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Super Yang-Mills theory at weak, intermediate, and strong coupling

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We consider three dimensional SU(N) $\mathcal{N}=1$ super-Yang-Mills theory compactified on the space-time $\mathbf{R}\times S^1\times S^1$. In particular, we compactify the light-cone coordinate x^- on a light-like circle via DLCQ, and wrap the remaining transverse coordinate $x^\perp$ on a spatial circle. By retaining only the first few excited modes in the transverse direction, we are able to solve for bound state wave functions and masses numerically by diagonalizing the discretized light-cone supercharge. This regularization of the theory is shown to preserve supersymmetry. We plot bound state masses as a function of the coupling, showing the transition in particle masses as we move from a weakly to a strongly coupled theory. We analyze both numerically and analytically massless states which exist only in the limit of strong or weak gauge coupling. In addition, we find massless states that persist for all values of the gauge coupling. An analytical treatment of these massless states is provided. Interestingly, in the strong coupling limit, these massless states become string-like. [S0556-2821(99)03608-5]

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I. INTRODUCTION

An outstanding challenge in quantum field theory is solving non-Abelian gauge theories at intermediate and strong coupling. Recently, there has been considerable progress in understanding the properties of strongly coupled gauge theories with supersymmetry [1–3]. In particular, there are a number of supersymmetric gauge theories that are believed to be inter-connected through a web of strong-weak coupling dualities. Although existing evidence for these dualities is encouraging, there is still an urgent need to address these issues at a more fundamental level. Ideally, we would like to solve for the bound states of these theories directly and at any coupling.

Of course, solving a field theory from first principles is typically an intractable task. Nevertheless, it has been known for some time that $1+1$ dimensional field theories *can* be solved from first principles via a straightforward application of discrete light cone quantization (DLCQ) (see [4] for a review). In more recent times, a large class of supersymmetric gauge theories in two dimensions was studied using a supersymmetric form of DLCQ (SDLCQ), which is known to preserve supersymmetry [5–11].

Evidently, it would be desirable to extend these DLCQ or SDLCQ algorithms to solve higher dimensional theories. One important difference between two dimensional and higher dimensional theories is the phase diagram induced by variations in the gauge coupling. The spectrum of a $1+1$ dimensional gauge theory scales trivially with respect to the gauge coupling, while a theory in higher dimensions has the potential of exhibiting a complex phase structure, which may include a strong-weak coupling duality. It is therefore interesting to study the phase diagram of gauge theories in $D\geqslant 3$ dimensions.

Towards this end, we consider three dimensional SU(N) $\mathcal{N}=1$ super-Yang-Mills theory compactified on the space-time $\mathbf{R}\times S^1\times S^1$. In particular, we compactify the light-cone coordinate x^- on a light-like circle via DLCQ, and wrap the remaining transverse coordinate $x^\perp$ on a spatial circle. By retaining only the first few excited modes in the transverse direction, we are able to solve for bound state wave functions and masses numerically by diagonalizing the discretized light-cone supercharge. We show that the supersymmetric formulation of the DLCQ procedure — which was studied in the context of two dimensional theories [5,11] — extends naturally in $2+1$ dimensions, resulting in an exactly supersymmetric spectrum.

The contents of this paper are organized as follows. In Sec. II, we formulate SU(N) $\mathcal{N}=1$ super-Yang-Mills theory defined on the compactified space-time $\mathbf{R}\times S^1\times S^1$. Explicit expressions are given for the light-cone supercharges, which are then discretized via the SDLCQ procedure. Quantization of the theory is then carried out by imposing canonical (anti)commutation relations for boson and fermion fields. In Sec. III, we present the results of our numerical diagonalizations by plotting bound state masses as a function of gauge coupling. We also study the bound state structure of the massless states in the theory. In Sec. IV, we provide an analytical treatment of certain massless states in the theory, and discuss the appearance of new massless states at strong coupling. We conclude our analysis with a discussion of our results in Sec. V.

II. LIGHT-CONE QUANTIZATION AND SDLCQ

We wish to study the bound states of $\mathcal{N}=1$ super-Yang-Mills theory in $2+1$ dimensions. Any numerical approach necessarily involves introducing a momentum lattice — i.e. parton momenta can only take on discretized values. The usual space-time lattice explicitly breaks supersymmetry; so if we wish to discretize our theory *and* preserve supersymmetry, then a judicious choice of lattice is required.

In $1+1$ dimensions, it is well known that the light-cone momentum lattice induced by the DLCQ procedure preserves supersymmetry if the supercharge rather than the Hamiltonian is discretized [5,11]. In $2+1$ dimensions, a supersymmetric prescription is also possible. We begin by introducing light-cone coordinates $x^\pm=(x^0\pm x^1)/\sqrt{2}$, and compactifying the x^- coordinate on a light-like circle. In this way, the conjugate light-cone momentum k^+ is discretized. To discretize the remaining (transverse) momentum $k^\perp = k^2$, we may compactify $x^\perp = x^2$ on a spatial circle. Of course, there is a significant difference between the discretized light-cone momenta k^+, and discretized transverse momenta $k_\perp$; namely, the light-cone momentum k^+ is always positive,[1] while $k_\perp$ may take on positive or negative values. The positivity of k^+ is a key property that is exploited in DLCQ calculations; for any given light-cone compactification, there are only a finite number of choices for k^+ — the total number depending on how finely we discretize the momenta.[2] In the context of two dimensional theories, this implies a finite number of Fock states [12].

In the case we are interested in — in which there is an additional transverse dimension — the number of Fock states is no longer finite, since there are an arbitrarily large number of transverse momentum modes defined on the transverse spatial circle. Thus, an additional truncation of the transverse momentum modes is required to render the total number of Fock states finite, and the problem numerically tractable.[3] In this work, we choose the simplest truncation procedure beyond retaining the zero mode; namely, only partons with transverse momentum $k_\perp = 0, \pm 2\pi/L$ will be allowed, where L is the size of the transverse circle.

Let us now apply these ideas in the context of a specific super-Yang-Mills theory. We start with $2+1$ dimensional $\mathcal{N}=1$ super-Yang-Mills theory defined on a space-time with one transverse dimension compactified on a circle:

$$S = \int d^2x \int_0^L dx_\perp \, \mathrm{tr}\left(-\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \mathrm{i}\bar{\Psi}\gamma^\mu D_\mu \Psi \right). \tag{1}$$

After introducing the light-cone coordinates $x^\pm = (1/\sqrt{2})(x^0 \pm x^1)$, decomposing the spinor Ψ in terms of chiral projections,

$$\psi = \frac{1+\gamma^5}{2^{1/4}}\Psi, \quad \chi = \frac{1-\gamma^5}{2^{1/4}}\Psi, \tag{2}$$

and choosing the light-cone gauge $A^+ = 0$, the action becomes

$$S = \int dx^+ dx^- \int_0^L dx_\perp \, \mathrm{tr}\left[\frac{1}{2}(\partial_- A^-)^2 + D_+\phi\,\partial_-\phi + \mathrm{i}\psi D_+\psi + \mathrm{i}\chi\partial_-\chi + \frac{\mathrm{i}}{\sqrt{2}}\psi D_\perp\phi + \frac{\mathrm{i}}{\sqrt{2}}\phi D_\perp\psi \right]. \tag{3}$$

A simplification of the light-cone gauge is that the non-dynamical fields A^- and χ may be explicitly solved from their Euler-Lagrange equations of motion:

$$A^- = \frac{g}{\partial_-^2} J = \frac{g}{\partial_-^2}(i[\phi, \partial_-\phi] + 2\psi\psi),$$

$$\chi = -\frac{1}{\sqrt{2}\partial_-} D_\perp \psi. \tag{4}$$

These expressions may be used to express any operator in terms of the physical degrees of freedom only. In particular, the light-cone energy, P^-, and momentum operators, $P^+, P^\perp$, corresponding to translation invariance in each of the coordinates $x^\pm$ and $x_\perp$ may be calculated explicitly:

$$P^+ = \int dx^- \int_0^L dx_\perp \, \mathrm{tr}[(\partial_-\phi)^2 + \mathrm{i}\psi\partial_-\psi], \tag{5}$$

$$P^- = \int dx^- \int_0^L dx_\perp \, \mathrm{tr}\left[-\frac{g^2}{2} J \frac{1}{\partial_-^2} J - \frac{\mathrm{i}}{2} D_\perp\psi \frac{1}{\partial_-} D_\perp\psi \right], \tag{6}$$

$$P_\perp = \int dx^- \int_0^L dx_\perp \, \mathrm{tr}[\partial_-\phi\,\partial_\perp\phi + \mathrm{i}\psi\partial_\perp\psi]. \tag{7}$$

The light-cone supercharge in this theory is a two component Majorana spinor, and may be conveniently decomposed in terms of its chiral projections:

$$Q^+ = 2^{1/4} \int dx^- \int_0^L dx_\perp \, \mathrm{tr}[\phi\partial_-\psi - \psi\partial_-\phi], \tag{8}$$

$$Q^- = 2^{3/4} \int dx^- \int_0^L dx_\perp \, \mathrm{tr}\left[2\partial_\perp\phi\,\psi + g(\mathrm{i}[\phi, \partial_-\phi] + 2\psi\psi)\frac{1}{\partial_-}\psi \right]. \tag{9}$$

The action (3) gives the following canonical (anti)commutation relations for propagating fields at equal x^+:

$$[\phi_{ij}(x^-, x_\perp), \partial_-\phi_{kl}(y^-, y_\perp)] = \frac{1}{2}\mathrm{i}\,\delta(x^- - y^-)\,\delta(x_\perp - y_\perp) \times \left(\delta_{il}\delta_{jk} - \frac{1}{N}\delta_{ij}\delta_{kl} \right), \tag{10}$$

$$\{\psi_{ij}(x^-, x_\perp), \psi_{kl}(y^-, y_\perp)\} = \frac{1}{2}\,\delta(x^- - y^-)\,\delta(x_\perp - y_\perp) \times \left(\delta_{il}\delta_{jk} - \frac{1}{N}\delta_{ij}\delta_{kl} \right). \tag{11}$$

Using these relations one can check the supersymmetry algebra:

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[1]Since we wish to consider the decompactified limit in the end, we omit zero modes. This is a necessary technical constraint in numerical calculations.

[2]The ''resolution'' of the discretization is usually characterized by a positive integer K, which is called the ''harmonic resolution'' [12,13]; for a given choice of K, the light-cone momenta k^+ are restricted to positive integer multiples of P^+/K, where P^+ is the total light-cone momentum of a state.

[3]This truncation procedure, which is characterized by some integer upper bound, is analogous to the truncation of k^+ imposed by the ''harmonic resolution'' K.

$$\{Q^+,Q^+\}=2\sqrt{2}P^+,\quad \{Q^-,Q^-\}=2\sqrt{2}P^-,$$
$$\{Q^+,Q^-\}=-4P_\perp. \tag{12}$$

We will consider only states which have vanishing transverse momentum, which is possible since the total transverse momentum operator is kinematical.[4] On such states, the light-cone supercharges Q^+ and Q^- anti-commute with each other, and the supersymmetry algebra is equivalent to the $\mathcal{N}=(1,1)$ supersymmetry of the dimensionally reduced (i.e., two dimensional) theory [5]. Moreover, in the $P_\perp=0$ sector, the mass squared operator M^2 is given by $M^2=2P^+P^-$.

As we mentioned earlier, in order to render the bound state equations numerically tractable, the transverse momentum of partons must be truncated. First, we introduce the Fourier expansion for the fields ϕ and ψ, where the transverse space-time coordinate $x^\perp$ is periodically identified:

$$\phi_{ij}(0,x^-,x_\perp)=\frac{1}{\sqrt{2\pi L}}\sum_{n^\perp=-\infty}^{\infty}\int_0^\infty\frac{dk^+}{\sqrt{2k^+}}[a_{ij}(k^+,n^\perp)e^{-\mathrm{i}k^+x^- -\mathrm{i}(2\pi n^\perp/L)x_\perp}+a^\dagger_{ji}(k^+,n^\perp)e^{\mathrm{i}k^+x^- +\mathrm{i}(2\pi n^\perp/L)x_\perp}],$$

$$\psi_{ij}(0,x^-,x_\perp)=\frac{1}{2\sqrt{\pi L}}\sum_{n^\perp=-\infty}^{\infty}\int_0^\infty dk^+[b_{ij}(k^+,n^\perp)e^{-\mathrm{i}k^+x^- -\mathrm{i}(2\pi n^\perp/L)x_\perp}+b^\dagger_{ji}(k^+,n^\perp)e^{\mathrm{i}k^+x^- +\mathrm{i}(2\pi n^\perp/L)x_\perp}].$$

Substituting these into the (anti)commutators (11), one finds

$$[a_{ij}(p^+,n_\perp),a^\dagger_{lk}(q^+,m_\perp)]=\delta(p^+-q^+)\,\delta_{n_\perp,m_\perp}\left(\delta_{il}\delta_{jk}-\frac{1}{N}\delta_{ij}\delta_{lk}\right), \tag{13}$$

$$\{b_{ij}(p^+,n_\perp),b^\dagger_{lk}(q^+,m_\perp)\}=\delta(p^+-q^+)\,\delta_{n_\perp,m_\perp}\left(\delta_{il}\delta_{jk}-\frac{1}{N}\delta_{ij}\delta_{lk}\right). \tag{14}$$

The supercharges now take the following form:

$$Q^+=\mathrm{i}2^{1/4}\sum_{n^\perp\in\mathbf{Z}}\int_0^\infty dk\sqrt{k}[b^\dagger_{ij}(k,n^\perp)a_{ij}(k,n^\perp)-a^\dagger_{ij}(k,n^\perp)b_{ij}(k,n^\perp)], \tag{15}$$

$$\begin{aligned}
Q^-=&\frac{2^{7/4}\pi\mathrm{i}}{L}\sum_{n^\perp\in\mathbf{Z}}\int_0^\infty dk\frac{n^\perp}{\sqrt{k}}[a^\dagger_{ij}(k,n^\perp)b_{ij}(k,n^\perp)-b^\dagger_{ij}(k,n^\perp)a_{ij}(k,n^\perp)]\\
&+\frac{\mathrm{i}2^{-1/4}g}{\sqrt{L\pi}}\sum_{n_i^\perp\in\mathbf{Z}}\int_0^\infty dk_1dk_2dk_3\,\delta(k_1+k_2-k_3)\,\delta_{n_1^\perp+n_2^\perp,n_3^\perp}\\
&\times\Bigg\{\frac{1}{2\sqrt{k_1k_2}}\frac{k_2-k_1}{k_3}[a^\dagger_{ik}(k_1,n_1^\perp)a^\dagger_{kj}(k_2,n_2^\perp)b_{ij}(k_3,n_3^\perp)-b^\dagger_{ij}(k_3,n_3^\perp)a_{ik}(k_1,n_1^\perp)a_{kj}(k_2,n_2^\perp)]\\
&+\frac{1}{2\sqrt{k_1k_3}}\frac{k_1+k_3}{k_2}[a^\dagger_{ik}(k_3,n_3^\perp)a_{kj}(k_1,n_1^\perp)b_{ij}(k_2,n_2^\perp)-a^\dagger_{ik}(k_1,n_1^\perp)b^\dagger_{kj}(k_2,n_2^\perp)a_{ij}(k_3,n_3^\perp)]\\
&+\frac{1}{2\sqrt{k_2k_3}}\frac{k_2+k_3}{k_1}[b^\dagger_{ik}(k_1,n_1^\perp)a^\dagger_{kj}(k_2,n_2^\perp)a_{ij}(k_3,n_3^\perp)-a^\dagger_{ij}(k_3,n_3^\perp)b_{ik}(k_1)a_{kj}(k_2,n_2^\perp)]\\
&+\left(\frac{1}{k_1}+\frac{1}{k_2}-\frac{1}{k_3}\right)[b^\dagger_{ik}(k_1,n_1^\perp)b^\dagger_{kj}(k_2,n_2^\perp)b_{ij}(k_3,n_3^\perp)+b^\dagger_{ij}(k_3,n_3^\perp)b_{ik}(k_1,n_1^\perp)b_{kj}(k_2,n_2^\perp)]\Bigg\}.
\end{aligned} \tag{16}$$

[4]Strictly speaking, on a transverse cylinder, there are separate sectors with total transverse momenta $2\pi n/L$; we consider only one of them, $n=0$.

We now perform the truncation procedure; namely, in all sums over the transverse momenta $n_i^\perp$ appearing in the above expressions for the supercharges, we restrict summation to the following allowed momentum modes: $n_i^\perp = 0, \pm 1$. More generally, the truncation procedure may be defined by $|n_i^\perp| \leqslant N_{max}$, where N_{max} is some positive integer. In this work, we consider the simple case $N_{max}=1$. Note that this prescription is symmetric, in the sense that there are as many positive modes as there are negative ones. In this way we retain parity symmetry in the transverse direction.

How does such a truncation affect the supersymmetry properties of the theory? Note first that an operator relation $[A,B]=C$ in the full theory is not expected to hold in the truncated formulation. However, if A is quadratic in terms of fields (or in terms of creation and annihilation operators), one can show that the relation $[A,B]=C$ implies

$$[A_{tr}, B_{tr}] = C_{tr}$$

for the truncated operators A_{tr}, B_{tr}, and C_{tr}. In our case, Q^+ is quadratic, and so the relations $\{Q^+_{tr}, Q^+_{tr}\} = 2\sqrt{2} P^+_{tr}$ and $\{Q^+_{tr}, Q^-_{tr}\} = 0$ are true in the $P_\perp = 0$ sector of the truncated theory. The $\{Q^-_{tr}, Q^-_{tr}\}$ however is not equal to $2\sqrt{2}P^-_{tr}$. So the diagonalization of $\{Q^-_{tr}, Q^-_{tr}\}$ will yield a different bound state spectrum than the one obtained after diagonalizing $2\sqrt{2}P^-_{tr}$. Of course the two spectra should agree in the limit $N_{max} \to \infty$. At any finite truncation, however, we have the liberty to diagonalize any one of these operators. This choice specifies our regularization scheme.

Choosing to diagonalize the light-cone supercharge, however, has an important advantage: *the spectrum is exactly supersymmetric for any truncation*. In contrast, the spectrum of the Hamiltonian becomes supersymmetric only in the $N_{max} \to \infty$ limit.[5]

To summarize, we have introduced a truncation procedure that facilitates a numerical study of the bound state problem, and preserves supersymmetry. The interesting property of the light-cone supercharge Q^- [Eq. (16)] is the presence of a gauge coupling constant as an independent variable, which does not appear in the study of two dimensional theories. In the next section, we will study how variations in this coupling affects the bound states in the theory.

III. NUMERICAL RESULTS: BOUND STATE SOLUTIONS

In order to implement the DLCQ formulation of the bound state problem — which is tantamount to imposing periodic boundary conditions $x^- = x^- + 2\pi R$ [13] — we simply restrict the light-cone momentum variables k_i appearing in the expressions for Q^+ and Q^- to the following discretized set of momenta: $\{(1/K)P^+, (2/K)P^+, (3/K)P^+, \ldots\}$. Here, P^+ denotes the total light-cone momentum of a state, and may be thought of as a fixed constant, since it is easy to form a Fock basis that is already diagonal with respect to the operator P^+ [12]. The integer K is called the ''harmonic resolution,'' and $1/K$ measures the coarseness of our discretization. The continuum limit is then recovered by taking the limit $K \to \infty$. Physically, $1/K$ represents the smallest positive unit of longitudinal momentum fraction allowed for each parton in a Fock state.

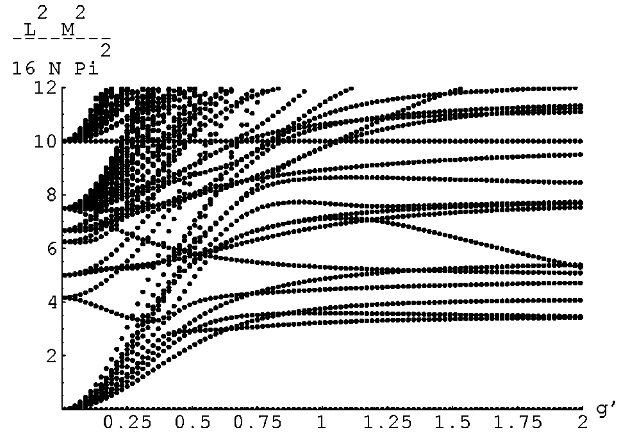

FIG. 1. Plot of bound state mass squared M^2 in units $16\pi^2 N/L^2$ as a function of the dimensionless coupling $0 \leqslant g' \leqslant 2$, defined by $(g')^2 = g^2 NL/16\pi^3$, at $N=1000$ and $K=5$. Boson and fermion masses are identical.

Of course, as soon as we implement the DLCQ procedure, which is specified unambiguously by the harmonic resolution K, and cutoff transverse momentum modes via the constraint $|n_i^\perp| \leqslant N_{max}$, the integrals appearing in the definitions for Q^+ and Q^- are replaced by finite sums, and so the eigenequation $2P^+P^-|\Psi\rangle = M^2|\Psi\rangle$ is reduced to a finite matrix diagonalization problem. In this last step we use the fact that P^- is proportional to the square of the light-cone supercharge[6] Q^-. In the present work, we are able to perform numerical diagonalizations for $K=2$, 3, 4 and 5 with the help of MATHEMATICA and a desktop PC. In Fig. 1, we plot the bound state mass squared M^2, in units $16\pi^2 N/L^2$, as a function of the dimensionless coupling $g' = g\sqrt{NL}/4\pi^{3/2}$, in the range $0 \leqslant g' \leqslant 2$. We consider the specific case $N = 1000$, although our algorithm can calculate masses for any choice of N, since it enters our calculations as an algebraic variable. Since there is an exact boson-fermion mass degeneracy, one needs only diagonalize the mass matrix M^2 corresponding to the bosons. For $K=5$, there are precisely 600 bosons and 600 fermions in the truncated light-cone Fock space; so the mass matrix that needs to be diagonalized has dimensions 600×600. At $K=4$, there are 92 bosons and 92 fermions, while at $K=3$, one finds 16 bosons and 16 fermions.

In Fig. 2, we plot the bound state spectrum in the range $0 \leqslant g' \leqslant 10$. It is apparent now that new massless states ap-

[5]If one chooses anti-periodic boundary conditions in the x^- coordinate for fermions, then there is no choice; one can only diagonalize the light-cone Hamiltonian. See [14] for more details on this approach.

[6]Strictly speaking, $P^- = (1/\sqrt{2})(Q^-)^2$ is an identity in the continuum theory and a *definition* in the compactified theory, corresponding to the SDLCQ prescription [5,11].

pear in the strong coupling limit $g' \to \infty$.

An interesting property of the spectrum is the presence of exactly massless states that persist for all values of the coupling g'. For $K=5$, there are 16 such states (8 bosons and 8 fermions). At $K=4$, one finds 8 states (4 bosons and 4 fermions) that are exactly massless for any coupling, while for $K=3$, there are 4 states (two bosons and two fermions) with this property. We will have more to say regarding these states in the next section, but here we note that the structure of these states become ''string-like'' in the strong coupling limit. This is illustrated in Fig. 3, where we plot the ''average length'' (or average number of partons) of each of these massless states.[7] This quantity is obtained by counting the number of partons in each Fock state that comprises a massless bound state, appropriately weighted by the modulus of the wave function squared. Clearly, at strong coupling, the average number of partons saturates the maximum possible value allowed by the resolution — in this case 5 partons. The same behavior is observed at lower resolutions. Thus, in the continuum limit $K \to \infty$, we expect the massless states in this theory to become string-like at strong coupling.

One interesting property of the model studied here is the manifest $\mathcal{N}=(1,1)$ supersymmetry in the $P^{\perp}=0$ momentum sector, by virtue of the supersymmetry relations (12). Moreover, if we consider retaining only the zero mode $n_i^{\perp}=0$, then the light-cone supercharge Q^- for the $2+1$ model is identical to the $1+1$ dimensional $\mathcal{N}=(1,1)$ supersymmetric Yang-Mills theory studied in [5,7,8], after a rescaling by the factor $1/g'$. (This is equivalent to expressing the mass squared M^2 in units $\tilde{g}^2 N/\pi$, where $\tilde{g}=g/\sqrt{L}$. The quantity $\tilde{g}$ is then identified as the gauge coupling in the $1+1$ theory.) We may therefore think of the additional transverse degrees of freedom in the $2+1$ model, represented by the modes $n^{\perp}=\pm 1$, as a modification of the $1+1$ model. A natural question that follows from this viewpoint is, how well does the $1+1$ spectrum approximate the $2+1$ spectrum after performing this rescaling? Before discussing the numerical results summarized in Table I, let us first attempt to predict what will happen at small coupling g'. In this case, the coefficients of terms in the rescaled Hamiltonian P^- that correspond to summing the transverse momentum squared $|k^{\perp}|^2$ of partons in a state will be large. So the low energy sector will be dominated by states with $n^{\perp}=0$, i.e., those states that appear in the Fock space of the $\mathcal{N}=(1,1)$ model in $1+1$ dimensions. This is indeed supported by the results in Table I.

For large coupling g', however, it is clear that the approximation breaks down. In fact, one can show that the tabulated masses in the rescaled $2+1$ model tend to zero in the strong coupling limit, which eliminates any scope for making comparisons between the two and three dimensional models.

Thus, the non-perturbative problem of solving dimensionally reduced models in $1+1$ dimensions can only provide information about bound state masses in the corresponding *weakly coupled* higher dimensional theory.

TABLE I. Values for the mass squared M^2, in units $\tilde{g}^2 N/\pi$, with $\tilde{g}^2=g^2/L$, for bound states in the dimensionally reduced $\mathcal{N}=(1,1)$ model, and the $2+1$ model studied here. The quantity $\tilde{g}$ is identified as the gauge coupling in the $1+1$ model. We set $K=3$, 4 and 5, and $N=1000$. Note that the comparison of masses between the $1+1$ model and the (re-scaled) $2+1$ model is good only at weak coupling g'.

	Comparison between $1+1$ and $2+1$ spectra			
-	$1+1$ model	Rescaled $2+1$ model		
K	-	$g'=.01$	$g'=.1$	$g'=1.0$
$K=5$	15.63	15.5	15.17	3.7
	18.23	17.6	17.9	3.5
	21.8	21.3	21.7	3.2
$K=4$	-	-	-	-
	18.0	17.99	17.6	3.56
	21.3	21.3	21.0	3.1
$K=3$	-	-	-	-
	-	-	-	-
	20.2	20.2	19.8	3.1

IV. ANALYTICAL RESULTS: THE MASSLESS SECTOR

In the previous section we presented the results of studying the bound state problem using numerical methods. In performing such a study we conveniently chose the simplest nontrivial truncation of the transverse momentum modes, namely, $n_{\perp}=0,\pm 1$. Surprisingly, such a simple scheme provided many interesting insights concerning the massless and massive sector. In particular we see that there are three types of massless states: those that are massless only at $g=0$ or $g=\infty$ (but not both) and those that are massless for any value of the coupling. In this section, we will analyze only the massless sector of the theory, and show that the observed properties of the spectrum with the truncation $n^{\perp}=0,\pm 1$, also persists if we include higher modes: $n^{\perp}=0,\pm 1, \pm 2,\ldots,\pm N_{max}$. We therefore consider the model with supercharges given by Eqs. (15) and (16), and restrict summation of transverse momentum modes via the constraint $|n^{\perp}| \leqslant N_{max}$.

For states carrying positive light-cone momentum, P^+ is never zero, and so massless states must satisfy the equation $P^-|\Psi\rangle=0$, which, using the relation $P^-=(1/\sqrt{2})(Q^-)^2$ and Hermiticity of Q^-, reduces to

$$Q^-|\Psi\rangle=0. \tag{17}$$

This is the equation we wish to study in detail.

We begin with an analysis of the weak coupling limit of the theory. This limit means that the dimensionless coupling constant is small: i.e. $g\sqrt{L} \ll 1$. We will consider the strong-weak coupling behavior of the theory on a cylinder with

[7]The ''noisiness'' in this plot for larger values of g' reflects the ambiguity of choosing a basis for the eigenspace, due to the exact mass degeneracy of the massless states.

fixed circumference L, and so it is convenient to choose the units in which $L=1$ for this discussion. The supercharge (16) consists of two parts: one is proportional to the coupling and the other is coupling independent:

$$Q^- = Q_\perp + g\tilde{Q}. \tag{18}$$

So at $g=0$ Eq. (17) reduces to $Q_\perp|\Psi\rangle = 0$, which means that $|\Psi\rangle$ may be viewed as a state in the Fock space of the two dimensional $\mathcal{N}=(1,1)$ super Yang-Mills theory, which may be obtained by dimensional reduction of the $2+1$ theory. Thus the massless states at $g=0$ are states with any combination of $a^\dagger(k,0)$ and $b^\dagger(k,0)$ modes, and no partons with nonzero transverse momentum.

What happens with these massless states when one switches on the coupling? To answer this question, we need some information about the behavior of states as functions of the coupling. We assume that wave functions are analytic in terms of g at least in the vicinity of $g=0$. This means that in this region any massless state $|\Psi\rangle$ may be written in the form

$$|\Psi\rangle = \sum_{n=0}^{\infty} g^n |n\rangle, \tag{19}$$

where states $|n\rangle$ are coupling independent. Then using relation (18), the g-dependent equation (17) may be written as an infinite system of relations between different $|n\rangle$:

$$Q_\perp|0\rangle = 0, \tag{20}$$

$$Q_\perp|n\rangle + \tilde{Q}|n-1\rangle = 0, \quad n>0. \tag{21}$$

The first of these equations was already used to exclude partons carrying non-zero transverse momentum, which is a property of the massless bound states at zero coupling. The second equation is non-trivial. Let us consider two different subspaces in the theory. The first of these subspaces consists of states with no creation operators for transverse modes which we will label 1. The other is the complement of this space in which the operator $Q_\perp$ is invertible and we label this space 2. Equation (21) defines the recurrence relation when $\tilde{Q}|n-1\rangle$ is in subspace 2:

$$|n\rangle = -Q_\perp^{-1}(\tilde{Q}|n-1\rangle|_2). \tag{22}$$

The consistency condition is that projection of $\tilde{Q}|n-1\rangle$ in subspace 1 is zero,

$$\tilde{Q}|n-1\rangle|_1 = 0. \tag{23}$$

This condition implies that not all states of the two dimensional theory, $g=0$, may be extended to such states at arbitrary g using Eq. (22). Taking $n=1$, Eq. (23) implies that $|0\rangle$ is a massless state of the dimensionally reduced theory. The numerical solutions, of course, show this correspondence between the $2+1$ and $1+1$ [5,7,8] massless bound states. Starting from a massless state of the two dimensional theory, and we construct states $|n\rangle$ using Eq. (22), and for which Eq. (23) is always satisfied. Then $|\Psi\rangle$ may be found from summing a geometric series:

$$|\Psi\rangle = \sum_{n=0}^{\infty} (-gQ_\perp^{-1}\tilde{Q})^n|0\rangle = \frac{1}{1+gQ_\perp^{-1}\tilde{Q}}|0\rangle. \tag{24}$$

So starting from the massless state of the two dimensional $\mathcal{N}=(1,1)$ model, one can always construct unique massless states in the three dimensional theory at least in the vicinity of $g=0$.

The state (24) turns out to be massless for any value of the coupling,

$$Q^-|\Psi\rangle = Q_\perp(1+gQ_\perp^{-1}\tilde{Q})\frac{1}{1+gQ_\perp^{-1}\tilde{Q}}|0\rangle = Q_\perp|0\rangle = 0, \tag{25}$$

though the state itself is dependent on g. Thus, we have shown that massless states of the three dimensional theory, at nonzero coupling, can be constructed from massless states of the corresponding model in two dimensions. All other states containing only two dimensional modes can also be extended to the eigenstates of the full theory. But such eigenstates are massless only at zero coupling. Assuming analyticity, one can then show that their masses grow linearly at g in the vicinity of zero. Such behavior also agrees with our numerical results.

To illustrate the general construction explained above we consider one simple example. Working in DLCQ at resolution $K=3$ we choose a special two dimensional massless state[8] [5,7,8]

$$|0\rangle = \mathrm{tr}[a^\dagger(1,0)a^\dagger(2,0)]|\mathrm{vac}\rangle. \tag{26}$$

Then in the SU(N) theory we find

$$\begin{aligned}\tilde{Q}|0\rangle = \frac{3}{2\sqrt{2}}\big(\mathrm{tr}\{a^\dagger(1,0)[b^\dagger(1,-1)a^\dagger(1,1)\\ -a^\dagger(1,1)b^\dagger(1,-1)+b^\dagger(1,1)a^\dagger(1,-1)\\ -a^\dagger(1,-1)b^\dagger(1,1)]\}\big)|\mathrm{vac}\rangle,\end{aligned} \tag{27}$$

$$\begin{aligned}|1\rangle = -Q_\perp^{-1}\tilde{Q}|0\rangle = -\frac{\sqrt{L}}{4\pi^{3/2}}\frac{3}{2\sqrt{2}}[a^\dagger(1,0)a^\dagger(1,-1)\\ \times a^\dagger(1,1) - a^\dagger(1,0)a^\dagger(1,1)a^\dagger(1,-1)]|\mathrm{vac}\rangle,\end{aligned} \tag{28}$$

$$\tilde{Q}|1\rangle = 0. \tag{29}$$

The last equation provides the consistency condition (23) for $n=2$, and it also shows that for this special example we have

[8]The state $|0\rangle$ denotes a massless state, while $|\mathrm{vac}\rangle$ represents the light-cone vacuum.

only two states $|0\rangle$ and $|1\rangle$, instead of a general infinite set. The matrix form of the operator $1+gQ_\perp^{-1}\tilde{Q}$ in the $|0\rangle,|1\rangle$ basis is

$$1+gQ_\perp^{-1}\tilde{Q}=\begin{pmatrix}1 & -g\\ 0 & 1\end{pmatrix}=\begin{pmatrix}1 & g\\ 0 & 1\end{pmatrix}^{-1}. \tag{30}$$

Then the solution of Eq. (24) is

$$|\Psi\rangle=|0\rangle+g|1\rangle=\mathrm{tr}[a^\dagger(1,0)a^\dagger(2,0)]|\mathrm{vac}\rangle + \frac{g\sqrt{L}}{4\pi^{3/2}}\frac{3}{2\sqrt{2}}[a^\dagger(1,0)a^\dagger(1,1)a^\dagger(1,-1) - a^\dagger(1,0)a^\dagger(1,-1)a^\dagger(1,1)]|\mathrm{vac}\rangle. \tag{31}$$

This state was observed numerically, and the dependence of the wave function on the coupling constant is precisely the one given by the last formula.

In principle, we can determine the wave functions of all massless states using this formalism. Our procedure has an important advantage over a direct diagonalization of the three dimensional supercharge. First, in order to find two dimensional massless states, one needs to diagonalize the corresponding supercharge [5]. However, the dimension of the relevant Fock space is much less than the three dimensional theory [at large resolution K, the ratio of these dimensions is of order $(N_{max}+1)^{\alpha K}$, $\alpha\sim 1/4$]. The extension of the two dimensional massless solution into a massless solution of the three dimensional theory requires diagonalizing a matrix which has a smaller dimension than the original problem in three dimensions. Thus, if one is only interested in the massless sector of the three dimensional theory, the most efficient way to proceed in DLCQ calculations is to solve the two dimensional theory, and then to upgrade the massless states to massless solutions in three dimensions.

Finally, we will make some comments on bound states at very strong coupling. Of course, we have states (24) which are massless at any coupling, but our numerical calculation shows there are additional states which become massless at $g=\infty$ (see Fig. 2). To discuss these states it is convenient to consider

$$\overline{Q^-}=\frac{1}{g}Q_\perp+\tilde{Q} \tag{32}$$

instead of Q^-, and perform the strong coupling expansion. Since we are interested only in massless states, the absolute normalization does not matter. We repeat all the arguments used in the weak coupling case: first, we introduce the space 1^* where $\tilde{Q}$ cannot be inverted and its orthogonal complement 2^*. Then any state from 1^* is massless at $g=\infty$, but assuming the expansion

$$|\Psi\rangle=\sum_{n=0}^{\infty}\frac{1}{g^n}|n\rangle^* \tag{33}$$

at large enough g, one finds the analogues of Eqs. (22) and (23):

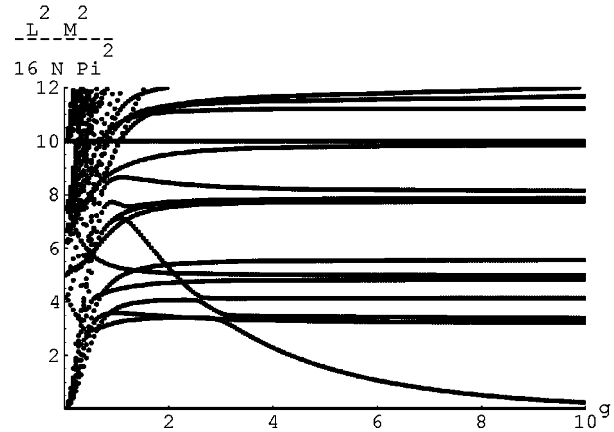

FIG. 2. Plot of bound state mass squared M^2 in units $16\pi^2N/L^2$ as a function of the dimensionless coupling $0\leqslant g'\leqslant 10$, defined by $(g')^2=g^2NL/16\pi^3$, at $N=1000$ and $K=5$. Note the appearance of a new massless state at strong coupling.

$$|n\rangle^*=-\tilde{Q}^{-1}(Q_\perp|n-1\rangle^*|_{2*}), \tag{34}$$

$$Q_\perp|n-1\rangle^*|_{1*}=0. \tag{35}$$

As in the small coupling case, there are two possibilities: either one can construct all states $|n\rangle^*$ satisfying the consistency conditions, or at least one of these conditions fails. The former case corresponds to the massless state in the vicinity of $g=\infty$, which can be extended to the massless states at all couplings. The states constructed in this way — and ones given by Eq. (24) — define the same subspace. In the latter case, the state is massless at $g=\infty$, but it acquires a mass at finite coupling. There is a big difference, however, between the weak and strong coupling cases. While the kernel of $Q_\perp$ consists of ''two dimensional'' states, the description of the states annihilated by $\tilde{Q}$ is a nontrivial dynamical problem. Since the massless states can be constructed starting from either $g=0$ or $g=\infty$, we do not have to solve this problem to build them. If, however, one wishes to show that massless states become long in the strong coupling limit (there is nu-

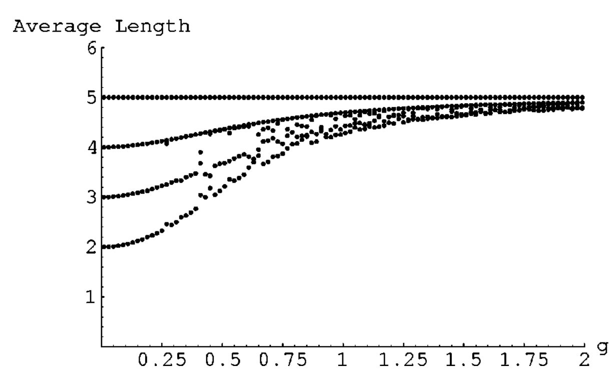

FIG. 3. Plot of average length for the eight massless bosonic states as a function of the dimensionless coupling g', defined by $(g')^2=g^2NL/16\pi^3$, at $N=1000$ and $K=5$. Note that the states attain the maximum possible length allowed by the resolution $K=5$ in the limit of strong coupling.

merical evidence for such behavior — see Fig. 3), the structure of 1^* space becomes important, and we leave this question for future investigation.

V. DISCUSSION

In this work, we considered the bound states of three dimensional SU(N) $\mathcal{N}=1$ super-Yang-Mills theory defined on the compactified space-time $\mathbf{R}\times S^1\times S^1$. In particular, we compactified the light-cone coordinate x^- on a light-like circle via DLCQ, and wrapped the remaining transverse coordinate $x^\perp$ on a spatial circle. We showed explicitly that the supersymmetric form of DLCQ (SDLCQ) employed in recent studies of two dimensional supersymmetric gauge theories extends naturally in $2+1$ dimensions, which resulted in an exactly supersymmetric spectrum. We also showed that the $\mathcal{N}=1$ supersymmetry is enhanced to $\mathcal{N}=(1,1)$ in a reference frame with vanishing total transverse momentum $P^\perp=0$. The supersymmetric theory considered here is actually super-renormalizable.[9]

By retaining only the first few excited modes in the transverse direction, we were able to solve for bound state wave functions and masses numerically by diagonalizing the discretized light-cone supercharge. The results of our numerical calculations for bound state masses are summarized in Figs. 1 and 2. The theory exhibits a stable spectrum at both small and large coupling. In Table I, we compared solutions of the $2+1$ dimensional theory to corresponding solutions in the dimensionally reduced $1+1$ theory, after an appropriate rescaling of the $1+1$ dimensional coupling constant, and observed that the lower dimensional theory provides a good approximation to the low energy spectrum of the higher dimensional theory at weak coupling only. Any scope for making comparisons breaks down, however, at intermediate and strong coupling. One also notes a smooth dependence of bound state masses in terms of the DLCQ harmonic resolution K, a fact that was observed previously in studies of related supersymmetric models [8].

Interestingly, we find exactly massless states that persist for all values of the gauge coupling. These states should be contrasted with those that are massless only at the extreme values $g=0$ or $g=\infty$ (but not both).

An analytical treatment of the massless states revealed a connection with the massless solutions of the corresponding dimensionally reduced model [5,7,8]. We have shown that the wave functions of the massless states that remain massless for all values of the gauge coupling are in one-to-one correspondence with the massless states of the $1+1$ dimensional theory. This fact seems to be non-trivial, since the concept of mass is defined differently in the two theories. From the three dimensional point of view, the states in the Hilbert space that are naturally associated with the two dimensional Fock space (i.e. those states made up from partons with zero transverse momentum) are massless at $g=0$.

The bound state structure of the massless states in the $2+1$ theory was also studied for different couplings, and summarized in Fig. 3, where we plotted the average number of partons for each of the massless solutions. We concluded that in the decompactified limit $K\rightarrow\infty$, these massless states must become string-like in the strong coupling limit.

Evidently, it would be interesting to relate these observations with the recent claim that strongly coupled super-Yang-Mills theory corresponds to string theory in an anti–de Sitter background [3]. Of course, the techniques we have employed in this study may be applied to any supersymmetric gauge theory defined on a suitably compactified space-time. This should facilitate a more general study of the strongly coupled dynamics of super-Yang-Mills theories, and in particular, allow one to scrutinize more carefully the string-like properties of Yang-Mills theories.

[9]Ultraviolet renormalization is never an issue here, since we have truncated the transverse momentum modes, which acts as a regulator. DLCQ regulates any longitudinal divergences for vanishing k^+.

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