Illusions of general relativity in Brans-Dicke gravity

Valerio Faraoni*
RGGR, Faculté des Sciences, Université Libre de Bruxelles, Campus Plaine CP231, Boulevard du Triomphe, 1050 Bruxelles, Belgium
(Received 18 August 1998; published 22 March 1999)

Contrary to common belief, the standard tenet of Brans-Dicke theory reducing to general relativity in the $\omega\to\infty$ limit is false when the trace of the matter energy-momentum tensor vanishes. The issue is clarified in a new approach using conformal transformations. The otherwise unaccountable limiting behavior of Brans-Dicke gravity is easily understood in terms of the conformal invariance of the theory when the sources of gravity have radiation-like properties. The rigorous computation of the asymptotic behavior of the Brans-Dicke scalar field is straightforward in this new approach. [S0556-2821(99)05208-X]

PACS number(s): 04.50.+h, 04.20.Jb

I. INTRODUCTION

Brans-Dicke (BD) theory is the prototype of gravitational theories alternative to Einstein's general relativity [1]. The essential feature of Brans-Dicke theory is the presence of a scalar field to describe gravitation together with the metric. In this sense, BD gravity is a modification of general relativity, in which the gravitational field is described by the metric tensor alone.

Currently there is a revival of interest in Brans-Dicke gravity and its generalizations, which are collectively known as scalar-tensor theories [1]. The reasons for the current interest are several. First, the association of scalar fields to the metric seems to be unavoidable in superstring theories [2]. Second, scalar-tensor theories are invariant under a restricted class of conformal transformations [3–7]; and this property is reminiscent of the conformal invariance of string theories in the string frame. Further motivation comes from the fact that BD gravity can be derived from a Kaluza-Klein theory [3] in which the scalar field is generated by the presence of compactified extra dimensions, an essential feature of all modern unified theories.

Finally, not the least reason for the renewed interest is the study of BD and scalar–tensor theories with respect to their cosmological applications, the extended and hyperextended inflationary scenarios [8,9]. Many authors [10,15] have considered the possibility that general relativity behaves as an attractor for scalar-tensor theories [17]. It is generally agreed that the convergence of BD gravity to general relativity can occur during the matter-dominated era, or even during the inflationary phase of the early universe. The convergence of scalar-tensor theories has been studied in Refs. [18,19]: a scalar-tensor theory converges to general relativity if [18,19]

$$\omega\to\infty, \quad \frac{1}{\omega^3}\frac{d\omega}{d\phi}\to 0. \tag{1.1}$$

This paper is restricted to consideration of the BD theory for the sake of simplicity.

It is a common belief that BD gravity reduces to general relativity when the BD parameter $\omega\to\infty$ (see e.g. Ref. [20]), and the BD field ϕ is believed to exhibit the asymptotic behavior

$$\phi=\phi_0+\mathrm{O}\left(\frac{1}{\omega}\right) \tag{1.2}$$

(where ϕ_0 is a constant) when $\omega\to\infty$. However, the standard tenet about the $\omega\to\infty$ limit has been shown to be false; a number of exact BD solutions have been reported not to tend to the corresponding general relativity solutions when $\omega\to\infty$ [21–29]. In addition, the asymptotic behavior of the BD field is not Eq. (1.2) but rather

$$\phi=\phi_0+\mathrm{O}\left(\frac{1}{\sqrt{\omega}}\right) \tag{1.3}$$

for these solutions. These occurrences are alarming since the standard belief that BD theory always reduces to general relativity in the large ω limit is the basis for setting lower limits on the ω-parameter using Solar System experiments [1] (the limit $\omega>500$ coming from time-delay experiments [30] is often quoted).

As an example, one can consider the static, spherically symmetric, vacuum Brans solution [31,32] given by

$$ds^2=-\mathrm{e}^{2\alpha}dt^2+\mathrm{e}^{2\beta}[dr^2+r^2(d\theta^2+\sin^2\theta\, d\varphi^2)], \tag{1.4}$$

$$\mathrm{e}^{2\alpha}=\left(\frac{1-B/r}{1+B/r}\right)^{2/\sigma}, \tag{1.5}$$

$$\mathrm{e}^{2\beta}=\left(1+\frac{B}{r}\right)^4\left(\frac{1-B/r}{1+B/r}\right)^{2(\sigma-C-1)/\sigma}, \tag{1.6}$$

$$\phi=\phi_0\left(\frac{1-B/r}{1+B/r}\right)^{-C/\sigma}, \tag{1.7}$$

where

$$\sigma=\left[(C+1)^2-C\left(1-\frac{\omega C}{2}\right)\right]^{1/2}, \tag{1.8}$$

*Email address: vfaraoni@ulb.ac.be

$$B=\frac{M}{2C^2\phi_0}\left(\frac{2\omega+4}{2\omega+3}\right)^{1/2},\quad C=-\frac{1}{2\omega},\tag{1.9}$$

and where M is the mass. This solution reduces to the Schwarzschild solution of Einstein's theory for $\omega\to\infty$ [24]. However, choices of the constant C different from the one in Eq. (1.9) are possible, and for arbitrary values of the parameter C the solution (1.4)–(1.9) does not reduce to the Schwarzschild solution when $\omega\to\infty$. In addition, the scalar field exhibits the asymptotic behavior (1.3) in these cases [21,28,33]. It is to be remarked that the values of the parameters (M,C,ω) in the Brans solution are not arbitrary; physical requirements impose constraints between the allowed values of these parameters. This is the content, for example, of Ref. [34], in which it was shown that the positivity of the tensor mass puts bounds on C and σ [34]. A complete understanding of the relationships between the parameters M, C and ω, and their respective ranges of admissible values is not yet available. To make the situation worse, the limit of the Brans solution, like that of any BD exact solution, depends on the coordinate system adopted (see Ref. [35] for a discussion of the coordinate-dependence, and Refs. [26,27] for a coordinate-independent approach to the problem). A detailed study of the Brans solution requires considerations specific to this particular solution, which is not the main topic of the present paper, and will be the subject of a future work.

Other examples of exact BD solutions studied in the literature which do not have the expected general relativistic limit for $\omega\to\infty$ include the static, spherically symmetric, electrovacuum solution of Ref. [36]; Nariai's [37] solution with the radiation equation of state; the cylindrically symmetric, electrovacuum solution of Ref. [38]; the vacuum O'Hanlon and Tupper [39] solution; the Bianchi I universe with radiation equation of state [40]; the static cosmological solution of Ref. [22]; the Einstein–de Sitter solution of Ref. [22]; and the solutions with cylindrical symmetry and $T\neq 0$ of Ref. [29]. See Ref. [41] for the weak field limit of BD solutions.

Recently, it was realized that the asymptotic behavior of BD solutions goes hand-in-hand with the vanishing of the trace $T=T^{\alpha}{}_{\alpha}$ of the matter stress-energy tensor $T_{\alpha\beta}$ [33]. This is a hint suggesting a new approach to the issue of the $\omega\to\infty$ limit of BD theory. The vanishing of the trace of the stress-energy tensor is associated to conformal invariance [42] and the closely related mathematical technique of conformal transformation. The latter has been widely used in recent years in the context of scalar-tensor theories, nonlinear gravitational theories, cosmology, non-minimally coupled scalar fields (see Refs. [43,44] for reviews). Further, conformal transformations leave the light cones unchanged; the propagation of light and the causal structure of spacetime are unaffected. It is a natural step to use conformal transformations in problems involving sources of gravity with radiation-like properties.

A new approach is explored in this paper by using the well known but seldom used conformal invariance of BD theory when $T_{\mu\nu}=0$. Initially, we notice that the symmetry enjoyed by the purely gravitational sector of the BD action also occurs when matter with $T=0$ is included into the action. Then the entire BD action is invariant under a one-parameter Abelian group $\{\mathcal{F}_{\alpha}\}$ of transformations $\mathcal{F}_{\alpha}$ consisting of a conformal rescaling of the metric and a suitable scalar field redefinition. A change $\omega\to\tilde{\omega}$ of the BD parameter is equivalent to a symmetry operation $\mathcal{F}_{\alpha}$ that moves BD theory within an equivalence class $\mathcal{E}$. The $\omega\to\infty$ limit is also seen as a parameter change that moves BD theory within the same equivalence class $\mathcal{E}$. General relativity is not invariant under the action of a transformation $\mathcal{F}_{\alpha}$, and therefore it cannot be obtained by taking the $\omega\to\infty$ limit, an operation that cannot bring a BD spacetime $(M,g^{(\omega)}_{\mu\nu},\phi^{(\omega)})$ outside the class $\mathcal{E}$. Obtaining general relativity from BD gravity may be an illusion.

On the other hand, when the trace of the stress-energy tensor does not vanish, BD gravity is not invariant under the transformations $\mathcal{F}_{\alpha}$, and a change in the ω-parameter or the $\omega\to\infty$ limit do not move a BD spacetime $(M,g^{(\omega)}_{\mu\nu},\phi^{(\omega)})$ within an equivalence class; general relativity can then be reobtained. The new approach based on conformal transformations allows one to derive the asymptotic behavior (1.3) of the BD scalar field when $T=0$ with a rigorous computation.

Previous works on the problem of the Einstein limit of BD theory focussed on particular BD solutions. In the present paper, instead, we present general results, without referring to special solutions.

This paper details the new approach to the problem of the Einstein's limit of Brans-Dicke gravity; the preliminary results and method which were outlined in a previous letter [45]. Section II develops the formalism related to the conformal invariance property of BD gravity. Then the symmetry property is applied to the problem of the $\omega\to\infty$ limit. The asymptotic behavior of the BD field is studied in Sec. IV, while Sec. V presents a discussion and the conclusions.

Throughout the paper, we use the metric signature $---+++$; the Riemann tensor is given in terms of the Christoffel symbols by $R_{\mu\nu\rho}{}^{\sigma}=\Gamma^{\sigma}_{\mu\rho,\nu}-\Gamma^{\sigma}_{\nu\rho,\mu}+\Gamma^{\alpha}_{\mu\rho}\Gamma^{\sigma}_{\alpha\nu}-\Gamma^{\alpha}_{\nu\rho}\Gamma^{\sigma}_{\alpha\mu}$, the Ricci tensor is $R_{\mu\rho}\equiv R_{\mu\nu\rho}{}^{\nu}$, and $R=g^{\alpha\beta}R_{\alpha\beta}$. ∇_{μ} is the covariant derivative operator, $\Box\equiv g^{\mu\nu}\nabla_{\mu}\nabla_{\nu}$, and we use units in which the speed of light and Newton's constant assume the value unity.

II. BRANS-DICKE THEORY AND CONFORMAL INVARIANCE

The starting point of our analysis is the BD action in the so-called Jordan conformal frame

$$S_{BD}=\frac{1}{16\pi}\int d^4x\sqrt{-g}\left[\phi R+\frac{\omega}{\phi}g^{\alpha\beta}\nabla_{\alpha}\phi\nabla_{\beta}\phi\right]+S_{matter},\tag{2.1}$$

where S_{matter} is the matter part of the action which is independent of the BD scalar field ϕ. The BD field equations are

$$R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R=\frac{8\pi}{\phi}T_{\mu\nu}+\frac{\omega}{\phi^2}\left(\nabla_\mu\phi\nabla_\nu\phi-\frac{1}{2}g_{\mu\nu}\nabla^\alpha\phi\nabla_\alpha\phi\right)$$

$$+\frac{1}{\phi}(\nabla_\mu\nabla_\nu\phi-g_{\mu\nu}\Box\phi), \tag{2.2}$$

$$\Box\phi=\frac{8\pi T}{3+2\omega}. \tag{2.3}$$

Let us consider the purely gravitational sector of the theory. Under the conformal transformation

$$g_{\mu\nu}\rightarrow\tilde{g}_{\mu\nu}=\Omega^2 g_{\mu\nu}, \tag{2.4}$$

where $\Omega(x^\alpha)$ is a non-vanishing smooth function, the Ricci curvature R and the Jacobian determinant $\sqrt{-g}$ appearing in the action (2.1) transform as [46–48]

$$\tilde{R}=\Omega^{-2}\left[R+\frac{6\Box\Omega}{\Omega}\right],\quad \sqrt{-\tilde{g}}=\Omega^4\sqrt{-g}. \tag{2.5}$$

The integrand in the purely gravitational part of the action (2.1) is

$$\mathcal{L}_{BD}\sqrt{-g}=\sqrt{-\tilde{g}}\left[\Omega^{-2}\phi\tilde{R}-\frac{6\phi\Box\Omega}{\Omega^5}\right.$$

$$\left.+\frac{\omega}{\Omega^2\phi}\tilde{g}^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi\right]. \tag{2.6}$$

The ansatz

$$\Omega=\phi^\alpha \tag{2.7}$$

with $\alpha\neq 1/2$ for the conformal factor Ω, and the redefinition of the scalar field

$$\phi\rightarrow\tilde{\phi}=\phi^{1-2\alpha}, \tag{2.8}$$

yield

$$\mathcal{L}_{BD}\sqrt{-g}=\sqrt{-\tilde{g}}\left[\tilde{\phi}\tilde{R}+\frac{\tilde{\omega}}{\tilde{\phi}}\tilde{g}^{\mu\nu}\nabla_\mu\tilde{\phi}\nabla_\nu\tilde{\phi}\right], \tag{2.9}$$

where

$$\tilde{\omega}=\frac{\omega-6\alpha(\alpha-1)}{(1-2\alpha)^2}. \tag{2.10}$$

Thus, the gravitational part of the BD action is left unchanged in form by the transformation $\mathcal{F}_\alpha$ consisting of the conformal rescaling (2.4), (2.7), and the change of the scalar field variable (2.8) for $\alpha\neq 1/2$. The transformations

$$\mathcal{F}_\alpha:(M,g^{(\omega)}_{\mu\nu},\phi^{(\omega)})\rightarrow(M,\tilde{g}^{(\tilde{\omega})}_{\mu\nu},\tilde{\phi}^{(\tilde{\omega})}) \tag{2.11}$$

mapping a BD spacetime $(M,g^{(\omega)}_{\mu\nu},\phi^{(\omega)})$ into another constitute a one-parameter Abelian group of symmetries with a singularity in the parameter dependence at $\alpha=1/2$. To prove this statement, one begins by noticing that the consecutive action of two maps $\mathcal{F}_\alpha$, $\mathcal{F}_\beta$ of the kind (2.4), (2.7), (2.8) is a map of the same kind:

$$\mathcal{F}_\alpha\circ\mathcal{F}_\beta=\mathcal{F}_\gamma, \tag{2.12}$$

where

$$\gamma(\alpha,\beta)=\alpha+\beta-2\alpha\beta. \tag{2.13}$$

Furthermore, $\alpha,\beta\neq 1/2$ implies $\gamma(\alpha,\beta)\neq 1/2$. For $\alpha<1/2$, the identity corresponds to the transformation with $\alpha=0$,

$$\mathcal{F}_0=\text{identity}. \tag{2.14}$$

The inverse $(\mathcal{F}_\alpha)^{-1}$ of the transformation $\mathcal{F}_\alpha$ is the map $\mathcal{F}_\delta$, where

$$\delta=-\frac{\alpha}{1-2\alpha} \tag{2.15}$$

for $\alpha<1/2$. Finally, since $\gamma(\alpha,\beta)=\gamma(\beta,\alpha)$, the group $\{\mathcal{F}_\alpha\}$ is commutative.

The group $\{\mathcal{F}_\alpha\}$ establishes an equivalence relation: two BD spacetimes $(M,g^{(\omega)}_{\mu\nu},\phi^{(\omega)})$, $(M,\tilde{g}^{(\tilde{\omega})}_{\mu\nu},\tilde{\phi}^{(\tilde{\omega})})$ are equivalent if they are related by a transformation $\mathcal{F}_\alpha$. All the spacetimes $(M,g_{\mu\nu},\phi)$ related by such a map constitute an equivalence class $\mathcal{E}$. This property is crucial in the understanding of the anomalous behavior of BD solutions when $\omega\rightarrow\infty$ and $T=0$, which is discussed in the next section.

III. APPLICATION TO THE $\omega\rightarrow\infty$ LIMIT OF BRANS-DICKE THEORY

In the previous section we considered the purely gravitational part of the BD Lagrangian (2.1). When ordinary (i.e., other than the BD scalar) matter is added to the BD action, the conformal invariance is generally broken. However the transformations $\mathcal{F}_\alpha$ are still symmetries of Brans-Dicke theory when the stress-energy tensor $T_{\mu\nu}$ has a vanishing trace. In fact, under the conditions $T_{\mu\nu}=T_{\nu\mu}$ and $T=0$, the conservation equation

$$\nabla^\nu T_{\mu\nu}=0 \tag{3.1}$$

containing the dynamical equations for the motion of matter, is conformally invariant [47]. We notice that, in the Jordan frame, the stress-energy tensor $T_{\mu\nu}$ does not depend on the scalar field ϕ, and hence it is not affected by the change of the ϕ-variable (2.8). Then the total BD action is invariant under the action of the group of transformations $\{\mathcal{F}_\alpha\}$ if $T=0$. This salient feature of invariance of the BD action in the presence of matter has not apparently been previously observed. From the physical perspective, the lack of conformal invariance corresponds to the presence of a length or mass scale in the theory. This happens in general relativity. Conformal invariance corresponds to the absence of a preferred length or mass scale in the theory, hence to scale invariance.

With the understanding afforded by this new observation, when $T=0$, a change of the BD parameter $\omega\rightarrow\tilde{\omega}$ is equiva-

lent to a transformation $\mathcal{F}_\alpha$ for a suitable value of the parameter α. A BD spacetime $(M, g_{\mu\nu}, \phi)$ is moved into the equivalence class $\mathcal{E}$ discussed in the previous section. In particular, one can consider a parameter change in which $\tilde{\omega} \gg 1$. This is made possible by the fact that the function $\tilde{\omega}(\alpha)$ given by Eq. (2.4) has a pole singularity at $\alpha = 1/2$ and it can assume arbitrarily large values there. Also the $\omega \to \infty$ limit can be seen as a parameter change $\omega \to \tilde{\omega}$, where $\tilde{\omega}$ grows without bound. The result is that this limit simply moves the BD spacetime $(M, g^{(\omega)}_{\mu\nu}, \phi^{(\omega)})$ within the equivalence class $\mathcal{E}$. General relativity, however is not conformally invariant [16]. This is the reason why GR cannot be obtained as the $\omega \to \infty$ limit of BD theory when $T = 0$. If matter with $T \neq 0$ is added to the BD gravitational Lagrangian, the conformal equivalence is broken.

This explanation of the anomalies in the $\omega \to \infty$ limit emerges in a simple and clear way in the new approach based on conformal transformations. This possibility relies upon the structure of the function $\tilde{\omega}(\alpha)$ given by Eq. (2.10), which deserves further comment. $\tilde{\omega}(\alpha)$ has four branches, symmetric about $\alpha = 1/2$, which is a pole singularity, and about $\omega = -3/2$. Since both the $\alpha < 1/2$ and the $\alpha > 1/2$ branches span the entire range $(-\infty, +\infty)$ of the parameter $\tilde{\omega}$, we restrict our considerations to only one of the two branches. In this paper, we choose the $\alpha < 1/2$ branch for ease of demonstration. Then $\tilde{\omega} = \omega$ at $\alpha = 0$, which corresponds to the identity $\mathcal{F}_0$, in the group of transformations (2.4), (2.7) and (2.8).

The $\alpha \to 1/2$ limit corresponds to the $\omega \to \infty$ limit of the BD parameter. It is indeed convenient to use the new parameter α instead of the usual ω (or $\tilde{\omega}$); and this is done in the next section. It is well known [43,44] that when $\alpha = 1/2$, the conformal transformation

$$g_{\mu\nu} \to \tilde{g}_{\mu\nu} = \phi\, g_{\mu\nu} \tag{3.2}$$

in conjunction with the BD scalar field redefinition

$$\tilde{\phi} = \int \frac{(3+2\omega)^{1/2}}{\phi}\, d\phi \tag{3.3}$$

recasts the theory in the so-called Einstein conformal frame (or ''Pauli frame''). In the Einstein frame, the gravitational part of the action becomes that of Einstein gravity plus a non-self-interacting scalar field as a material source,

$$S = \int d^4x \sqrt{-\tilde{g}} \left[\frac{\tilde{R}}{16\pi} - \frac{1}{2} \tilde{g}^{\mu\nu} \nabla_\mu \tilde{\phi} \nabla_\nu \tilde{\phi} \right]. \tag{3.4}$$

In the Einstein frame, one cannot contemplate solutions of the vacuum Einstein equations $R_{\mu\nu} = 0$, because the scalar field $\tilde{\phi}$ cannot be eliminated. In addition, the scalar $\tilde{\phi}$ exhibits an anomalous coupling to the energy-momentum tensor of ordinary matter if $T \neq 0$ ([44,43] and references therein).

The transformation (3.2), (3.3) is well known since the original BD paper [31]; and has been generalized and applied a number of times to scalar-tensor and non-linear gravity theories. In the Einstein frame, the ω parameter disappears and there is no $\omega \to \infty$ limit.

Finally, we note that the $\omega = -3/2$ BD theory corresponds to the $\alpha \to \pm\infty$ limit and is a fixed point of the transformation $\mathcal{F}_\alpha$ given by Eqs. (2.4), (2.5). In fact, for $\alpha = \infty$ we obtain $\tilde{\omega} = \omega = -3/2$ from Eq. (2.10). Although the BD field equations (2.2), (2.3) are not defined in the form presented here for $\omega = -3/2$, the corresponding theory is sometimes studied.

The formalism of conformal transformations allows a general treatment of the $\omega \to \infty$ limit of BD theory without resorting to special exact solutions. In the next section, we show that the new approach allows a straightforward computation of the asymptotic behavior of the BD field, which is the root of the problems in the $\omega \to \infty$ limit.

IV. ASYMPTOTIC BEHAVIOR OF THE BD SCALAR FOR $\omega \to \infty$

It is generally difficult to obtain a series expansion of the BD scalar field ϕ in powers of $1/\omega$ for $\omega \to \infty$. This is the reason why the asymptotic behavior of ϕ has been derived only as an order of magnitude estimate [20,33], or exactly only for special solutions. Contrary to the standard tenet that $\phi = \text{constant} + O(\omega^{-1})$ as $\omega \to \infty$, the scaling $\phi = \text{const} + O(\omega^{-1/2})$ has been obtained when the trace T of the matter stress-energy tensor vanishes [33].

Instead of using the BD parameter ω, we consider the new parameter α obtained by inverting Eq. (2.10),

$$\alpha = \frac{1}{2} \left(1 \pm \frac{\sqrt{3}}{\sqrt{3 + 2\tilde{\omega}}} \right) \tag{4.1}$$

for $\tilde{\omega} > -3/2$, keeping in mind that the situation is symmetric for $\tilde{\omega} < -3/2$. The limit $\tilde{\omega} \to \infty$ corresponds to $\alpha \to 1/2$, and Eq. (2.8) yields

$$\tilde{\phi} = 1 \mp \left(\frac{3}{2\tilde{\omega}} \right)^{1/2} \ln \phi \tag{4.2}$$

as $\tilde{\omega} \to \infty$. Since the ''old'' BD scalar field ϕ corresponds to the fixed value $\omega = 0$ of the parameter, its value is not affected by the limit $\tilde{\omega} \to \infty$; then the ''new'' BD field $\tilde{\phi}$ has the asymptotic behavior (1.3).

The second term in the right hand side of Eq. (2.2) does not go to zero in the $\tilde{\omega} \to \infty$ limit because $\nabla_\mu \tilde{\phi} = \mp (3/2\tilde{\omega})^{1/2} \nabla_\mu \ln \phi$ and

$$\tilde{A} \equiv \frac{\tilde{\omega}}{\tilde{\phi}^2} \left(\nabla_\mu \tilde{\phi} \nabla_\nu \tilde{\phi} - \frac{1}{2} g_{\mu\nu} \nabla^\alpha \tilde{\phi} \nabla_\alpha \tilde{\phi} \right)$$

$$\to \frac{3}{2} \left(\nabla_\mu \phi \nabla_\nu \phi - \frac{1}{2} g_{\mu\nu} \nabla^\alpha \phi \nabla_\alpha \phi \right), \tag{4.3}$$

and Eq. (2.2) does not reduce to the Einstein equation with the same $T_{\mu\nu}$ as $\tilde{\omega}\to\infty$. In this sense, the asymptotic behavior of the BD scalar ϕ when $\omega\to\infty$ determines whether a metric which solves the BD equations (2.2), (2.3) converges to a solution of the Einstein equations.

The quantity

$$A\equiv\frac{\omega}{\phi^2}\left(\nabla_\mu\phi\nabla_\nu\phi-\frac{1}{2}g_{\mu\nu}\nabla^\alpha\phi\nabla_\alpha\phi\right) \tag{4.4}$$

cannot be identically vanishing: in fact, assuming that $A=0$, one has two possibilities. (i) $\nabla_\alpha\phi\nabla^\alpha\phi=0$; then $\partial_\mu\phi=0$ and ϕ is identically constant, which does not correspond to a BD solution. (ii) $\nabla_\alpha\phi\nabla^\alpha\phi\neq 0$: in this case one defines the vector

$$u^\mu\equiv\frac{\nabla^\mu\phi}{|\nabla_\alpha\phi\nabla^\alpha\phi|^{1/2}} \tag{4.5}$$

which has unit norm $u_\mu u^\mu=1$. The vanishing of A corresponds to $g_{\mu\nu}=2u_\mu u_\nu$. The trace of the latter equation gives $u_\mu u^\mu=2$, which contradicts the normalization of u^μ.

When matter represented by a stress-energy tensor $T_{\mu\nu}$ with non-vanishing trace is present, the invariance under the group $\{\mathcal{F}_\alpha\}$ is broken, and the conformal transformation approach cannot be applied. Then, only the order of magnitude estimate (1.2) instead of Eq. (1.3) is available [20] [we still lack a rigorous derivation of Eq. (1.2) when $T\neq 0$].

V. DISCUSSION AND CONCLUSIONS

When Brans-Dicke theory fails to reproduce general relativity it is disturbing as this contradicts the standard belief exposed in the textbooks, and indeed it is the basis for placing lower limits on the BD parameter ω using Solar System experiments. Repeated observations have been made in the literature that many exact solutions of BD theory fail to give back the corresponding general relativistic solution in the $\omega\to\infty$ limit when the trace T of the matter energy-momentum tensor vanishes [21–25, 27–29]. However, the connection between the vanishing trace and the problematic of obtaining general relativity as the $\omega\to\infty$ limit of BD theory was tentatively established only in Ref. [33].

It is rather a natural step to look at the conformal symmetry property of BD theory when matter with $T=0$ is added to the BD gravitational action, and to apply conformal transformation techniques. This new approach is useful as it permits an enhanced comprehension of the problems associated with the $\omega\to\infty$ limit of BD theory.

The $\omega\to\infty$ limit along with a parameter change $\omega\to\tilde{\omega}$ can be seen as a transformation which moves a BD spacetime $(M,g^{(\omega)}_{\mu\nu},\phi^{(\omega)})$ within an equivalence class that does not contain general-relativistic spacetimes. Moreover, a new parameter is introduced which is more appropriate than the usual ω parameter. The asymptotic behavior of the BD scalar field was previously obtained by using merely an order of magnitude estimate, and was verified only for particular exact solutions. Now, the behavior of ϕ as $\omega\to\infty$ can be computed using the new approach.

The condition $T\neq 0$ is not a necessary and sufficient condition for BD exact solutions to reduce to the corresponding solutions of the Einstein equations, contrary to what was stated in Ref. [33]. In fact, certain solutions corresponding to $T\neq 0$ are known, which fail to reduce to the corresponding general relativistic solutions when $\omega\to\infty$ [29]. What has been proved in this paper is that *solutions with $T=0$ generically fail to reduce to the corresponding solutions of general relativity when $\omega\to\infty$* (apart from the trivial case of the Minkowski metric corresponding to $\phi=$constant). An explanation which is independent of particular exact solutions has been given for this behavior.

Of course, the results of this paper do not exclude that solutions associated to a nonvanishing trace $T\neq 0$ fail to have the expected general-relativistic limit, for reasons different from the ones described in this paper, and examples of such situations have been reported in the literature [21,28,29].

Regarding the application of BD and scalar tensor theories to cosmological scenarios, using the new approach of this paper it becomes easy to understand why the general relativity-as-an-attractor behavior of scalar-tensor theories [10–15] has been discovered to occur during the matter-dominated era or during inflation, but not during the radiation era. In fact, during the latter epoch, the radiation equation of state $P=\rho/3$ makes the trace of the stress-energy tensor T vanish, and even if $\omega\to\infty$ it would be impossible to recover general relativity as a limiting solution and as an attractor. Indeed it has been shown that general relativity is very peculiar in the space of scalar-tensor theories and that a scalar-tensor theory does not always contain an attractor mechanism towards general relativity [13,15].

The approach presented here is not a panacea, however and its limitations must be balanced with its proper application. It is useful only when $T^\mu{}_\mu=0$ and it does not exhaust the understanding of the BD theory. The situation can be quite complicated; to obtain some general insight of what happens in the limit of a spacetime as one parameter varies consider, for example, the partial differential equation

$$L(a)f(x^\alpha)=0, \tag{5.1}$$

where $L(a)$ is a partial differential operator depending on the parameter a. Let L_0 be the limit of $L(a)$ as $a\to 0$, and let f_0 be the limit

$$f_0=\lim_{a\to 0}f(x^\alpha). \tag{5.2}$$

If ψ is a solution of the equation $L_0 f=0$, then in general one has $\psi\neq f_0$. Although the $\omega\to\infty$ limit of the BD field equations usually yields the Einstein equations when $T\neq 0$, it is not trivial that a BD exact solution tends to the corresponding solution of the Einstein equations in the same limit. This property of the BD field equations has not yet been investigated in the literature.

The $\omega\to\infty$ limit of a BD solution is even more ambiguous when there is more than one parameter involved. This is

the case of BD exact solutions which often depend on more parameters than the corresponding solution of the Einstein equations [49]. If n parameters $a_1, a_2, \ldots, a_n$ are present in a solution $f(a_1, \ldots, a_j, \ldots, a_n, x)$, the limits

$$\lim_{a_j \to 0} \lim_{a_i \to 0} f \tag{5.3}$$

and

$$\lim_{a_i \to 0} \lim_{a_j \to 0} f, \tag{5.4}$$

in general, do not coincide. Often the general relativistic solution can be obtained only for particular combinations of the parameters. Examples are given in Refs. [33,29,50,28].

From a more general point of view, the limit of spacetimes when a parameter varies may not be well defined even within the context of general relativity. The limit of a particular solution of the Einstein equations, when it exists, depends on the coordinate system adopted and hence it may not be unique [35]. For example, the limit of the Schwarzschild solution as the mass diverges is the Minkowski space or a Kasner space [35]. A coordinate-independent approach based on the Cartan scalars has been pursued in the context of general relativity [26] and applied to the $\omega \to \infty$ of BD theory [27]. It emerges that the limit of BD solutions to general-relativistic solutions corresponding to the same stress-energy tensor is not unique, or the limit may not yield a GR solution at all [27]. These issues are worth further investigation in the future.

ACKNOWLEDGMENTS

The author is grateful to S.P. Bergliaffa and to M. Susperregi for pointing out Refs. [29] and [15].

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