}^{\infty}\frac{dk}{2\pi}e^{-ik(t-t')}\frac{1}{k^2+\Omega_0^2}$$
$$\equiv\int_{-\infty}^{\infty}\frac{dk}{2\pi}e^{-ik(t-t')}G_0(k),$$

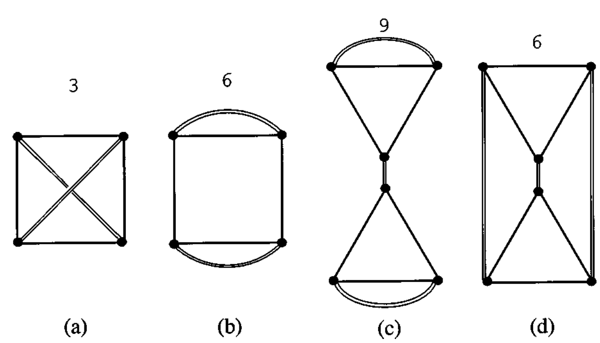

FIG. 5. Two-loop graphs for the constant classical solution. The numbers upper in the figures represent those of multiplicity.

$$\overline{\Delta}(t,t')=\int_{-\infty}^{\infty}\frac{dk}{2\pi}e^{-ik(t-t')}\frac{k^2+4\Omega_0^2}{k^2+\hat{\Omega}^2}\equiv\int_{-\infty}^{\infty}\frac{dk}{2\pi}e^{-ik(t-t')}\Delta_0(k), \tag{3.31}$$

where

$$\hat{\Omega}^2\equiv 4\Omega_0^2+\frac{g^2}{2\Omega_0}, \tag{3.32}$$

since $\overline{\Omega}$ is now Ω_0 under $T\to\infty$. With this notation, each graph Figs. 5(a)–5(d) can be calculated and expressed as follows:

$$(\mathrm{a})=\frac{g^4}{8}\int\frac{dldpdk}{(2\pi)^3}\Delta_0(p)\Delta_0(k)G_0(l)\times G_0(l+k)G_0(l+p)G_0(l+k+p)=\frac{g^4}{64\Omega_0^5}\frac{\rho+10}{\rho^2(\rho+1)(\rho+2)}, \tag{3.33}$$

$$(\mathrm{b})=\frac{g^4}{4}\int\frac{dldpdk}{(2\pi)^3}\Delta_0(p)\Delta_0(k)\times G_0(l)^2G_0(l+k)G_0(l+p)=\frac{g^4}{64\Omega_0^5}\frac{\rho+10}{\rho^2(\rho+1)}, \tag{3.34}$$

$$(\mathrm{c})=-\frac{g^6}{8}\Delta_0(0)\left[\int\frac{dldp}{(2\pi)^2}\Delta_0(p)G_0(l)^2G_0(l+p)\right]^2=-\frac{g^6}{128\Omega_0^8}\frac{(\rho+6)^2}{\rho^4(\rho+2)^2}, \tag{3.35}$$

$$(\mathrm{d})=-\frac{g^6}{12}\int\frac{dldpdkdq}{(2\pi)^4}\Delta_0(p)\Delta_0(q)\Delta_0(p+q)\times G_0(l)G_0(l+p)G_0(l+p+q)\times G_0(k)G_0(k+q)G_0(k+q+p)=-\frac{g^6}{64\Omega_0^8}\frac{(\rho^2+8\rho+4)}{\rho^4(\rho+1)(\rho+2)^2}. \tag{3.36}$$

Here we have introduced a parameter

$$\rho\equiv\sqrt{\frac{\hat{\Omega}^2}{\Omega_0^2}}. \tag{3.37}$$

The ground state energy

$$E_0=-\lim_{T\to\infty}\frac{1}{T}\ln Z(T) \tag{3.38}$$

is therefore

$$E_0^{\text{tree}}=\frac{\Omega_0}{2}-\frac{g^2}{32\Omega_0^2}, \tag{3.39}$$

$$E_0^{1-\text{loop}}=\frac{\Omega_0}{2}(\rho-1)-\frac{g^2}{32\Omega_0^2}, \tag{3.40}$$

$$E_0^{2-\text{loop}}=\frac{\Omega_0}{2}(\rho-1)-\frac{g^2}{32\Omega_0^2}-\frac{g^4}{64\Omega_0^5}\frac{(\rho+3)(\rho+10)}{\rho^2(\rho+1)(\rho+2)}+\frac{g^6}{128\Omega_0^8}\frac{\rho^3+15\rho^2+64\rho+44}{\rho^4(\rho+1)(\rho+2)^2}. \tag{3.41}$$

Let us analyze the following two cases.

Case (i): the anharmonic oscillator. The solution of Eq. (3.24) is given [13] by

$$\Omega_0=\begin{cases}\dfrac{2\omega}{\sqrt{3}}\cos\left[\dfrac{1}{3}\cos^{-1}\left(\dfrac{3\sqrt{3}g^2}{8\omega^3}\right)\right], & 0\leqslant\dfrac{g^2}{8}\leqslant\dfrac{\omega^3}{3\sqrt{3}},\\[2ex] \sqrt[3]{\dfrac{g^2}{8}+\sqrt{\dfrac{g^4}{64}-\dfrac{\omega^6}{27}}}+\sqrt[3]{3\dfrac{g^2}{8}-\sqrt{\dfrac{g^4}{64}-\dfrac{\omega^6}{27}}}, & \dfrac{\omega^3}{3\sqrt{3}}\leqslant\dfrac{g^2}{8}.\end{cases} \tag{3.42}$$

Putting $\omega^2\mapsto 1$ we calculate the ratio of $E_0^{L\text{-loop}}(L=0,1,2)$ to the exact numerical value in Table III.

Case (ii): the double well. The solution of Eq. (3.24) is

$$\Omega_0=\sqrt[3]{\sqrt{\frac{g^4}{64}+\frac{|\omega^2|^3}{27}}+\frac{g^2}{8}}-\sqrt[3]{\sqrt{\frac{g^4}{64}+\frac{|\omega^2|^3}{27}}-\frac{g^2}{8}}. \tag{3.43}$$

Putting $\omega^2\mapsto -1$ we again compare the result to the exact numerical value in Table IV.

For the anharmonic oscillator case, the auxiliary field method can fit the data within a 13% error in the one-loop expansion and a 3% error in the two-loop expansion, which we consider to be excellent. For the double-well case, the method gives us a $\sim$10% error except in the region $O(10^{-1})<g^2<O(1)$, where as in the zero-dimensional case, there might be a need for a three-loop correction to

TABLE III. The anharmonic oscillator.

$g^2/8$	Exact	Tree Tree/exact	One loop (One loop)/exact	Two loop (Two loop)/exact
10^{-3}	0.50075	0.50025 0.9990	0.50075 1	0.50075 1
10^{-2}	0.50726	0.50248 0.9906	0.50737 1.0002	0.50725 0.9999
10^{-1}	0.55915	0.52290 0.9352	0.56435 1.009	0.55775 0.9975
1	0.80377	0.65268 0.8038	0.85522 1.0640	0.78548 0.9772
10	1.5050	1.1080 0.7362	1.6729 1.1116	1.4738 0.9793
10^2	3.1314	2.2356 0.7139	3.5280 1.1267	3.0865 0.9857
10^3	6.6942	4.7445 0.7088	7.5659 1.1302	6.6112 0.9876

TABLE IV. The double well.

$g^2/8$	Exact	Tree Tree/exact	One loop (One loop)/exact	Two loop (Two loop)/exact
10^{-3}	−61.794	−62.500 1.0114	−61.794 1	−61.794 1
10^{-2}	−5.5532	−6.245 1.1246	−5.5575 1.0008	−5.5541 1.0002
10^{-1}	−0.15413	−0.57593 3.7368	−0.02326 0.1509	−0.08479 0.5501
1	0.51478	0.25 0.4856	0.66421 1.2903	0.41605 0.8082
10	1.3716	0.92366 0.6734	1.5839 1.1548	1.2112 0.8830
10^2	3.0695	2.1501 0.7005	3.4867 1.1359	2.7543 0.8973
10^3	6.6655	4.7048 0.7059	7.5467 1.1322	6.0004 0.9002

improve the result. Apart from this, it would be still a good approximation for a very large coupling region.

IV. DISCUSSION

The auxiliary field method, defined by the introduction of an auxiliary field and by utilizing its loop expansion, can give excellent results for the large coupling region $O(10^{-3})<g^2<O(10^3)$; even a component of the original variable is single. However, in the quantum double-well case, there needs to be higher order corrections than the two-loop correction in the region $O(10^{-2})<g^2<O(1)$. A maximum deviation in the ground state energy of a two-loop calculation reaches a value 18 times the exact value and has the wrong sign at $g^2\sim 0.15$. We have calculated the first excited energy E_1 up to one loop,

$$\Delta E \equiv E_1^{\text{1-loop}} - E_0^{\text{1-loop}} = \frac{2\Omega_0^3}{3\Omega_0^2+\omega^2}\left(1-\frac{g^2}{8\Omega_0^3}+\frac{3g^2}{4\sqrt{2}\Omega_0^2}\sqrt{\frac{1}{3\Omega_0^2+\omega^2}}\right), \tag{4.1}$$

and found a level crossing around this region of the coupling constant. Apparently, for this coupling, the approximation breaks down. However, it is cumbersome to go beyond the one-loop calculation in quantum field theory as well as in quantum mechanics. The approximation scheme should be simple and transparent. We therefore look for another solution rather than a time-independent solution; that is, we must solve Eq. (3.11) more carefully. The structure of the dominant contribution to the path integral has recently been clarified by means of approximations such as the valley method [14]. With these in mind, work in this direction is in progress.

As for applications, the formula is applicable almost to any situation. Our interest is the dynamical structure of QCD, which has recently been revealed through consideration of gauge invariance by Lavelle and McMullan [15] and others [16], for example. It is thus tempting to introduce this method into QCD, which is also our future program.

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