Superconformal mechanics, black holes, and nonlinear realizations

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The $OSp(2|2)$-invariant planar dynamics of a $D=4$ superparticle near the horizon of a large mass extreme black hole is described by an $N=2$ superconformal mechanics, with the $SO(2)$ charge being the superparticle's angular momentum. The *non-manifest* superconformal invariance of the superpotential term is shown to lead to a shift in the $SO(2)$ charge by the value of its coefficient, which we identify as the orbital angular momentum. The full $SU(1,1|2)$ invariant dynamics is found from an extension to $N=4$ superconformal mechanics. [S0556-2821(99)01608-2]

I. INTRODUCTION AND THE $OSp(1|2)$ MODEL

The dynamics of a particle described by the action

$$I = \int dt \left[\frac{1}{2} m\dot{x}^2 - \frac{g}{2x^2} \right] \tag{1}$$

is invariant under the group $SL(2,\mathbb{R})$, one of the generators being the Hamiltonian H. The group $SL(2,\mathbb{R})$ is the conformal group in a ''spacetime'' of one dimension (time), and so the action I is that of a one-dimensional conformal ''field'' theory, i.e., a model of conformal mechanics. The model was introduced, and its quantum properties investigated, in [1]. Recently, it was shown that it describes the radial motion of a particle of mass m and charge q near the horizon of an extreme (i.e., $M=|Q|$) Reissner-Nordström (RN) black hole in a limit in which $|q|/m$ tends to unity at the same time as the black hole mass M tends to infinity, with $M^2(m-|q|)$ remaining finite [2]. The coupling constant g is then found to be

$$g = 8M^2(m-|q|) + 4l(l+1)/m, \tag{2}$$

where l is the particle's orbital angular momentum.[1]

It was also shown in [2] that the radial motion of a superparticle in the same background, but with zero angular momentum, is described, in the same limit, by an $OSp(1|2)$-invariant *superconformal* mechanics. However, because the fermionic gauge symmetries of the superparticle require $m=|q|$, and because $l=0$ is assumed, the coupling constant g of this model vanishes, and the potential term is therefore absent. This is a reflection of the exact balance of

the gravitational and electric forces on a *static* superparticle in an extreme RN black hole background. It was further pointed out in [2] that the full superparticle dynamics must be invariant under the larger $SU(1,1|2)$ superconformal group because this is the isometry group of the $AdS_2 \times S^2$ near-horizon supergeometry. This full dynamics will of course describe not only the radial motion of the superparticle but also its motion on the 2-sphere. However, there is nothing to prevent us from considering only the radial motion, which will be the equation of motion of an $SU(1,1|2)$-invariant generalization of Eq. (1).

In addition to considering only the radial equation of motion of the superparticle we can also consider a restriction on the full dynamics in which the particle is assumed to move within an equatorial plane or the further restriction to purely radial motion (i.e., $l=0$). These restrictions correspond to a reduction of the superconformal symmetry to some subgroup of $SU(1,1|2)$, in fact to the sequence of subgroups

$$SU(1,1|2) \supset SU(1,1|1) \cong OSp(2|2) \supset OSp(1|2). \tag{3}$$

In the first restriction, to $SU(1,1|1)$, the $SU(2)$ group of rotations is reduced to the $U(1)$ group of rotations in the plane. The corresponding superconformal mechanics is the $SU(1,1|1)$ generalization of Eq. (1) constructed and analyzed in [3,4]. As the above discussion suggests, the $U(1)$ charge of this model is directly related to the angular momentum of a superparticle; the precise relation will be given below. That the subsequent restriction to $OSp(1|2)$ describes purely radial motion was justified in detail in [2]. Of principal interest here are the $OSp(2|2)$ and $SU(1,1|2)$ models because they allow $l \neq 0$ and hence $g \neq 0$.

The $OSp(2|2)$ superconformal mechanics was initially presented as a particular model of $N=2$ supersymmetric quantum mechanics [3,4]. Its superspace action is a functional of a single world line superfield $x(t, \eta_1, \eta_2)$, where η_i $(i=1,2)$ are anticommuting partners to the world line time coordinate t. This action is[2]

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[1]When $l \neq 0$ the particle's motion is not purely radial, of course, but by ''radial motion'' above we mean the equation for the radial position of the particle.

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[2]The constant l is related to the constant f of [4] by $f=2l$.

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$$I = -i \int dt \, d^2\eta \{2mi D_1 x D_2 x + 4l \log x\}, \qquad (4)$$

where

$$D_{\eta_i} \equiv D_i = \frac{\partial}{\partial \eta_i} - \frac{i}{2}\eta_i\frac{\partial}{\partial t}, \quad i,j=1,2, \quad \{D_i,D_j\}=-i\delta_{ij}\partial_t, \qquad (5)$$

are the super-world-line covariant spinor derivatives.

The action (4) is manifestly invariant under the two world line supersymmetries, with corresponding Noether charges Q_i, but is also invariant under the other two supersymmetries of $OSp(2|2)$, with Noether charges S_i. The full set of Noether charges includes those corresponding to dilatations (D), proper conformal transformations (K), and the $so(2)$ charge B. For $l=0$ these Noether charges obey the (anti)commutation relations of the $osp(2|2) \cong su(1,1|1)$ algebra. The non-zero (anti)commutators are

$$[H,D]=iH, \quad [K,D]=-iK, \quad [H,K]=2iD,$$

$$\{Q_i,Q_j\}=\delta_{ij}H, \quad \{S_i,S_j\}=\delta_{ij}K,$$

$$\{Q_i,S_j\}=\delta_{ij}D+\frac{1}{2}\epsilon_{ij}B,$$

$$[D,Q_i]=-\frac{i}{2}Q_i, \quad [D,S_i]=\frac{i}{2}S_i,$$

$$[K,Q_i]=-iS_i, \quad [H,S_i]=iQ_i,$$

$$[B,Q_i]=-i\epsilon_{ij}Q_j, \quad [B,S_i]=-i\epsilon_{ij}S_j, \quad i,j=1,2. \qquad (6)$$

When $l \neq 0$ one finds the same algebra but B is no longer the $U(1)$ Noether charge associated with the $U(1)$ invariance of Eq. (4). This is not due to any change in this Noether charge, which continues to be the same fermion bilinear as before [given by Eq. (65) below]. Let us use $\hat{B}$ to denote this fermion bilinear. Then, B in Eqs. (6) is given by $B=\hat{B}+2l$; so $B=\hat{B}$ when $l=0$ but not otherwise. The main aim of this paper is to provide a mathematical explanation for why this shift of the $U(1)$ charge occurs and a physical explanation of its significance.

The Q_i-supersymmetries are linearly realized by the action (4). The S_i-supersymmetries are non-linearly realized, the variables $(D_i x)|$ being the corresponding Goldstone fermions (where, as usual, $|$ is short for $|_{\eta_i=0}$). Thus, the above supersymmetric mechanics is one in which supersymmetry is partially broken. In fact, the supersymmetry is "half-broken," as is to be expected from its superparticle origin, and $x(t,\eta)$ is the Nambu-Goldstone superfield. Since the terms in the action of lowest dimension should be determined entirely by (super)symmetry, we may use the method of non-linear realizations of spacetime (super)symmetries [5] to construct them. This was done in [6] for a class of $SU(1,1|n)$-invariant $N=2n$ superconformal mechanics models that include the bosonic model and the $N=2,4$ mod-

els of interest here. As a simple illustration of the method we shall now show how the $OSp(1|2)$-invariant $N=1$ superconformal mechanics of [2] can be found in this way.

The superalgebra $osp(1|2)$ is spanned by (H,K,D,Q,S). We choose as an "unbroken" subalgebra that spanned by H and Q, to which we associate the independent variables $\zeta^M = (t,\eta)$. These parametrize a real $(1,1)$-dimensional super world line. The $OSp(1|2)$ group element on the super world line is written as

$$g(t,\eta)=e^{-itH}e^{i\eta Q}e^{i\lambda(t,\eta)S}e^{iz(t,\eta)D}e^{i\omega(t,\eta)K}, \qquad (7)$$

where the dependent variables z and ω associated with the "broken" generators D and K are commuting world line superfields, and λ is an anticommuting world line superfield associated with the "broken" supercharge S. It will prove convenient to introduce the (non-exact) differential

$$d\tau \equiv dt - \frac{i}{2}\eta \, d\eta, \qquad (8)$$

because we then have

$$d = d\tau \, \partial_t + d\eta \, D_\eta, \qquad (9)$$

where

$$D_\eta \equiv \frac{\partial}{\partial \eta} - \frac{i}{2}\eta\frac{\partial}{\partial t} \equiv \partial_\eta - \frac{i}{2}\eta\partial_t \qquad (10)$$

is the super-world-line covariant spinor derivative satisfying $2D_\eta^2 = -i\partial_t$. A calculation now yields[3]

$$ig^{-1}dg = d\tau \, e^{-z}H - (d\tau \lambda + d\eta)e^{-z/2}Q$$

$$- [d\tau(2\omega e^{-z}+\dot{z}) + d\eta(D_\eta z - i\lambda)]D$$

$$- \left[d\tau\left(\dot{\omega}-\omega\dot{z}-e^{-z}\omega^2+\frac{i}{2}\lambda\dot{\lambda}e^z\right)\right.$$

$$\left. + d\eta\left(D_\eta\omega-\omega D_\eta z+i\omega\lambda-\frac{i}{2}e^z\lambda D_\eta\lambda\right)\right]K$$

$$- [d\tau(\dot{\lambda}e^{z/2}-\omega e^{-z/2}\lambda) + d\eta(D_\eta e^{z/2}-\omega e^{-z/2})]S. \qquad (11)$$

We can rewrite this in the form[4]

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[3]We assume for the purposes of this calculation, and those to follow, that η and λ anticommute with Q and S. The opposite assumption, that they commute, leads to a change of sign of all fermion bilinears. Since this sign is not fixed by physical considerations, we are free to make either choice for present purposes. We leave the reader to decide whether one or the other choice is required for mathematical consistency.

[4]We shall use a script $\mathcal{D}$ to denote the *group* covariant derivatives $\mathcal{D}_A$, $\mathcal{D}_M$ used in this paper, while D_η or $D_{\eta_i} \equiv D_i$ will refer to the *super-world-line* derivatives.

$$ig^{-1}dg = d\xi^M E_M{}^A[H_A - (\mathcal{D}_A z)D - (\mathcal{D}_A \omega)K - (\mathcal{D}_A \lambda)S], \tag{12}$$

where $d\xi^M \equiv (d\tau, d\eta)$ and $H_A = (H_0, H_1) \equiv (H, Q)$. We find that

$$E_M{}^A = \begin{pmatrix} e^{-z} & -\lambda e^{-z/2} \\ 0 & -e^{-z/2} \end{pmatrix}, \quad E_A{}^M = \begin{pmatrix} e^z & -\lambda e^z \\ 0 & -e^{z/2} \end{pmatrix}, \tag{13}$$

and, for example,

$$\mathcal{D}_A \omega \equiv \begin{pmatrix} \mathcal{D}_0 \omega \\ \mathcal{D}_1 \omega \end{pmatrix} = \begin{pmatrix} e^z\left(\dot{\omega} - \omega \dot{z} - e^{-z}\omega^2 + \dfrac{i}{2}\lambda \dot{\lambda} e^z - \lambda D_\eta \omega + \lambda \omega D_\eta z \right) \\ -e^{z/2}\left(D_\eta \omega - \omega D_\eta z + i\omega\lambda - \dfrac{i}{2}e^z \lambda D_\eta \lambda \right) \end{pmatrix}. \tag{14}$$

Manifestly $OSp(1|2)$-invariant superspace actions have the form

$$I = \int dt\, d\eta\, (\mathrm{sdet}\, E)\mathcal{L}(\mathcal{D}_A z, \mathcal{D}_A \omega, \mathcal{D}_A \lambda), \tag{15}$$

where $\mathcal{L}$ is an anticommuting world line scalar. The invariance is manifest in the sense that $\mathrm{sdet}E\,\mathcal{L}$ transforms as a scalar density.[5] The superspace structure group is chosen so as to leave invariant the relation $2D_\eta^2 = -i\partial_t$. It follows that the fermion and boson components of the covariant derivatives are independent tensors (in fact, scalars in this case). The lowest dimension Lagrangian is therefore proportional to the $\mathcal{D}_1 \omega$ component. All the other choices lead to higher-derivative component actions.[6] Since

$$\mathrm{sdet}\, E = -e^{-z/2}, \tag{16}$$

we have, discarding a total derivative (the $D_\eta \omega$ term),

$$I = -im \int dt\, d\eta\left(\omega D_\eta z - i\omega\lambda + \frac{i}{2}e^z \lambda D_\eta \lambda \right). \tag{17}$$

The ω and λ equations yield

$$\omega = -\frac{1}{2}e^z \dot{z}, \quad \lambda = -iD_\eta z, \tag{18}$$

which are equivalent to the manifestly $OSp(1|2)$-invariant constraints $\mathcal{D}_A z = 0$, which could have been imposed *ab initio*. Either way, the action then reduces to

$$I(z) = \frac{i}{4}m \int dt\, d\eta\, e^z \dot{z} D_\eta z. \tag{19}$$

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[5] Let the infinitesimal transformation of the coordinates $\zeta^M = (t, \eta)$ be $\delta\zeta^M = (\delta t, \delta\eta)$. Then, a scalar density L is one for which $\delta L = (\delta\zeta^M L)\overleftarrow{\partial}_M$.

[6] Assuming that $\mathcal{D}_A z = 0$ is imposed as a constraint to eliminate ω and λ as independent superfields, because ω would otherwise be an independent field with wrong-sign kinetic terms.

The equation of motion is equivalent, when combined with the constraint $\mathcal{D}_A z = 0$, to the manifestly $OSp(1|2)$-invariant equation $\mathcal{D}_A \omega = 0$ (and these imply $\mathcal{D}_A \lambda = 0$). Setting

$$z = \log x^2, \tag{20}$$

performing the superspace integral, and then setting the fermions to zero we recover the Lagrangian (1) with $g=0$. By retaining the fermions we recover the $N=1$ superconformal mechanics of [2].

Note that it is not possible to construct an $OSp(1|2)$-extension of the g/x^2 potential. This might be possible if we were to suppose that all supersymmetries are non-linearly realized, but the resulting action would involve variables other than the components of the superfield (t, η), and it would not be expressible in superfield form. To find a suitable supersymmetric generalization of the potential term we must consider the further extension to $N=2$ or $N=4$. Similar techniques to those just described were used in [6] to obtain the field equations of the $N=2$ superconformal mechanics model of [3,4] in manifestly $OSp(2|2) \cong SU(1,1|1)$ invariant form, as a special case of a construction valid for $SU(1,1|n)$, but no attempt was made to demonstrate the *manifest* invariance of the superspace *action*. There is a good reason for this: as we shall see here, the superpotential term of the $SU(1,1|1)$ model *cannot be expressed in a manifestly invariant form*. This possibility arises because manifest invariance is only a sufficient condition for invariance, not a necessary one.

The existence of actions which are invariant but not manifestly so has often been noted in connection with Wess-Zumino (WZ) terms associated with central extensions of a (super)algebra. The WZ term, expressed as an indefinite integral, is the variable conjugate to the central generator [7,8]. In our case, we obtain the superpotential term in the action in a similar way as the variable conjugate to the $U(1)$ charge B, *even though this charge is not central*. The fact that the superpotential term in the superspace action [4] cannot be written in manifestly invariant superspace form leads to a modification of the algebra of Noether charges. This is in close analogy to the modification of the supertranslation currents for the super p-branes as a consequence of the non-manifest supersymmetry of the WZ terms in their actions [9].

The analogy is not complete, however, because in the case under study here the modification can be removed by a redefinition of the $U(1)$ charge. It is this redefinition that leads to the l-dependent expression $B = \hat{B} + 2l$ for the $U(1)$ charge that we mentioned previously. The mathematical explanation for this l-dependence is therefore the non-manifest nature of the superconformal invariance of the superpotential term in the action. Its physical significance is best seen in the context of an embedding of the $SU(1,1|1)$ model into the $SU(1,1|2)$ superconformal model, because of the interpretation of the (superspace) field equation of the latter as the radial equation for a superparticle near an extreme RN black hole.

Many technical aspects of the discussion to follow of the $N=2$ and $N=4$ superconformal mechanics models are similar to those in [6], which we became aware of after submission to the archives of an earlier version of this paper. However, the thrust of our argument is quite different, centering as it does on our improved understanding of the nature and significance of the superpotential term and the black hole interpretation.

II. $N=2$ SUPERCONFORMAL MECHANICS

We now turn to the $OSp(2|2)$-invariant $N=2$ superconformal mechanics of [3,4]. The anticommutation relations of the Lie superalgebra $Osp(2|2)$ are those of Eqs. (6). We select $(H,Q_i)(i=1,2)$ as the "unbroken" generators associated with the real super-world-line coordinates $\zeta^M = (t, \eta^i)$. As before, it is convenient to define $\tau \equiv [dt - (i/2)d\eta^i \eta_i]$ because we then have

$$d = d\tau\, \partial_t + \eta^i D_i, \qquad (21)$$

where D_i are the supercovariant derivatives of Eqs. (5).

We may write the $OSp(2|2)$ group element as

$$g(t,\eta) = e^{-itH} e^{i\eta^i Q_i} e^{i\lambda^i(t,\eta)S_i} e^{iz(t,\eta)D} e^{i\omega(t,\eta)K} e^{ia(t,\eta)B}. \qquad (22)$$

Defining $d\xi^M \equiv (d\tau, d\eta_i)$ and $H_A = (H_0, H_i) \equiv (H, Q_i)$ we can rewrite this as

$$ig^{-1}dg = d\xi^M E_M{}^A[H_A - (\mathcal{D}_A z)D - (\mathcal{D}_A \omega)K$$
$$- (\mathcal{D}_A \lambda_i)S^i - (\mathcal{D}_A a)B], \qquad (23)$$

where $E_M{}^A$ is the world line supervielbein and $\mathcal{D}_A$ is the group-covariant derivative. A calculation yields

$$E_M{}^A = \begin{pmatrix} e^{-z} & -e^{-z/2}\lambda^T R(a) \\ 0 & -e^{-z/2}R(a) \end{pmatrix}, \qquad (24)$$

where λ^T means the transpose of λ^i as a two-vector and $R(a)$ is a 2×2 rotation matrix:

$$R(a) = \begin{pmatrix} \cos a & -\sin a \\ \sin a & \cos a \end{pmatrix}. \qquad (25)$$

Note that

$$\mathrm{sdet}\, E = -1, \qquad (26)$$

so that manifestly invariant actions have the form

$$I = \int dt\, d^2\eta\, \mathcal{L}(\mathcal{D}_A \phi), \qquad (27)$$

where $\phi = (z, \omega, a)$ denotes the set of world line superfields. The covariant derivatives can be written as

$$\mathcal{D}_A = E_A{}^M \mathcal{D}_M, \qquad (28)$$

where $E_A{}^M$ is the inverse supervielbein,

$$E_A{}^M = \begin{pmatrix} e^z & -e^z \lambda^T \\ 0 & -e^{z/2}R^T(a) \end{pmatrix}, \qquad (29)$$

and $\mathcal{D}_M = (\mathcal{D}_\tau, \mathcal{D}_{\eta^i})$ are the components of the covariant derivatives on the (still non-coordinate) basis $(d\tau, d\eta^i)$. The transformation properties of $\mathcal{D}_M \phi$ are not as simple as those of $\mathcal{D}_A \phi$ (which are super-world-line scalars) but they have a simpler form. The expressions of $\mathcal{D}_M \phi$ are found to be

$$\mathcal{D}_M z = (2\omega e^{-z} + \dot{z}, \quad D_i z - i\lambda_i),$$

$$\mathcal{D}_M \omega = \left(\dot{\omega} - \omega\dot{z} - e^{-z}\omega^2 - \frac{i}{2}e^z \dot{\lambda}_i \lambda^i, \quad D_i\omega - \omega D_i z + i\omega\lambda_i - \frac{i}{2}e^z D_i \lambda_j \lambda^j \right),$$

$$\mathcal{D}_M \lambda_j = \left(e^{z/2}\dot{\lambda}_k R^k{}_j - e^{-z/2}\omega\lambda_k R^k{}_j, \quad e^{z/2}D_i\lambda_k R^k{}_j - e^{-z/2}\omega R_{ij} + \frac{i}{2}e^{z/2}\lambda_i\lambda_k R^k{}_j \right),$$

$$\mathcal{D}_M a = \left(\dot{a} - \frac{i}{2}\lambda_1\lambda_2, \quad D_i a - \frac{i}{2}\epsilon_{ij}\lambda^j \right). \qquad (30)$$

To proceed, we begin by imposing the manifestly invariant constraint

$$\mathcal{D}_A z = 0, \tag{31}$$

which is equivalent to $\mathcal{D}_M z = 0$ and is solved, algebraically, by

$$\omega = -\frac{1}{2} e^z \dot{z}, \quad \lambda_i = -i D_i z. \tag{32}$$

As in the previous cases, we could arrange for these equalities to arise as equations of motion for ω and λ_i, but in this case it is simpler to impose Eq. (31) as a constraint. As a direct consequence of Eq. (32), we then find that the $A = i$ components of $\mathcal{D}_A \lambda_j$ satisfy

$$\mathcal{D}_{(i} \lambda_{j)} = 0; \tag{33}$$

so the *manifestly* superconformal invariant, and $SO(2)$ invariant, Lagrangian of lowest dimension must be a linear combination of $\mathcal{D}_0 a$ and $\varepsilon^{ij} \mathcal{D}_i \lambda_j$. If we insist that our action describe the dynamics of a particle in a *one-dimensional* space, with (real) coordinate $z(t) = z(t, \eta_i)|$, then we cannot make use of $\mathcal{D}_0 a$. In this case, and using that $\mathcal{D}_A \lambda_j$ for $A = i$ is given by $e^z D_i \lambda_j - \omega \delta_{ij} + (i/2) e^z \lambda_i \lambda_j$, we get

$$\mathcal{L} \propto \varepsilon^{ij} \mathcal{D}_i \lambda_j = -2 e^z \left(D_1 \lambda_2 + \frac{i}{2} \lambda_1 \lambda_2 \right). \tag{34}$$

Then, using the constraint (32), adjusting the proportionality constant, and integrating by parts, we arrive at the action

$$I_{kin} = \frac{m}{2} \int dt \, d^2 \eta \, e^z D_1 z D_2 z, \tag{35}$$

which is the first part of Eq. (4) with $z = \log x^2$. Let the components of the $z(t, \eta_i)$ superfield be defined by $z(t) = z|, \lambda_i = -i D_i z|$ and $F' = -i D_1 D_2 z|$. Then, defining new variables x, χ, F by

$$x = e^{z/2}, \quad \chi_i = -\sqrt{\frac{m}{2}} e^{z/2} \lambda_i, \quad F = 2F', \tag{36}$$

and performing the η^i integrations, we arrive at a component action with the Lagrangian

$$\mathcal{L}_{kin} = \left[\frac{m}{2} \dot{x}^2 + \frac{i}{2} (\chi_1 \dot{\chi}_1 + \chi_2 \dot{\chi}_2) + \frac{1}{8} m x^2 F^2 + \frac{i}{2} F \chi_1 \chi_2 \right]. \tag{37}$$

After elimination of F by its algebraic equation of motion the bosonic Lagrangian reduces to that of Eq. (1), and so we have now constructed an $OSp(2|2)$-invariant extension of the $g = 0$ conformal mechanics. All other *manifestly* invariant actions must involve either higher-derivatives, higher powers of first derivatives or (non-auxiliary) bosonic variables other than $x(t)$. Thus, any $OSp(2|2)$-invariant generalization of the $g \neq 0$ conformal mechanics *cannot be described by a manifestly invariant action*. This does not exclude the possibility of an action that is invariant but not

manifestly invariant. The existence of such ''non-manifest'' invariants has usually been associated with the possibility of a central extension of the Lie (super)algebra of the symmetry (super)group. In such cases the action is a WZ term (see, for example, [10]). A number of super-world-line examples of this were discussed in [8]. In our case, however, there can be no central extension because the relevant cohomology of the $osp(2|2)$ algebra is trivial. One might therefore be tempted to conclude that there can be no further $OSp(2|2)$-invariants and hence that there is no $OSp(2|2)$-invariant extension of the g/x^2 potential of conformal mechanics. But this would be wrong, as we now explain.

A further $OSp(2|2)$-invariant may be found by the method of [8]. We first note that the bosonic and spinor components of $\mathcal{D}_A a = (\mathcal{D}_0 a, \mathcal{D}_i a)$ are independent super-world-line scalar fields because invariance of the relation $\{D_i, D_j\} = -i \delta_{ij} \partial_t$ requires the structure group of the frame bundle to be just $SO(2)$. The group covariant derivatives $\mathcal{D}_A$ transform as a $SO(2)$ doublet for $A = i = 1,2$; so the manifestly $OSp(2|2)$ invariant constraint $\mathcal{D}_i a = 0$ is also $SO(2)$ invariant, and equivalent to

$$D_i a - \frac{i}{2} \epsilon_{ij} \lambda^j = 0. \tag{38}$$

Since $\lambda_i = -i D_i z$ [Eq. (32)], this new constraint implies that

$$D_1 a = \frac{1}{2} D_2 z, \tag{39}$$

and hence that

$$\dot{a} = i D_1 D_2 z. \tag{40}$$

This can be integrated to give

$$a(t) = a| = i \int^t dt' D_1 D_2 z(t', \eta)| = i \int^t dt' d^2 \eta \, z(t', \eta). \tag{41}$$

The variable field $a(t) = a|$ can thus be viewed as a superspace action in the form of an indefinite integral. This new action is superconformal invariant, up to a surface term, because Eq. (40) implies that

$$\delta(i D_1 D_2 z) = \partial_t \delta a. \tag{42}$$

The left hand side is the variation of the component Lagrangian of the new superspace action whereas the right hand side is a total time derivative. By itself, this is not quite sufficient to establish the desired result. According to Eq. (40), the component Lagrangian is itself a total time derivative, and so it is hardly surprising that the same is true of its variation. Of course, Eq. (40) tells us nothing about the component Lagrangian; instead it provides us with information about the independent superfield a. However, while a is an independent superfield, its variation δa is not. In fact δa is a function of the superfield z and its derivatives, and is independent of a. Thus, we indeed learn from Eq. (42) that the variation of the component Lagrangian of the superspace action (41) is

a total time derivative, and hence that this action is invariant up to a surface term. We have now deduced that the action

$$I=I_{kin}-2il\int dt\,d^2\eta\,z, \tag{43}$$

where I_{kin} is given in Eq. (35), is $OSp(2|2)$ invariant for arbitrary real constant l. Setting $z=\log x^2$ this action is seen to be precisely that of Eq. (4). The superpotential term is not *manifestly* invariant because z does not transform as a scalar density. A calculation shows that z fails to transform as a scalar density by a term that, being *linear* in η, does not contribute to the variation of the superspace integral. Despite the non-manifest superconformal invariance of the action (43) the z superfield equation *can* be expressed in the manifestly superconformal invariant form

$$\varepsilon^{ij}\mathcal{D}_i\lambda_j=4l/m \tag{44}$$

(recall that $\mathcal{D}_i$ are the $A=i$ components of $\mathcal{D}_A$).

The component Lagrangian including the contribution of the superpotential term (obtained by performing the superspace integrations) is

$$\mathcal{L}=\frac{1}{2}m\dot{x}^2+\frac{i}{2}(\chi_1\dot{\chi}_1+\chi_2\dot{\chi}_2)+\frac{1}{8}mx^2F^2+F\left(l+\frac{i}{2}\chi_1\chi_2\right). \tag{45}$$

Elimination of F now yields

$$\mathcal{L}=\frac{1}{2}m\dot{x}^2+\frac{i}{2}(\chi_1\dot{\chi}_1+\chi_2\dot{\chi}_2)-\frac{2}{mx^2}l(l+i\chi_1\chi_2). \tag{46}$$

Setting the fermions to zero we recover the bosonic Lagrangian of Eq. (1) with

$$g=4l^2/m. \tag{47}$$

Thus, we have found an $OSp(2|2)$ invariant extension of conformal mechanics. It is just the model constructed in [3,4]. There is an apparent discrepancy with Eq. (2) but this will be resolved after we have looked at the $SU(1,1|2)$ model.

III. $N=4$ SUPERCONFORMAL MECHANICS

We turn now to the $SU(1,1|2)$-invariant $N=4$ superconformal mechanics describing the full superparticle radial dynamics. The $su(1,1|2)$ superalgebra is spanned by the $sl(2;\mathbb{R})$ generators (H,K,D), the $SU(2)$ generators $J_a(a=1,2,3)$, and the $SU(2)$ doublet supersymmetry charges (Q^i,S^i) and the Hermitian conjugates $(\bar{Q}_i,\bar{S}_i)$. The superalgebra has the following non-vanishing (anti)commutation relations:

$$[H,D]=iH, \quad [K,D]=-iK,$$

$$[H,K]=2iD, \quad [J_a,J_b]=i\varepsilon_{abc}J_c,$$

$$\{Q^i,\bar{Q}_j\}=2\delta^i_j H, \quad \{S^i,\bar{S}_j\}=2\delta^i_j K,$$

$$\{Q^i,\bar{S}_j\}=2(\sigma_a)_j{}^i J_a+2i\delta^i_j D,$$

$$\{\bar{Q}_i,S^j\}=2(\sigma_a)_i{}^j J_a-2i\delta^j_i D,$$

$$[D,Q^i]=-\frac{i}{2}Q^i, \quad [D,\bar{Q}_i]=-\frac{i}{2}\bar{Q}_i,$$

$$[D,S^i]=\frac{i}{2}S^i, \quad [D,\bar{S}_i]=\frac{i}{2}\bar{S}_i,$$

$$[K,Q^i]=S^i, \quad [K,\bar{Q}_i]=-\bar{S}_i,$$

$$[H,S^i]=Q^i, \quad [H,\bar{S}_i]=-\bar{Q}_i,$$

$$[J_a,Q^i]=-\frac{1}{2}Q^j(\sigma_a)_j{}^i, \quad [J_a,\bar{Q}_j]=\frac{1}{2}(\sigma_a)_j{}^k\bar{Q}_k,$$

$$[J_a,S^i]=-\frac{1}{2}S^j(\sigma_a)_j{}^i, \quad [J_a,\bar{S}_j]=\frac{1}{2}(\sigma_a)_j{}^k\bar{S}_k. \tag{48}$$

We take the super-world-line-valued supergroup element to be

$$g(t,\eta,\bar{\eta})=e^{-itH}e^{i(\eta_i Q^i+\bar{\eta}^i\bar{Q}_i)}e^{i(\lambda_i S^i+\bar{\lambda}^i\bar{S}_i)}$$
$$\times e^{izD}e^{i\omega K}e^{i\phi J_1}e^{i\theta J_2}e^{i\psi J_3}, \tag{49}$$

where $\lambda,\bar{\lambda}$, z, ω, ϕ,θ, ψ depend on $(t,\eta,\bar{\eta})$. The anticommuting coordinates η^i and $\bar{\eta}_i$ are related by complex conjugation, i.e., $(\eta^i)*=\bar{\eta}_i$. We shall again define

$$d\tau=dt-i(\eta_i d\bar{\eta}^i+\bar{\eta}^i d\eta_i), \tag{50}$$

which leads to

$$d=d\tau\,\partial_t+d\eta_i D^i+d\bar{\eta}^i\bar{D}_i, \tag{51}$$

where

$$D^i_\eta\equiv D^i=\frac{\partial}{\partial\eta_i}-i\bar{\eta}^i\frac{\partial}{\partial t}, \quad \bar{D}_{\eta\,i}\equiv\bar{D}_i=\frac{\partial}{\partial\bar{\eta}^i}-i\eta_i\frac{\partial}{\partial t} \tag{52}$$

are the superspace covariant derivatives satisfying $\{D^i,\bar{D}_j\}=-2i\delta^i_j\partial_t$. It should also be noted that $\bar{D}_i=-(D^i)*$.

The left-invariant 1-form can be written as

$$ig^{-1}dg=d\xi^M[E_M{}^A H_A-(\mathcal{D}_M z)D-(\mathcal{D}_M\omega)K-(\mathcal{D}_M\lambda)^i S_i$$
$$-(\mathcal{D}_M\bar{\lambda})_i\bar{S}^i-(\mathcal{D}_M\phi)J_1-(\mathcal{D}_M\theta)J_2-(\mathcal{D}_M\psi)J_3], \tag{53}$$

where

$$d\xi^M=(d\tau,d\eta^i,d\bar{\eta}_i), \quad H_A=(H,Q_i,\bar{Q}^i). \tag{54}$$

A calculation yields

$$E_M{}^A = \begin{pmatrix} e^{-z} & ie^{-z/2}\lambda_l s_j{}^l & -ie^{-z/2}\bar{\lambda}^l(s^{-1})_l{}^j \\ 0 & -e^{-z/2}s_j{}^i & 0 \\ 0 & 0 & -e^{-z/2}(s^{-1})_i{}^j \end{pmatrix}. \quad (55)$$

$$E_A{}^M = \begin{pmatrix} e^z & ie^z\lambda_j & -ie^z\bar{\lambda}^j \\ 0 & -e^{z/2}(s^{-1})_j{}^i & 0 \\ 0 & 0 & -e^{z/2}s_i{}^j \end{pmatrix}. \quad (56)$$

We shall need the inverse supervielbein

Further calculation yields

$$\mathcal{D}_M z = (2\omega e^{-z} + \dot{z}, \quad D^i z + 2\bar{\lambda}^i, \quad \text{c.c.}),$$

$$\mathcal{D}_M \omega = \Big(\dot{\omega} - \omega\dot{z} - e^{-z}\omega^2 + i(\lambda\dot{\bar{\lambda}} + \bar{\lambda}\dot{\lambda})e^z - \frac{1}{36}(\bar{\lambda}\sigma\lambda)^2 e^z,$$

$$D^i\omega - \omega D^i z - 2\omega\bar{\lambda}^i - i(\lambda D^i\bar{\lambda} + \bar{\lambda}D^i\lambda)e^z - \frac{i}{3}e^z(\bar{\lambda}\sigma_a\lambda)(\sigma_a\bar{\lambda})^i, \quad \text{c.c.}\Big)$$

$$\mathcal{D}_M \lambda_i = \Big(\Big(\dot{\lambda}_k e^{z/2} - \omega e^{-z/2}\lambda_k + \frac{i}{6}e^{z/2}(\bar{\lambda}\sigma_a\lambda)(\lambda\sigma_a)_k \Big) s_i{}^k,$$

$$\Big(D^j\lambda_k e^{z/2} - i\omega e^{-z/2}\delta_k^j - \frac{1}{2}e^{z/2}(\bar{\lambda}\sigma_a)^j(\lambda\sigma_a)_k - \frac{1}{2}e^{z/2}\bar{\lambda}^j\lambda_k \Big) s_i{}^k,$$

$$\Big(\bar{D}_j\lambda_k e^{z/2} - \frac{1}{2}e^{z/2}(\lambda\sigma_a)_j(\lambda\sigma_a)_k + \frac{1}{2}e^{z/2}\lambda_j\lambda_k \Big) s_i{}^k \Big) \quad (57)$$

and

$$\mathcal{D}_M\phi = (\dot{\phi}\cos\theta\cos\psi - \dot{\theta}\sin\psi + i[Ad(s^{-1})]_{a1}\bar{\lambda}\sigma_a\lambda, \quad D^i\phi\cos\theta\cos\psi - D^i\theta\sin\psi - 2i[Ad(s^{-1})]_{a1}(\bar{\lambda}\sigma_a)^i, \quad \text{c.c.}),$$

$$\mathcal{D}_M\theta = (\dot{\theta}\cos\psi + \dot{\phi}\cos\theta\sin\psi + i[Ad(s^{-1})]_{a2}\bar{\lambda}\sigma_a\lambda, \quad D^i\theta\cos\psi + D^i\phi\cos\theta\sin\psi - 2i[Ad(s^{-1})]_{a2}(\bar{\lambda}\sigma_a)^i, \quad \text{c.c.}),$$

$$\mathcal{D}_M\psi = (\dot{\psi} - \dot{\phi}\sin\theta + i[Ad(s^{-1})]_{a3}\bar{\lambda}\sigma_a\lambda, \quad D^i\psi - D^i\phi\sin\theta - 2i[Ad(s^{-1})]_{a3}(\bar{\lambda}\sigma_a)^i, \quad \text{c.c.}), \quad (58)$$

where the explicit forms of $s_i{}^k$ and $[Ad(s^{-1})]$ are

$$s_i{}^k = (e^{i\psi\sigma_3/2}e^{i\theta\sigma_2/2}e^{i\phi\sigma_1/2})_i{}^k,$$

$$[Ad(s^{-1})] = \begin{pmatrix} \cos\theta\cos\psi & -\cos\phi\sin\psi + \sin\phi\sin\theta\cos\psi & \sin\phi\sin\psi + \cos\phi\sin\theta\cos\psi \\ \cos\theta\sin\psi & \cos\phi\cos\psi + \sin\phi\sin\theta\sin\psi & -\sin\phi\cos\psi + \cos\phi\sin\theta\sin\psi \\ -\sin\theta & \sin\phi\cos\theta & \cos\phi\cos\theta \end{pmatrix}. \quad (59)$$

The supercovariant derivatives $\mathcal{D}_A z$, etc., can now be found from the formula $\mathcal{D}_A = E_A{}^M \mathcal{D}_M$. We begin, as before, by imposing the manifestly invariant constraint $\mathcal{D}_A z = 0$, which yields

$$\omega = -\frac{1}{2}e^z\dot{z}, \quad \lambda_i = \frac{1}{2}\bar{D}_i z. \quad (60)$$

This leaves z as the only independent superfield. Manifest $SU(1,1|2)$ invariants will be expressed as full superspace integrals of the form $\int dt d^4\eta\, \text{sdet}E\mathcal{L}$ where $\mathcal{L}$ is a super-world-line scalar, but there is now no Lagrangian built from covariant derivatives of z that has a dimension low enough to

yield a kinetic term containing an $\dot{x}^2$ term. This problem could be circumvented by imposing the complex constraint

$$\varepsilon_{ij}\mathcal{D}^i\bar{\lambda}^j = 0. \quad (61)$$

The linearization of this constraint yields $\varepsilon_{ij}D^iD^j z = 0$. This is the reduction to $D=1$ of the $D=4$ ''linear'' superfield constraint, which is solved in terms of a conserved vector. The reduction to $D=1$ of a conserved vector is a triplet $X^i{}_j(X^i{}_i=0)$ and a singlet X subject to the constraint $\dot{X}=0$. The latter constraint means that the equation of motion for X (obtained by variation of an action in which X is treated as an

unconstrained superfield) is actually the time-derivative of the true equation of motion. Thus, the true field equation is the once-integrated X-equation in which there is an arbitrary integration constant. This analysis will apply equally to the full constraints (60) except that their solution in terms of $X^i{}_j$ and X will be more involved.[7] We thus deduce that the remaining equations of the $SU(1,1|2)$ superconformal mechanics have the form of an $SU(2)$ triplet equation for z and a singlet equation involving an arbitrary constant. Both must be constructed from the supercovariant derivatives $\mathcal{D}_A\lambda$ and complex conjugates in order to be manifestly $SU(1,1|2)$-invariant equations of the appropriate dimension. There is only one candidate for the triplet equation:

$$\mathcal{D}_{(i}\lambda_{j)}=0. \tag{62}$$

The singlet equation is

$$\mathcal{D}^i\lambda_i+\bar{\mathcal{D}}_i\bar{\lambda}^i=8l/m; \tag{63}$$

as anticipated, it involves an arbitrary integration constant. Choosing the constant as above, one finds that the bosonic field equation is precisely equivalent to that derived from Eq. (1), again with $g=4l^2/m$.

IV. SUPERPARTICLE/BLACK HOLE INTERPRETATION

We claimed in the Introduction that the $N=2$ and $N=4$ models of superconformal mechanics describe a particular limit of the (planar or full) radial dynamics of a superparticle near the horizon of an extreme RN black hole. In order to justify this claim we must first account for the discrepancy between the formula (47) for g with the formula (2) found from the superparticle. To do so we must take into account quantum mechanics. The Hamiltonian corresponding to the Lagrangian (46) is

$$H=\frac{p^2}{2m}+\frac{4l(l+\hat{B})}{2mx^2}, \tag{64}$$

where

$$\hat{B}=\frac{i}{2}[\chi_1,\chi_2]. \tag{65}$$

The phase space Lagrangian is

$$\mathcal{L}=p\dot{x}+\frac{i}{2}\delta^{ij}\chi_i\dot{\chi}_j-H, \tag{66}$$

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[7] It was shown in [11] how such non-linear constraints may be solved, at least in principle.

so that the canonical (anti)commutation relations of the quantum theory are

$$[x,p]=i, \quad \{\chi_i,\chi_j\}=-2\delta_{ij}. \tag{67}$$

The anticommutation relations are realized by the operators $\chi_1=i\sigma_1,\chi_2=i\sigma_2$, in which case $\hat{B}=\sigma_3$. We see from this that the eigenvalues of $\hat{B}$ as an operator in the quantum theory are ±1. On the $+1$ eigenspace we have

$$H=\frac{p^2}{2m}+\frac{g}{2x^2} \tag{68}$$

with

$$g=4l(l+1)/m, \tag{69}$$

which is the $m=q$ case of Eq. (2). On the -1 eigenspace of B we can take $l\to-l$ to arrive at the same result. Thus, our results are consistent with those obtained in [2] once quantum effects are included (as they implicitly were in [2]). We thus confirm the identification of the constant l in the superconformal mechanics model as the orbital angular momentum of a particle near the horizon of a large mass extreme RN black hole.

The operator $\hat{B}$ is the Noether charge (called B in [4]). When $l\neq0$ this is not to be identified with the $U(1)$ charge B in the superalgebra (6). Instead, we have

$$B=\hat{B}+2l. \tag{70}$$

That this is a consequence of the non-manifest invariance of the superpotential term can be seen as follows. The action

$$2l\int dt[\dot{a}(t)-iD_1D_2z|] \tag{71}$$

is manifestly invariant; so the Noether charges $\mathcal{N}$ computed from *this* action by the prescription $\delta I=\int\dot{c}\cdot\mathcal{N}$, where c is a set of parameters promoted to function of time, must close to the algebra of Eqs. (6). But a shifts by a constant under $U(1)$ so that the Noether charge for $I=I_{kin}+$ Eq. (71) is now the B of Eq. (70). Dropping the $\dot{a}$ term from Eq. (71) leaves us with the actual, but non-manifest, invariant superconformal mechanics action without the $2l$ contribution to the $U(1)$ charge. Note that the only additional contribution to the Noether charge from $\dot{a}$ in Eq. (71) comes from the group variation $a'(t')-a(t)$ of the first component of the superfield $a(t,\eta_i)$ which is only affected by the $U(1)$ transformations.

As we have shown, the $OSp(2|2)$ invariant superconformal mechanics is a truncation of an $SU(1,1|2)$ invariant model. The same is true of the superalgebras; if one sets $Q^2=0$ and $Q^1=Q$, and similarly for S^i, and also $J_1=J_2=0$, then one arrives at the algebra of $SU(1,1|1)$ in which (Q,S) is the complex $SU(1,1)$ doublet of supercharges and J_3 is the $U(1)$ charge. We can now write $Q=Q^1+iQ^2$ where Q^i are the real supercharges of the isomorphic

$osp(2|2)$ superalgebra and similarly for S. Comparison with the $osp(2|2)$ (anti)commuatiton relations given earlier then leads to the identification $2J_3 = B$. Hence,

$$J_3 = l + \frac{1}{2}\hat{B}. \tag{72}$$

If we restrict the dynamics of the particle described by the $SU(1,1|2)$ model to motion in an equatorial plane, then J_3 is the particle's angular momentum. We see from Eq. (72) that this angular momentum has an orbital component l, arising from the presence of the potential term in the action, and a spin component, arising from the fermion variables.

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