Cosmic microwave background anisotropies: Nonlinear dynamics

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We develop a new approach to local nonlinear effects in cosmic microwave background anisotropies, and discuss the qualitative features of these effects. New couplings of the baryonic velocity to radiation multipoles are found, arising from nonlinear Thomson scattering effects. We also find a new nonlinear shear effect on small angular scales. The full set of evolution and constraint equations is derived, including the nonlinear generalizations of the radiation multipole hierarchy, and of the dynamics of multi-fluids. These equations govern radiation anisotropies in any inhomogeneous spacetime, but their main application is to second-order effects in a universe that is close to the Friedmann models. Qualitative analysis is given here, and quantitative calculations are taken up in further papers. [S0556-2821(99)02508-4]

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I. INTRODUCTION

Recent and upcoming advances in observations of the cosmic microwave background (CMB) radiation are fueling the construction of increasingly sophisticated and detailed models to predict the anisotropy on small angular scales. Such models require highly specific input in order to produce numerical results, and they involve intricate problems of computation. As a complement to such specific predictive models, it is also useful to pursue a more qualitative and analytical investigation of CMB anisotropies. A general qualitative analysis does not rely on detailed assumptions about the origin of primordial fluctuations, the density parameters of the background, re-ionization and structure formation history, etc. Instead, the aim is to better understand the underlying physical and geometric factors in the dynamics of radiation anisotropies, and hopefully to uncover new results and insights. In this paper, we follow such an approach, and develop a new analysis of local nonlinear effects in CMB anisotropies. We are able to give a physically transparent qualitative analysis of how inhomogeneities and relative motions produce nonlinear effects in CMB anisotropies. We derive the nonlinear generalization of Thomson scattering, and we find a new nonlinear shear effect on small scales.

We use a 1+3 covariant approach (i.e., a ''covariant Lagrangian'' approach) to CMB anisotropies, based on the choice of a physically determined 4-velocity vector field u^a. This allows us to derive the exact nonlinear equations for physical quantities as measured by observers moving with that 4-velocity. Then the nonlinear equations provide a covariant basis for investigating second-order effects, as well as for linearizing about a Friedmann-Lemaitre-Robertson-Walker (FLRW) background. The basic theoretical ingredients are (a) the covariant Lagrangian dynamics of Ehlers [1] and Ellis [2], and the perturbation theory of Hawking [3] and Ellis and Bruni [4] which is derived from it; (b) the 1+3 covariant kinetic theory formalism of Ellis, Treciokas and Matravers [5,6] (which builds on work by Ehlers, Geren and Sachs [7], Treciokas and Ellis [8] and Thorne [9]); and (c) the 1+3 covariant analysis of temperature anisotropies due to Maartens, Ellis and Stoeger [10].

The well-developed study of CMB anisotropies is based on the pioneering results in CMB physics (Sachs and Wolfe [11], Rees and Sciama [12], Peebles and Yu [13], Sunyaev and Zeldovich [14], Grishchuk and Zeldovich [15], and others), and on the development of gauge-invariant perturbation theory, particularly by Bardeen [16] and Kodama and Sasaki [17] (building on the work of Lifshitz [18]). There are comprehensive and detailed models—see e.g. Hu and Sugiyama [19–21], Ma and Bertschinger [22], Seljak *et al.* [23–26] and Durrer and Kahniashvili [27]. These provide the basis for sophisticated predictions and comparisons with the observations of recent, current and future satellite and ground-based experiments. The hope is that this inter-play between theory and observation (including the large-scale galactic distribution and other observations), in the context of inflationary cosmology, will produce accurate values for the various parameters that characterize the standard models, thus allowing theorists to discriminate between competing models (see for example [28,29]).

While these papers have provided a near-exhaustive treatment of many of the issues involved in CMB physics, there are a number of reasons for pursuing a complementary 1+3 covariant approach, as developed in [10,30–38].

First, the covariant approach by its very nature incorporates nonlinear effects. This approach starts from an inhomogeneous and anisotropic universe, without *a priori* restrictions on the degree of inhomogeneity and anisotropy, and then applies the linearization limit when required. The 1+3 covariant equations governing CMB anisotropies are thus applicable in fully nonlinear generality. These equations can then be specialized in various ways in addition to a standard FLRW linearization. Second-order effects in an almost-FLRW universe probably form the most important possibility, given the increasing accuracy and refinement of observations. The study of CMB anisotropies in homogeneous Bianchi universes with large anisotropy is another possibility that flows directly from general nonlinear equations. Such applications will be the subject of future papers in the program. The current paper is concerned with setting up the general dynamical equations and identifying the qualitative nature of nonlinear effects. (The general algebraic

equations are derived in [37].)

Second, the 1+3 covariant approach is based entirely on quantities with a direct and transparent physical and geometric interpretation, and the fundamental quantities describing anisotropy and inhomogeneity are all automatically gauge-invariant when a suitable covariant choice of fundamental 4-velocity has been made. As a consequence the approach leads to results with unambiguous physical meaning (provided the fundamental 4-velocity field is chosen in a physically unique and appropriate way; we discuss the various options below).

This approach has been developed in the context of density perturbations [4,39–54] and gravitational wave perturbations [3,55–59]. (See also [60] for a recent review.) In relation to CMB anisotropies, the covariant Lagrangian approach was initiated[1] by Stoeger, Maartens and Ellis [30], who proved the following result: *if all comoving observers in an expanding universe region measure the anisotropy of the CMB after last scattering to be small, then the universe is almost FLRW in that region.*[2] No *a priori* assumptions are made on the spacetime geometry or on the source and nature of CMB anisotropies, so that this result provides a general theoretical underpinning for CMB analysis in perturbed FLRW universes. It effectively consitutes a proof of the stability of the corresponding exact-isotropy result of Ehlers, Geren and Sachs [7]. The weak Copernican principle implicit in the assumption that all fundamental observers see small anisotropy is in principle partially testable via the Sunyaev-Zeldovich effect (see [32] and references therein). The qualitative result was extended to a quantitative set of limits on the anisotropy and inhomogeneity of the universe imposed by the observed degree of CMB anisotropy, independently of any assumptions on cosmic dynamics or perturbations before recombination [10,31–33].

More recently, this approach to CMB anisotropies in an almost FLRW universe has been extended by Dunsby [34], who derived a 1+3 covariant version of the Sachs-Wolfe formula, and by Challinor and Lasenby [35,36], who performed a comprehensive 1+3 covariant analysis of the imprint of scalar perturbations on the CMB, confirming the results of other approaches from this viewpoint and bringing new insights and clarifications via the covariant approach. In [35], they also discuss qualitatively the imprint of tensor perturbations on the CMB, in the covariant approach (see [38] for quantitative results).

This paper is closely related to, and partly dependent upon, all of these previous 1+3 covariant analyses. It extends and generalizes aspects of these papers, using and developing the covariant nonlinear Einstein-Boltzmann-hydrodynamic formalism. We analyze the nonlinear dynamics of radiation anisotropies, with the main application being second-order effects in an almost FLRW universe. We identify and describe the qualitative features of such effects. This lays the basis for a generalization of results on well-known second-order effects, such as the Rees-Sciama and Vishniac effects (see e.g. [19]), and on recent second-order corrections of the Sachs-Wolfe effect [63,64]. Developing a quantitative analysis on the basis of the equations and qualitative analysis given here is the subject of further work. Ultimately this involves the solution of partial differential equations, which requires in particular a choice of coordinates, breaking covariance. However, the 1+3 covariant approach means that all the equations and variables have a direct and transparent physical meaning.

In Sec. II, the covariant Lagrangian formalism for relativistic cosmology is briefly summarized. Section III develops an exact 1+3 covariant treatment of multi-fluids and their relative velocities, building on [52]. In Sec. IV, the covariant Lagrangian approach to kinetic theory is outlined. Section V develops a nonlinear treatment of Thomson scattering, which identifies new couplings of the baryonic relative velocity to the radiation multipoles. We derive the hierarchy of exact covariant multipole equations which arise from the Boltzmann equation. This section uses and generalizes a combination of the results of Ellis *et al.* [5] on the multipoles of the Boltzmann equation in general, Maartens *et al.* [10] on a covariant description of temperature fluctuations, and Challinor and Lasenby [36] on Thomson scattering. The equations constitute a covariant and nonlinear generalization of previous linearized treatments. In Sec. VI, we consider qualitative implications of the nonlinear equations. We identify the role of the kinematic quantities in the nonlinear terms, and comment on the implications for second-order effects, which include a new nonlinear shear correction to CMB anisotropies on small angular scales. We also give the multipole equations for the case where the radiation anisotropy is small, but spacetime anisotropy and inhomogeneity are unrestricted.[3]

Finally, we give the linearized form of the multipole equations, regaining the equations of Challinor and Lasenby [36]. This provides a covariant Lagrangian version of the more usual metric-based formalism of gauge-invariant perturbations (see e.g. [65,20,22,23]). In a further paper [66], the linearized equations derived here are expanded in covariant scalar modes, and this is used to determine analytic properties of CMB linear anisotropy formation.

We follow the notation and conventions of [2,5,10], with the improvements and developments introduced by [67,48]. In particular, the units are such that c, $8\pi G$ and k_B are equal to 1; the signature is $(-+++)$; spacetime indices are $a,b,\ldots = 0,1,2,3$; the curvature tensors are $R^a{}_{bcd} = -\partial_d \Gamma^a{}_{bc} + \cdots, R_{ab} = R^c{}_{acb}$ and $R = R^a{}_a$; the Ricci identity is $\nabla_{[a}\nabla_{b]}u_c = \frac{1}{2}R_{abcd}u^d$; A_l denotes the index string $a_1 a_2 \cdots a_l$; e^{A_l} denotes the tensor product $e^{a_1}e^{a_2}\cdots e^{a_l}$;

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[1]A 1+3 covariant approach to CMB anisotropy was independently outlined by Bonanno and Romano [61] in general terms, using a flux-limited diffusion theory, but the detailed implications of small CMB anisotropy were not pursued.

[2]Note the importance of expansion: a *static* isotropic cosmology with arbitrarily large inhomogeneity can be constructed in which all observers see an isotropic CMB [62].

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[3]This case will apply before decoupling, in order to be consistent with the almost-FLRW result quoted above.

(square) round brackets enclosing indices denote the (anti-)symmetric part; and angular brackets denote the projected symmetric and tracefree (PSTF) part (defined below). The spatially projected part of the covariant derivative is denoted by D_a, following [67].[4] The approximate equality symbol, as in $J \approx 0$, indicates equality up to first (linear) order in an almost-FLRW spacetime.

II. COVARIANT LAGRANGIAN FORMALISM IN RELATIVISTIC COSMOLOGY

The Ehlers-Ellis $1+3$ formalism [1,2,68] is a covariant Lagrangian approach; i.e., every quantity has a natural interpretation in terms of observers comoving with the fundamental 4-velocity u^a (where $u^a u_a = -1$). Provided this is defined uniquely in an invariant manner, all related quantities have a direct physical or geometric meaning, and may in principle be measured in the instantaneous rest space of the comoving fundamental observers. Any coordinate system or tetrad can be used when specific calculations are made. These features are a crucial part of the strengths of the formalism and of the perturbation theory that is derived from it. We will follow the streamlining and development of the formalism given by Maartens [67], the essence of which is to make explicit use of irreducible quantities and derivatives, and to develop the identities which these quantities and derivatives obey (see also [48,57,58,51,52]).

The basic algebraic tensors are (a) the projector $h_{ab} = g_{ab} + u_a u_b$, where g_{ab} is the spacetime metric, which projects into the instantaneous rest space of comoving observers and (b) the projected alternating tensor $\varepsilon_{abc} = \eta_{abcd} u^d$, where $\eta_{abcd} = -\sqrt{|g|} \delta^0_{[a} \delta^1_b \delta^2_c \delta^3_{d]}$ is the spacetime alternating tensor. Thus

$$\eta_{abcd} = 2u_{[a}\varepsilon_{b]cd} - 2\varepsilon_{ab[c}u_{d]}, \quad \varepsilon_{abc}\varepsilon^{def} = 3! h_{[a}{}^d h_b{}^e h_{c]}{}^f.$$

The PSTF parts of vectors and rank-2 tensors are

$$V_{\langle a \rangle} = h_a{}^b V_b, \quad S_{\langle ab \rangle} = \left\{ h_{(a}{}^c h_{b)}{}^d - \frac{1}{3} h^{cd} h_{ab} \right\} S_{cd},$$

with higher rank formulas given in [37]. The skew part of a projected rank-2 tensor is spatially dual to the projected vector $S_a = \frac{1}{2}\varepsilon_{abc}S^{[bc]}$, and then any projected rank-2 tensor has the irreducible covariant decomposition

$$S_{ab} = \frac{1}{3}S h_{ab} + \varepsilon_{abc}S^c + S_{\langle ab \rangle},$$

where $S = S_{cd}h^{cd}$ is the spatial trace. In the $1+3$ covariant formalism, all quantities are either scalars, projected vectors or PSTF tensors. The equations governing these quantities involve a covariant vector product and its generalization to PSTF rank-2 tensors:

$$[V,W]_a = \varepsilon_{abc}V^b W^c, \quad [S,Q]_a = \varepsilon_{abc}S^b{}_d Q^{cd}.$$

The covariant derivative ∇_a defines $1+3$ covariant time and spatial derivatives

$$\dot{J}^{a\cdots}{}_{\cdots b} = u^c \nabla_c J^{a\cdots}{}_{\cdots b},$$

$$D_c J^{a\cdots}{}_{\cdots b} = h_c{}^d h^a{}_e \cdots h_b{}^f \nabla_d J^{e\cdots}{}_{\cdots f}.$$

Note that $D_c h_{ab} = 0 = D_d \varepsilon_{abc}$, while $\dot{h}_{ab} = 2u_{(a}\dot{u}_{b)}$ and $\dot{\varepsilon}_{abc} = 3u_{[a}\varepsilon_{bc]d}\dot{u}^d$. The projected derivative D_a further splits irreducibly into a $1+3$ covariant spatial divergence and curl [67]

$$\text{div } V = D^a V_a, \quad (\text{div } S)_a = D^b S_{ab},$$

$$\text{curl } V_a = \varepsilon_{abc}D^b V^c, \quad \text{curl } S_{ab} = \varepsilon_{cd(a}D^c S_{b)}{}^d,$$

and a $1+3$ covariant spatial distortion [57]

$$D_{\langle a}V_{b \rangle} = D_{(a}V_{b)} - \frac{1}{3}(\text{div } V)h_{ab},$$

$$D_{\langle a}S_{bc \rangle} = D_{(a}S_{bc)} - \frac{2}{5}h_{(ab}(\text{div } S)_{c)}.$$

Note that div curl is *not* in general zero, for vectors or rank-2 tensors (see [67,48,51,58] for the relevant formulas). The covariant irreducible decompositions of the derivatives of scalars, vectors and rank-2 tensors are given in exact (nonlinear) form by [57]

$$\nabla_a \psi = -\dot{\psi}u_a + D_a \psi, \tag{1}$$

$$\nabla_b V_a = -u_b\{\dot{V}_{\langle a \rangle} + A_c V^c u_a\} + u_a\left\{\frac{1}{3}\Theta V_b + \sigma_{bc}V^c \right.$$
$$\left. + [\omega,V]_b\right\} + \frac{1}{3}(\text{div } V)h_{ab} - \frac{1}{2}\varepsilon_{abc}\text{curl } V^c$$
$$+ D_{\langle a}V_{b \rangle}, \tag{2}$$

$$\nabla_c S_{ab} = -u_c\{\dot{S}_{\langle ab \rangle} + 2u_{(a}S_{b)d}A^d\} + 2u_{(a}\left\{\frac{1}{3}\Theta S_{b)c} \right.$$
$$\left. + S_{b)}{}^d(\sigma_{cd} - \varepsilon_{cde}\omega^e)\right\} + \frac{3}{5}(\text{div } S)_{\langle a}h_{b)c}$$
$$- \frac{2}{3}\varepsilon_{dc(a}\text{curl } S_{b)}{}^d + D_{\langle a}S_{bc \rangle}. \tag{3}$$

The algebraic correction terms in Eqs. (2) and (3) arise from the relative motion of comoving observers, as encoded in the kinematic quantities: the expansion $\Theta = D^a u_a$, the 4-acceleration $A_a \equiv \dot{u}_a = A_{\langle a \rangle}$, the vorticity[5] $\omega_a = -\frac{1}{2}\text{curl } u_a$, and the shear $\sigma_{ab} = D_{\langle a}u_{b \rangle}$. Thus, by Eq. (2),

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[4]In [4,43,45,35,36] it is denoted $^{(3)}\nabla_a$, while in [10,34] it is $\hat{\nabla}_a$.

[5]The vorticity tensor $\omega_{ab} = \varepsilon_{abc}\omega^c$ is often used, but we prefer to use the irreducible vector ω_a. The sign conventions, following [1,2], are such that in the Newtonian limit, $\vec{\omega} = -\frac{1}{2}\vec{\nabla} \times \vec{v}$. Note that $D^b \omega_{ab} = \text{curl } \omega_a$.

$$\nabla_b u_a = -A_a u_b + \frac{1}{3}\Theta h_{ab} + \varepsilon_{abc}\omega^c + \sigma_{ab}.$$

The irreducible parts of the Ricci identities produce commutation identities for the irreducible derivative operators. In the simplest case of scalars,

$$\mathrm{curl}\, \mathrm{D}_a \psi \equiv \varepsilon_{abc}\mathrm{D}^{[b}\mathrm{D}^{c]}\psi = -2\dot\psi\omega_a, \qquad (4)$$

$$\mathrm{D}_a\dot\psi - h_a{}^b(\mathrm{D}_b\psi)^{\boldsymbol{\cdot}} = -\dot\psi A_a + \frac{1}{3}\Theta\mathrm{D}_a\psi + \sigma_a{}^b\mathrm{D}_b\psi$$

$$+[\omega,\mathrm{D}\psi]_a. \qquad (5)$$

Identity (4) reflects the crucial relation of vorticity to non-integrability; non-zero ω_a implies that there are no constant-time 3-surfaces everywhere orthogonal to u^a, since the instantaneous rest spaces cannot be patched together smoothly.[6] Identity (5) is the key to deriving evolution equations for spatial gradients, which covariantly characterize inhomogeneity [4]. Further identities are given in [40,67,69,57,48].

The kinematic quantities govern the relative motion of neighboring fundamental world lines, and describe the universal expansion and its local anisotropies. The dynamic quantities describe the sources of the gravitational field, and directly determine the Ricci curvature locally via Einstein's field equations. They are the (total) energy density $\rho = T_{ab}u^a u^b$, isotropic pressure $p = \frac{1}{3}h_{ab}T^{ab}$, energy flux $q_a = -T_{\langle a\rangle b}u^b$, and anisotropic stress $\pi_{ab} = T_{\langle ab\rangle}$, where T_{ab} is the total energy-momentum tensor. The locally free gravitational field, i.e. the part of the spacetime curvature not directly determined locally by dynamic sources, is given by the Weyl tensor C_{abcd}. This splits irreducibly into the gravito-electric and gravito-magnetic fields

$$E_{ab} = C_{acbd}u^c u^d = E_{\langle ab\rangle}, \qquad H_{ab} = \frac{1}{2}\varepsilon_{acd}C^{cd}{}_{be}u^e = H_{\langle ab\rangle},$$

which provide a covariant Lagrangian description of tidal forces and gravitational radiation.

A FLRW (background) universe, with its unique preferred 4-velocity u^a, is covariantly characterized as follows:

Dynamics $\mathrm{D}_a\rho = 0 = \mathrm{D}_a p$, $q_a = 0$, $\pi_{ab} = 0$.
Kinematics $\mathrm{D}_a\Theta = 0$, $A_a = 0 = \omega_a$, $\sigma_{ab} = 0$.
Gravito-electric–magnetic field $E_{ab} = 0 = H_{ab}$.

The Hubble rate is $H = \frac{1}{3}\Theta = \dot a/a$, where $a(t)$ is the scale factor and t is cosmic proper time. In spatially homogeneous but anisotropic universes (Bianchi and Kantowski-Sachs models), the quantities $q_a, \pi_{ab}, \sigma_{ab}, E_{ab}$ and H_{ab} in the preceding list may be non-zero.

The Ricci identity for u^a and the Bianchi identities $\nabla^d C_{abcd} = \nabla_{[a}(-R_{b]c} + \frac{1}{6}Rg_{b]c})$ produce the fundamental evolution and constraint equations governing the above co-variant quantities [1,2]. Einstein's equations are incorporated[7] via the algebraic replacement of the Ricci tensor R_{ab} by $T_{ab} - \frac{1}{2}T_c{}^c g_{ab}$. These equations, in exact (nonlinear) form and for a general source of the gravitational field, are [57] the following:

Evolution:

$$\dot\rho + (\rho+p)\Theta + \mathrm{div}\, q$$
$$= -2A^a q_a - \sigma^{ab}\pi_{ab}, \qquad (6)$$

$$\dot\Theta + \frac{1}{3}\Theta^2 + \frac{1}{2}(\rho+3p) - \mathrm{div}\, A$$
$$= -\sigma_{ab}\sigma^{ab} + 2\omega_a\omega^a + A_a A^a, \qquad (7)$$

$$\dot q_{\langle a\rangle} + \frac{4}{3}\Theta q_a + (\rho+p)A_a + \mathrm{D}_a p + (\mathrm{div}\,\pi)_a$$
$$= -\sigma_{ab}q^b + [\omega,q]_a - A^b\pi_{ab}, \qquad (8)$$

$$\dot\omega_{\langle a\rangle} + \frac{2}{3}\Theta\omega_a + \frac{1}{2}\mathrm{curl}\, A_a = \sigma_{ab}\omega^b, \qquad (9)$$

$$\dot\sigma_{\langle ab\rangle} + \frac{2}{3}\Theta\sigma_{ab} + E_{ab} - \frac{1}{2}\pi_{ab} - \mathrm{D}_{\langle a}A_{b\rangle}$$
$$= -\sigma_{c\langle a}\sigma_{b\rangle}{}^c - \omega_{\langle a}\omega_{b\rangle} + A_{\langle a}A_{b\rangle}, \qquad (10)$$

$$\dot E_{\langle ab\rangle} + \Theta E_{ab} - \mathrm{curl}\, H_{ab} + \frac{1}{2}(\rho+p)\sigma_{ab}$$
$$+ \frac{1}{2}\dot\pi_{\langle ab\rangle} + \frac{1}{2}\mathrm{D}_{\langle a}q_{b\rangle} + \frac{1}{6}\Theta\pi_{ab}$$
$$= -A_{\langle a}q_{b\rangle} + 2A^c\varepsilon_{cd\langle a}H_{b\rangle}{}^d$$
$$+ 3\sigma_{c\langle a}E_{b\rangle}{}^c - \omega^c\varepsilon_{cd\langle a}E_{b\rangle}{}^d - \frac{1}{2}\sigma^c{}_{\langle a}\pi_{b\rangle c}$$
$$- \frac{1}{2}\omega^c\varepsilon_{cd\langle a}\pi_{b\rangle}{}^d, \qquad (11)$$

$$\dot H_{\langle ab\rangle} + \Theta H_{ab} + \mathrm{curl}\, E_{ab} - \frac{1}{2}\mathrm{curl}\,\pi_{ab}$$
$$= 3\sigma_{c\langle a}H_{b\rangle}{}^c - \omega^c\varepsilon_{cd\langle a}H_{b\rangle}{}^d$$
$$- 2A^c\varepsilon_{cd\langle a}E_{b\rangle}{}^d - \frac{3}{2}\omega_{\langle a}q_{b\rangle} + \frac{1}{2}\sigma^c{}_{\langle a}\varepsilon_{b\rangle cd}q^d. \qquad (12)$$

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[6]In this case, which has no Newtonian counterpart, the D_a operator is not intrinsic to a 3-surface, but it is still a well-defined spatial projection of ∇_a in each instantaneous rest space.

[7]Note that one constraint Einstein equation is not explicitly contained in this set—see [2,70].

Constraint:

$$\text{div}\,\omega = A^a \omega_a, \tag{13}$$

$$(\text{div}\,\sigma)_a - \text{curl}\,\omega_a - \frac{2}{3}\mathrm{D}_a\Theta + q_a = -2[\omega, A]_a, \tag{14}$$

$$\text{curl}\,\sigma_{ab} + \mathrm{D}_{\langle a}\omega_{b\rangle} - H_{ab} = -2A_{\langle a}\omega_{b\rangle}, \tag{15}$$

$$(\text{div}\,E)_a + \frac{1}{2}(\text{div}\,\pi)_a - \frac{1}{3}\mathrm{D}_a\rho + \frac{1}{3}\Theta q_a$$

$$= [\sigma, H]_a - 3H_{ab}\omega^b$$

$$+ \frac{1}{2}\sigma_{ab}q^b - \frac{3}{2}[\omega, q]_a, \tag{16}$$

$$(\text{div}\,H)_a + \frac{1}{2}\text{curl}\,q_a - (\rho+p)\omega_a$$

$$= -[\sigma, E]_a - \frac{1}{2}[\sigma, \pi]_a$$

$$+ 3E_{ab}\omega^b - \frac{1}{2}\pi_{ab}\omega^b. \tag{17}$$

If the universe is close to an FLRW model, then quantities that vanish in the FLRW limit are $O(\epsilon)$, where ϵ is a dimensionless smallness parameter, and the quantities are suitably normalized (e.g. $\sqrt{\sigma_{ab}\sigma^{ab}}/H < \epsilon$, etc.). The above equations are covariantly and gauge-invariantly linearized [4] by dropping all terms $O(\epsilon^2)$, and by replacing scalar coefficients of $O(\epsilon)$ terms by their background values. This linearization reduces all right hand sides of the evolution and constraint equations to zero.

III. 1+3 COVARIANT NONLINEAR ANALYSIS OF MULTI-FLUIDS

The formalism described above applies for *any* covariant choice of u^a. If the physics picks out only one u^a, then that becomes the natural and obvious 4-velocity to use. In a complex multi-fluid situation, however, there are various possible choices. The different particle species in cosmology will each have distinct 4-velocities; we could choose any of these as the fundamental frame, and other choices such as the center of mass frame are also possible. This allows a variety of covariant choices of 4-velocities, each leading to a slightly different 1+3 covariant description. One can regard a choice between these different possibilities as a partial gauge-fixing (but determined in a covariant and physical way). Any differences between such 4-velocities will be $O(\epsilon)$ in the almost-FLRW case and will disappear in the FLRW limit,[8] as is required in a consistent 1+3 covariant and gauge-

invariant linearization about an FLRW model (see [4,35] for further discussion).

In addition to the issue of linearization, one can also ask more generally what the impact of a change of fundamental frame is on the kinematic, dynamic and gravito-electric–magnetic quantities. If an initial choice u^a is replaced by a new choice $\tilde{u}^a$, then

$$\tilde{u}^a = \gamma(u^a + v^a) \quad \text{where} \quad v_a u^a = 0, \quad \gamma = (1 - v_a v^a)^{-1/2}, \tag{18}$$

where v^a is the (covariant) velocity of the new frame relative to the original frame. The exact transformations of all relevant quantities are given in the Appendix, and are taken from [52]. To linear order, the transformations take the form

$$\tilde{\Theta} \approx \Theta + \text{div}\,v, \quad \tilde{A}_a \approx A_a + \dot{v}_a + Hv_a,$$

$$\tilde{\omega}_a \approx \omega_a - \frac{1}{2}\text{curl}\,v_a, \quad \tilde{\sigma}_{ab} \approx \sigma_{ab} + \mathrm{D}_{\langle a}v_{b\rangle},$$

$$\tilde{\rho} \approx \rho, \quad \tilde{p} \approx p, \quad \tilde{q}_a \approx q_a - (\rho+p)v_a,$$

$$\tilde{\pi}_{ab} \approx \pi_{ab}, \quad \tilde{E}_{ab} \approx E_{ab}, \quad \tilde{H}_{ab} \approx H_{ab}.$$

Suppose now that a choice of fundamental frame has been made. (For the purposes of this paper, we will not need to specify such a choice.) Then we need to consider the velocities of each species which source the gravitational field, relative to the fundamental frame. If the 4-velocities are close, i.e. if the frames are in non-relativistic relative motion, then $O(v^2)$ terms may be dropped from the equations, except if we include nonlinear kinematic, dynamic and gravito-electric–magnetic effects, in which case, for consistency, we must retain $O(\epsilon^0 v^2)$ terms such as ρv^2, which are of the same order of magnitude in general as $O(\epsilon^2)$ terms. (See [72].) If the universe is close to FLRW, then $O(\epsilon^0 v^2)$ terms may be neglected, together with $O(\epsilon v)$ and $O(\epsilon^2)$ terms.

In summary, there are two different linearizations:

(a) Linearizing in relative velocities (i.e. assuming all species have nonrelativistic bulk motion relative to the fundamental frame), without linearizing in the kinematic, dynamic and gravito-electric–magnetic quantities that covariantly characterize the spacetime.

(b) FLRW linearization, which implies the special case of (a) obtained by also linearizing in the kinematic, dynamic and gravito-electric–magnetic quantities.

Clearly (a) is more general, and we can take it to be the physically relevant nonlinear regime, i.e. the case where only nonrelativistic average velocities[9] are considered, but no other assumptions are made about the physical or geometric quantities. In case (a), no restrictions are imposed on non-velocity terms, and we neglect only terms $O(\epsilon v^2, v^3)$. In case (b), we neglect terms $O(\epsilon^2, \epsilon v, v^2)$. Covariant second-

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[8]A similar situation occurs in relativistic thermodynamics, where suitable 4-velocities are close to the equilibrium 4-velocity, and hence to each other [71].

[9]Of course, this implies no restrictions on the velocities of individual particles within any species.

order effects against a FLRW background are included within (a), when we neglect terms $O(\epsilon^3)$. (Note that gauge invariance is a far more subtle problem at second order than at first order: see Bruni *et al.* [73].)

The dynamic quantities in the evolution and constraint equations (6)–(17) are the total quantities, with contributions from all dynamically significant particle species. Thus

$$T^{ab} = \sum_I T_I^{ab} = \rho u^a u^b + p h^{ab} + 2q^{(a}u^{b)} + \pi^{ab}, \quad (19)$$

$$T_I^{ab} = \rho_I u_I^a u_I^a + p_I h_I^{ab} + 2q_I^{(a}u_I^{b)} + \pi_I^{ab}, \quad (20)$$

where I labels the species. We include radiation photons ($I = R$), baryonic matter ($I = B$) modelled as a perfect fluid, cold dark matter ($I = C$) modelled as dust over the era of interest for CMB anisotropies, neutrinos ($I = N$) (assumed to be massless), and a cosmological constant ($I = V$).[10] Note that the dynamic quantities $\rho_I, \ldots$ in Eq. (20) are as measured in the I-frame, whose 4-velocity is given by

$$u_I^a = \gamma_I(u^a + v_I^a), \quad v_I^a u_a = 0. \quad (21)$$

Thus we have

$$p_C = 0 = q_C^a = \pi_C^{ab}, \quad q_B^a = 0 = \pi_B^{ab}, \quad (22)$$

$$p_R = \tfrac{1}{3}\rho_R, \quad p_N = \tfrac{1}{3}\rho_N, \quad (23)$$

where we have chosen the unique 4-velocity in the cold dark matter and baryonic cases which follows from modelling these fluids as perfect. The cosmological constant is characterized by

$$p_V = -\rho_V = -\Lambda, \quad q_V^a = 0 = \pi_V^{ab}, \quad v_V^a = 0.$$

The conservation equations for the species are best given in the overall u^a-frame, in terms of the velocities v_I^a of species I relative to this frame. Furthermore, the evolution and constraint equations of Sec. II are all given in terms of the u^a-frame. Thus we need the expressions for the partial dynamic quantities as measured in the overall frame. The velocity formula inverse to Eq. (21) is

$$u^a = \gamma_I(u_I^a + v_I^{*a}), \quad v_I^{*a} = -\gamma_I(v_I^a + v_I^2 u^a), \quad (24)$$

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[10]A more general treatment, incorporating all the sources which are currently believed to be potentially significant, would also include a dynamic scalar field that survives after inflation (''quintessence'') and hot dark matter in the form of massive neutrinos (see [28] for a survey with further references). Our main aim is not a detailed and comprehensive model with numerical predictions, but a qualitative discussion focusing on the underlying dynamic and geometric effects at the nonlinear and linear levels that are brought out clearly by a 1+3 covariant approach. In principle our approach is readily generalized to include other sources of the gravitational field.

where $v_I^{*a}u_{Ia} = 0$ and $v_I^{*a}v_{Ia}^* = v_I^a v_{Ia}$. Using this relation together with the general transformation equations (A8)–(A11), or directly from the above equations, we find the following exact (nonlinear) equations for the dynamic quantities of species I as measured in the overall u^a-frame:

$$\rho_I^* = \rho_I + \{\gamma_I^2 v_I^2(\rho_I + p_I) + 2\gamma_I q_I^a v_{Ia} + \pi_I^{ab} v_{Ia} v_{Ib}\}, \quad (25)$$

$$p_I^* = p_I + \tfrac{1}{3}\{\gamma_I^2 v_I^2(\rho_I + p_I) + 2\gamma_I q_I^a v_{Ia} + \pi_I^{ab} v_{Ia} v_{Ib}\}, \quad (26)$$

$$q_I^{*a} = q_I^a + (\rho_I + p_I)v_I^a + \{(\gamma_I - 1)q_I^a - \gamma_I q_I^b v_{Ib} u^a + \gamma_I^2 v_I^2(\rho_I + p_I)v_I^a + \pi_I^{ab}v_{Ib} - \pi_I^{bc}v_{Ib}v_{Ic}u^a\}, \quad (27)$$

$$\pi_I^{*ab} = \pi_I^{ab} + \{-2u^{(a}\pi_I^{b)c}v_{Ic} + \pi_I^{bc}v_{Ib}v_{Ic}u^a u^b\} + \{-\tfrac{1}{3}\pi_I^{cd}v_{Ic}v_{Id}h^{ab} + \gamma_I^2(\rho_I + p_I)v_I^{\langle a}v_I^{b\rangle} + 2\gamma_I v_I^{\langle a}q_I^{b\rangle}\}. \quad (28)$$

These expressions are the nonlinear generalization of well-known linearized results (see e.g. [45,71]). FLRW linearization implies that $v_I \ll 1$ for each I, and we neglect all terms which are $O(v_I^2)$ or $O(\epsilon v_I)$. This removes all terms in braces, dramatically simplifying the expressions:

$$\rho_I^* \approx \rho_I, \ p_I^* \approx p_I, \ q_I^{*a} \approx q_I^a + (\rho_I + p_I)v_I^a, \ \pi_I^{*ab} \approx \pi_I^{ab}.$$

To linear order, there is no difference in the dynamic quantities when measured in the I-frame or the fundamental frame, apart from a simple velocity correction to the energy flux. But in the general nonlinear case, this is no longer true.

The total dynamic quantities are simply given by

$$\rho = \sum_I \rho_I^*, \ p = \sum_I p_I^*, \ q^a = \sum_I q_I^{*a}, \ \pi^{ab} = \sum_I \pi_I^{*ab}.$$

Note that Eqs. (25)–(28) have been written to make clear the linear parts, so that the irreducible nature is not explicit. Irreducibility (in the u^a-frame) is revealed on using the relations

$$q_I^{\langle a\rangle} \equiv h^a{}_b q_I^b = q_I^a - q_I^b v_{Ib} u^a,$$

$$\pi_I^{\langle a\rangle\langle b\rangle} \equiv h^a{}_c h^b{}_d \pi_I^{cd} = \pi_I^{ab} - 2u^{(a}\pi_I^{b)c}v_{Ic} + \pi_I^{cd}v_{Ic}v_{Id}u^a u^b.$$

The exact equations show in detail the specific couplings and contributions of all partial dynamic quantities in the total quantities. For example, it is clear that in spatially homogeneous but anisotropic models, the partial energy fluxes q_I^a contribute to the total energy density, pressure and anisotropic stress at first order in the velocities v_I, while the partial anisotropic stresses π_I^{ab} contribute to the total energy flux at first order in v_I.

The total and partial 4-velocities define corresponding number 4-currents:

$$N^a = nu^a + j^a = \sum_I N_I^a, \quad N_I^a = n_I u_I^a + j_I^a, \quad (29)$$

where n and n_I are the number densities, j^a and j_I^a are the number fluxes, and $j_a u^a = 0 = j_{Ia} u_I^a$. It follows that

$$n = \sum_I n_I^* = \sum_I n_I + \sum_I \{(\gamma_I - 1)n_I + j_I^a v_{Ia}\}, \quad (30)$$

$$j^a = \sum_I j_I^{*a} = \sum_I (j_I^a + n_I v_I^a) + \sum_I \{(\gamma_I - 1)n_I v_I^a$$
$$- v_I^b j_{Ib} u^a\}, \quad (31)$$

where the quantities with an asterisk are as measured in the u^a-frame. Linearization removes the terms in braces, regaining the expressions in [45,71].

Four-velocities may be chosen in a number of covariant and physical ways. The main choices are [75,71] (a) the energy (Landau-Lifshitz) frame, defined by vanishing energy flux, and (b) the particle (Eckart) frame, defined by vanishing particle number flux. For a given single fluid, these frames coincide in equilibrium, but in general they are different. For each partial u_I^a, any change in choice $u_I^a \to \tilde{u}_I^a$ leads to transformations in the partial dynamic quantities, which are given by Eqs. (A8)–(A11) in the Appendix. For the fundamental u^a, a change in choice leads in addition to transformations of the kinematic quantities, given by Eqs. (A4)–(A7), and of the gravito-electric or magnetic field, given by equations (A12), (A13).

A convenient choice for each partial four-velocity u_I^a is the energy frame, i.e. $q_I^a = 0$ for each I (this is the obvious choice in the cases $I = C,B$). As measured in the fundamental frame, the partial energy fluxes do not vanish, i.e. $q_I^{*a} \neq 0$, and the total energy flux is given by

$$q^a = \sum_I [(\rho_I + p_I)v_I^a + \pi_I^{ab}v_{Ib} + O(\epsilon v_I^2, v_I^3)]. \quad (32)$$

With this choice, using the above equations, we find the following expressions for the dynamic quantities of matter as measured in the fundamental frame. For cold dark matter,

$$\rho_C^* = \gamma_C^2 \rho_C, \quad p_C^* = \frac{1}{3}\gamma_C^2 v_C^2 \rho_C, \quad (33)$$

$$q_C^{*a} = \gamma_C^2 \rho_C v_C^a, \quad \pi_C^{*ab} = \gamma_C^2 \rho_C v_C^{\langle a} v_C^{b\rangle}. \quad (34)$$

For baryonic matter,

$$\rho_B^* = \gamma_B^2(1 + w_B v_B^2)\rho_B, \quad p_B^* = \left[w_B + \frac{1}{3}\gamma_B^2 v_B^2(1 + w_B)\right]\rho_B, \quad (35)$$

$$q_B^{*a} = \gamma_B^2(1 + w_B)\rho_B v_B^a, \quad \pi_B^{*ab} = \gamma_B^2(1 + w_B)\rho_B v_B^{\langle a} v_B^{b\rangle}, \quad (36)$$

where $w_B \equiv p_B/\rho_B$. In the case of radiation and neutrinos, we will evaluate the dynamic quantities relative to the u^a-frame directly via kinetic theory, in the next section.

The total energy-momentum tensor is conserved, i.e. $\nabla_b T^{ab} = 0$, which is equivalent to the evolution equations (6) and (8). The partial energy-momentum tensors obey

$$\nabla_b T_I^{ab} = J_I^a = U_I^* u^a + M_I^{*a}, \quad (37)$$

where U_I^* is the rate of energy density transfer to species I as measured in the u^a-frame, and $M_I^{*a} = M_I^{*\langle a\rangle}$ is the rate of momentum density transfer to species I, as measured in the u^a-frame. Cold dark matter and neutrinos are decoupled during the period of relevance for CMB anisotropies, while radiation and baryons are coupled through Thomson scattering. Thus

$$J_C^a = 0 = J_N^a, \quad J_R^a = -J_B^a = U_T u^a + M_T^a, \quad (38)$$

where the Thomson rates are

$$U_T = n_E \sigma_T \left(\frac{4}{3}\rho_R^* v_B^2 - q_R^{*a} v_{Ba}\right) + O(\epsilon v_B^2, v_B^3), \quad (39)$$

$$M_T^a = n_E \sigma_T \left(\frac{4}{3}\rho_R^* v_B^a - q_R^{*a} + \pi_R^{*ab} v_{Bb}\right) + O(\epsilon v_B^2, v_B^3), \quad (40)$$

as given by Eq. (63), derived in Sec. V. Here n_E is the free electron number density, and σ_T is the Thomson cross section. Note that to linear order, there is no energy transfer, i.e. $U_T \approx 0$.

Using Eqs. (33)–(36) in Eq. (37), we find that, for cold dark matter,

$$\dot{\rho}_C + \Theta\rho_C + \rho_C \text{div } v_C = -(\rho_C v_C^2)^{\cdot} - \frac{4}{3}v_C^2 \Theta\rho_C - v_C^a D_a \rho_C$$
$$- 2\rho_C A_a v_C^a + O(\epsilon v_C^2, v_C^3), \quad (41)$$

$$\dot{v}_C^a + \frac{1}{3}\Theta v_C^a + A^a = A_b v_C^b u^a - \sigma^a{}_b v_C^b + [\omega, v_C]^a$$
$$- v_C^b D_b v_C^a + O(\epsilon v_C^2, v_C^3), \quad (42)$$

and for baryonic matter

$$\dot{\rho}_B + \Theta(1+w_B)\rho_B + (1+w_B)\rho_B \operatorname{div} v_B = -[(1+w_B)\rho_B v_B^2]^{\cdot} - \frac{4}{3}v_B^2\Theta(1+w_B)\rho_B - v_B^a D_a[(1+w_B)\rho_B] - 2(1+w_B)\rho_B A_a v_B^a$$

$$- n_E\sigma_T\left(\frac{4}{3}\rho_R^* v_B^2 - q_R^{*a} v_{Ba}\right) + O(\epsilon v_B^2, v_B^3), \tag{43}$$

$$(1+w_B)\dot{v}_B^a + \left(\frac{1}{3} - c_B^2\right)\Theta v_B^a + (1+w_B)A^a + \rho_B^{-1}D^a p_B + \rho_B^{-1}n_E\sigma_T(\rho_R^* v_B^a - q_R^{*a})$$

$$= (1+w_B)A_b v_B^b u^a - (1+w_B)\sigma^a{}_b v_B^b + (1+w_B)[\omega, v_B]^a - (1+w_B)v_B^b D_b v_B^a$$

$$+ c_B^2(1+w_B)(\operatorname{div} v_B)v_B^a - \rho_B^{-1}n_E\sigma_T\pi_R^{*ab}v_{Bb} + O(\epsilon v_B^2, v_B^3), \tag{44}$$

where $c_B^2 \equiv \dot{p}_B/\dot{\rho}_B$ (this equals the adiabatic sound speed only to linear order). These conservation equations generalize those given in [36] to the nonlinear case. FLRW linearization reduces the right hand sides of these equations to zero, dramatically simplifying the equations. The conservation equations for the massless species (radiation and neutrinos) are given below. Note from Eq. (42) that if the cold dark matter frame is chosen as the fundamental frame, then the 4-acceleration vanishes; i.e., $v_C^a = 0$ implies $A_a = 0$. This is the choice of fundamental frame advocated in [36].

IV. COVARIANT LAGRANGIAN KINETIC THEORY

Relativistic kinetic theory (see e.g. [74–78]) provides a self-consistent microscopically based treatment where there is a natural unifying framework in which to deal with a gas of particles in circumstances ranging from hydrodynamic to free-streaming behavior. The photon gas undergoes a transition from hydrodynamic tight coupling with matter, through the process of decoupling from matter, to non-hydrodynamic free streaming. This transition is characterized by the evolution of the photon mean free path from effectively zero to effectively infinity. The range of behavior can appropriately be described by kinetic theory with Thomson scattering [79,80], and the baryonic matter with which radiation interacts can reasonably be described hydrodynamically during these times. (The basic physics of radiation and matter and density perturbations in cosmology was developed in the works of Sachs and Wolfe [11], Silk [81], Peebles and Yu [13], Weinberg [82], and others.)

In the covariant Lagrangian approach of [5] (see also [7,8]), the photon 4-momentum p^a (where $p^a p_a = 0$) is split as

$$p^a = E(u^a + e^a), \quad e^a e_a = 1, \quad e^a u_a = 0, \tag{45}$$

where $E = -u_a p^a$ is the energy and $e^a = p^{\langle a \rangle}/E$ is the direction, as measured by a comoving (fundamental) observer. Then the photon distribution function is decomposed into covariant harmonics via the expansion [5,9]

$$f(x,p) = f(x,E,e) = F + F_a e^a + F_{ab}e^a e^b + \cdots$$

$$= \sum_{l \geq 0} F_{A_l}(x,E)e^{\langle A_l \rangle}, \tag{46}$$

where $e^{A_l} \equiv e^{a_1}e^{a_2}\cdots e^{a_l}$, and $e^{\langle A_l \rangle}$ provides a representation of the rotation group [37]. The covariant multipoles are irreducible since they are PSTF, i.e.

$$F_{a\cdots b} = F_{\langle a \cdots b \rangle} \Leftrightarrow F_{a\cdots b} = F_{(a\cdots b)}, \quad F_{a\cdots b}u^b = 0$$

$$= F_{a\cdots bc}h^{bc}.$$

They encode the anisotropy structure of the distribution in the same way as the usual spherical harmonic expansion

$$f = \sum_{l \geq 0} \sum_{m=-l}^{+l} f_l^m(x,E)Y_l^m(\vec{e}),$$

but here (a) the F_{A_l} are covariant, and thus independent of any choice of coordinates in momentum space, unlike the f_l^m, and (b) F_{A_l} is a rank-l tensor field on spacetime for each fixed E, and directly determines the l-multipole of radiation anisotropy after integration over E. The multipoles can be recovered from the distribution function via [5,37]

$$F_{A_l} = \Delta_l^{-1}\int f(x,E,e)e_{\langle A_l \rangle}d\Omega, \quad \text{with} \quad \Delta_l = 4\pi\frac{(l!)^2 2^l}{(2l+1)!}, \tag{47}$$

where $d\Omega = d^2 e$ is a solid angle in momentum space. A further useful identity is [5]

$$\int e^{A_l}d\Omega = \frac{4\pi}{l+1}\begin{cases} 0 & l \text{ odd}, \\ h^{(a_1 a_2}h^{a_3 a_4}\cdots h^{a_{l-1}a_l)} & l \text{ even}. \end{cases} \tag{48}$$

The first 3 multipoles arise from the radiation energy-momentum tensor, which is

$$T_R^{ab}(x) = \int p^a p^b f(x,p) \mathrm{d}^3 p$$

$$= \rho_R^* u^a u^b + \frac{1}{3}\rho_R^* h^{ab} + 2 q_R^{*(a} u^{b)} + \pi_R^{*ab},$$

where $\mathrm{d}^3 p = E \mathrm{d}E \mathrm{d}\Omega$ is the covariant volume element on the future null cone at event x. It follows that the dynamic quantities of the radiation (in the u^a-frame) are

$$\rho_R^* = 4\pi \int_0^\infty E^3 F \mathrm{d}E, \quad q_R^{*a} = \frac{4\pi}{3}\int_0^\infty E^3 F^a \mathrm{d}E,$$

$$\pi_R^{*ab} = \frac{8\pi}{15}\int_0^\infty E^3 F^{ab} \mathrm{d}E. \qquad (49)$$

From now on, we *drop* the asterisks from the radiation dynamic quantities relative to the fundamental frame, since we do not need to relate them to their values in the radiation frame.

We extend these dynamic quantities to all multipole orders by defining[11] [5]

$$\Pi_{a_1 \cdots a_l} = \int E^3 F_{a_1 \cdots a_l} \mathrm{d}E, \qquad (50)$$

so that $\Pi = \rho_R/4\pi, \Pi^a = 3 q_R^a/4\pi$ and $\Pi^{ab} = 15\pi_R^{ab}/8\pi$.

The Boltzmann equation is

$$\frac{\mathrm{d}f}{\mathrm{d}v} \equiv p^a \frac{\partial f}{\partial x^a} - \Gamma^a{}_{bc} p^b p^c \frac{\partial f}{\partial p^a} = C[f], \qquad (51)$$

where $p^a = \mathrm{d}x^a/\mathrm{d}v$ and $C[f]$ is the collision term, which determines the rate of change of f due to emission, absorption and scattering processes. This term is also decomposed into covariant harmonics:

$$C[f] = \sum_{l \geqslant 0} b_{A_l}(x,E) e^{A_l} = b + b_a e^a + b_{ab} e^a e^b + \cdots, \qquad (52)$$

where the multipoles $b_{A_l} = b_{\langle A_l \rangle}$ encode covariant irreducible properties of the particle interactions. Then the Boltzmann equation is equivalent to an infinite hierarchy of covariant multipole equations

$$L_{A_l}(x,E) = b_{A_l}[F_{A_m}](x,E),$$

where L_{A_l} are the multipoles of $\mathrm{d}f/\mathrm{d}v$, and will be given in the next section. These multipole equations are tensor field equations on spacetime for each value of the photon energy E (but note that energy changes along each photon path). Given the solutions $F_{A_l}(x,E)$ of the equations, the relation

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[11]Because photons are massless, we do not need the complexity of the moment definitions used in [5]. In [36], $J_{A_l}^{(l)}$ is used, where $J_{A_l}^{(l)} = \Delta_l \Pi_{A_l}$. From now on, all energy integrals will be understood to be over the range $0 \leqslant E \leqslant \infty$.

(46) then determines the full photon distribution $f(x,E,e)$ as a scalar field over phase space.

Over the period of importance for CMB anisotropies, i.e. considerably after electron-positron annihilation, the average photon energy is much less than the electron rest mass and the electron thermal energy may be neglected, so that the Compton interaction between photons and electrons (the dominant interaction between radiation and matter) may reasonably be described in the Thomson limit. (See [72] for refinements.) We will also neglect the effects of polarization (see e.g. [24]). For Thomson scattering

$$C[f] = \sigma_T n_E E_B[\bar{f}(x,p) - f(x,p)], \qquad (53)$$

where $E_B = -p_a u_B^a$ is the photon energy relative to the baryonic (i.e. baryon-electron) frame u_B^a, and $\bar{f}(x,p)$ determines the number of photons scattered into the phase space volume element at (x,p). The differential Thomson cross section is proportional to $1 + \cos^2\alpha$, where α is the angle between initial and final photon directions in the baryonic frame. Thus $\cos\alpha = e_B^a e'_{Ba}$ where e'_{Ba} is the initial and e_B^a is the final direction, so that

$$p'^a = E_B(u_B^a + e_B'^a), \quad p^a = E_B(u_B^a + e_B^a),$$

where we have used $E_B' = E_B$, which follows since the scattering is elastic. Here u_B^a is given by Eq. (21), where v_B^a is the velocity of the baryonic frame relative to the fundamental frame u^a, with $v_B^a u_a = 0$. Then $\bar{f}$ is given by [36,72]

$$\bar{f}(x,p) = \frac{3}{16\pi} \int f(x,p')[1 + (e_B^a e'_{Ba})^2] \mathrm{d}\Omega_B'. \qquad (54)$$

The exact forms of the photon energy and direction in the baryonic frame follow on using Eqs. (21) and (A2):

$$E_B = E \gamma_B (1 - v_B^a e_a), \qquad (55)$$

$$e_B^a = \frac{1}{\gamma_B(1 - v_B^c e_c)}[e^a + \gamma_B^2(v_B^b e_b - v_B^2)u^a$$

$$+ \gamma_B^2(v_B^b e_b - 1)v_B^a]. \qquad (56)$$

Anisotropic scattering will source polarization, and small errors are introduced by assuming that the radiation remains unpolarized [83]. A fully consistent and general treatment requires the incorporation of polarization. However, for simplicity, and in line with many previous treatments, we will neglect polarization effects.

V. NONLINEAR MULTIPOLE HIERARCHY

The full Boltzmann equation in photon phase space contains more information than necessary to analyze radiation anisotropies in an inhomogeneous universe. For that purpose, when the radiation is close to blackbody we do not require the full spectral behavior of the distribution multipoles, but only the energy-integrated multipoles. The monopole leads to the average temperature, while the higher order multipoles

determine the temperature fluctuations. The $1+3$ covariant and gauge-invariant definition of the average temperature T is given by [10]

$$\rho_R(x) = 4\pi \int E^3 F(x,E)\mathrm{d}E = rT(x)^4, \qquad (57)$$

where r is the radiation constant. If f is close to a Planck distribution, then T is the thermal blackbody average temperature. But note that no notion of *background* temperature is involved in this definition. There is an all-sky average implied in Eq. (57). Fluctuations across the sky are measured by integrating the higher multipoles (a precise definition is given below); i.e., the fluctuations are determined by the $\Pi_{a_1 \cdots a_l}(l \geq 1)$ defined in Eq. (50).

The form of $C[f]$ shows that covariant equations for the temperature fluctuations arise from decomposing the energy-integrated Boltzmann equation

$$\int E^2 \frac{\mathrm{d}f}{\mathrm{d}v}\mathrm{d}E = \int E^2 C[f]\mathrm{d}E \qquad (58)$$

into $1+3$ covariant multipoles. We begin with the right hand side, which requires the covariant form of the Thomson scattering term (54). Since the baryonic frame will move non-relativistically relative to the fundamental frame in all cases of physical interest, it is sufficient to linearize only in v_B, and not in the other quantities. Thus we drop terms in $O(\epsilon v_B^2, v_B^3)$ but do not neglect terms that are $O(\epsilon^0 v_B^2, \epsilon v_B)$ or $O(\epsilon^2)$ relative to the FLRW limiting background. In other words, we make no restrictions on the geometric and physical quantities that covariantly characterize the spacetime, apart from assuming a nonrelativistic relative average velocity for matter. The resulting expression will in particular be applicable for covariant second-order effects in FLRW backgrounds (recognizing that polarization effects should be included for a complete treatment) or for first-order effects in Bianchi backgrounds.

For brevity, we will use the notation

$$\mathcal{O}[3] \equiv O(\epsilon v_B^2, v_B^3),$$

noting that this does *not* imply any second-order restriction on the dynamic, kinematic and gravito-electric–magnetic quantities. It follows from Eqs. (48) and (54) that

$$4\pi \int \bar{f} E_B^3 \mathrm{d}E_B = (\rho_R)_B + \frac{3}{4}(\pi_R^{ab})_B e_{B^a} e_{B^b}, \qquad (59)$$

where the dynamic radiation quantities are evaluated in the *baryonic* frame. This approach relies on the frame transformations given in the Appendix, and allows us to evaluate the Thomson scattering integral more directly and clearly than other approaches. In the process, we are also generalizing to

include nonlinear effects. We use Eqs. (A8) and (A11) to transform back to the fundamental frame:[12]

$$(\rho_R)_B = \rho_R\left[1 + \frac{4}{3}v_B^2\right] - 2q_R^a v_{B^a} + \mathcal{O}[3],$$

$$(\pi_R^{ab})_B = \pi_R^{ab} + 2v_{B^c}\pi_R^{c\langle a}u^{b\rangle} - 2q_R^{\langle a}v_B^{b\rangle} + \tfrac{4}{3}\rho_R v_B^{\langle a}v_B^{b\rangle}$$
$$+ \mathcal{O}[3].$$

Now

$$\int E^2 C[f]\mathrm{d}E = n_E\sigma_T[1 + 3v_B^c e_c + (v_B^c e_c)^2$$

$$- \tfrac{3}{2}v_B^2]\int E_B^3\bar{f}\mathrm{d}E_B - n_E\sigma_T[1 - v_B^c e_c$$

$$+ \tfrac{1}{2}v_B^2]\int fE^3\mathrm{d}E + \mathcal{O}[3]. \qquad (60)$$

In addition, we need the following identity, valid for any projected vector v^a:

$$v^a e_a f = \tfrac{1}{3}F_a v^a + [Fv_a + \tfrac{2}{5}F_{ab}v^b]e^a + \left[F_{\langle a}v_{b\rangle}\right.$$

$$\left. + \tfrac{3}{7}F_{abc}v^c\right]e^{\langle a}e^{b\rangle} + \cdots$$

$$= \sum_{l\geq 0}\left[F_{\langle A_{l-1}}v_{a_l\rangle} + \left(\frac{l+1}{2l+3}\right)F_{A_l a}v^a\right]e^{\langle A_l\rangle}. \qquad (61)$$

(Here and subsequently, we use the convention that $F_{A_l} = 0$ for $l < 0$.) This identity may be proved using Eq. (48) and the identity (see [5], p. 470)

$$V_{\langle b}S_{A_l\rangle} = V_{(b}S_{A_l)} - \left(\frac{l}{2l+1}\right)V^c S_{c(A_{l-1}}h_{a_l b)} \quad \text{where}$$

$$S_{A_l} = S_{\langle A_l\rangle}. \qquad (62)$$

Using the above equations, we find that[13]

$$4\pi \int E^2 C[f]\mathrm{d}E = n_E\sigma_T[\tfrac{4}{3}\rho_R v_B^2 - q_R^a v_{Ba}]$$

$$- n_E\sigma_T[3q_R^a - 4\rho_R v_B^a - 3\pi_R^{ab}v_{Bb}]e_a$$

$$- n_E\sigma_T\left[\frac{27}{4}\pi_R^{ab} - \frac{3}{2}q_R^{\langle a}v_B^{b\rangle} - \frac{12}{7}\pi\Pi^{abc}v_{B^c}\right.$$

$$\left. - 3\rho_R v_B^{\langle a}v_B^{b\rangle}\right]e_{\langle a}e_{b\rangle} - n_E\sigma_T\left[4\pi\Pi^{abc}\right.$$

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[12]As noted in Sec. III, we retain the $O(v_B^2)$ term in $(\rho_R)_B$ since ρ_R is zero order.

[13]Challinor has independently derived the same result [38].

$$-\frac{45}{4}\pi_R^{\langle ab}v_B^{c\rangle}$$

$$\left.-\frac{16}{9}\pi\Pi^{abcd}v_{Bd}\right]e_{\langle a}e_b e_{c\rangle}+\cdots$$

$$+\mathcal{O}[3].\tag{63}$$

Now it is clear from Eqs. (59) and (60) that the first four multipoles are affected by Thomson scattering differently than the higher multipoles. This is confirmed by the form of Eq. (63). Defining the energy-integrated scattering multipoles

$$K_{A_l}=\int E^2 b_{A_l}dE,$$

we find from Eq. (63) that

$$K=n_E\sigma_T\left[\frac{4}{3}\Pi v_B^2-\frac{1}{3}\Pi^a v_{Ba}\right]+\mathcal{O}[3],\tag{64}$$

$$K^a=-n_E\sigma_T\left[\Pi^a-4\Pi v_B^a-\frac{2}{5}\Pi^{ab}v_{Bb}\right]+\mathcal{O}[3],\tag{65}$$

$$K^{ab}=-n_E\sigma_T\left[\frac{9}{10}\Pi^{ab}-\frac{1}{2}\Pi^{\langle a}v_B^{b\rangle}-\frac{3}{7}\Pi^{abc}v_{Bc}\right.$$
$$\left.-3\Pi v_B^{\langle a}v_B^{b\rangle}\right]+\mathcal{O}[3],\tag{66}$$

$$K^{abc}=-n_E\sigma_T\left[\Pi^{abc}-\frac{3}{2}\Pi^{\langle ab}v_B^{c\rangle}-\frac{4}{9}\Pi^{abcd}v_{Bd}\right]$$
$$+\mathcal{O}[3],\tag{67}$$

and, for $l>3$,

$$K^{A_l}=-n_E\sigma_T\left[\Pi^{A_l}-\Pi^{\langle A_{l-1}}v_B^{a_l\rangle}-\left(\frac{l+1}{2l+3}\right)\Pi^{A_l a}v_{Ba}\right]+\mathcal{O}[3].\tag{68}$$

Equations (64)–(68) are a nonlinear generalization of the results given by Challinor and Lasenby [36]. They show the new *coupling of baryonic bulk velocity to the radiation multipoles, arising from local nonlinear effects in Thomson scattering.* If we linearize fully, i.e. neglect all terms containing v_B except the $\rho_R v_B^a$ term in the dipole K^a, which is first order, then our equations reduce to those in [36]. The generalized nonlinear equations apply to the analysis of second-order effects on a FLRW background, to first-order effects on a spatially homogeneous but anisotropic background, and more generally, to any situation where the baryonic frame is non-relativistic relative to the fundamental u^a-frame.

Next we require the multipoles of df/dv. These can be read directly from the general expressions first derived in [5], which are exact, $1+3$ covariant and also include the case of massive particles. For clarity and completeness, we outline

an alternative, $1+3$ covariant derivation (the derivation in [5] uses tetrads). We require the identity [7,8]

$$\frac{dE}{dv}=-E^2\left[\frac{1}{3}\Theta+A_a e^a+\sigma_{ab}e^a e^b\right],\tag{69}$$

which follows directly from $E=-p^a u_a$, $p^b\nabla_b p^a=0$ and $\nabla_b u_a=-A_a u_b+D_b u_a$. Then

$$\frac{d}{dv}[F_{a_1\cdots a_l}(x,E)e^{a_1}\cdots e^{a_l}]$$

$$=\frac{d}{dv}[E^{-l}F_{a_1\cdots a_l}(x,E)p^{a_1}\cdots p^{a_l}]$$

$$=E\left\{\left[\frac{1}{3}\Theta+A_b e^b+\sigma_{bc}e^b e^c\right](lF_{a_1\cdots a_l}-EF'_{a_1\cdots a_l})\right.$$

$$\times e^{a_1}\cdots e^{a_l}+(u^{a_1}+e^{a_1})\cdots(u^{a_l}+e^{a_l})$$

$$\left.\times[\dot{F}_{a_1\cdots a_l}+e^b\nabla_b F_{a_1\cdots a_l}]\right\},$$

where a prime denotes $\partial/\partial E$. The first term is readily put into irreducible PSTF form using the identity (62) with $V_a=A_a$, and its extension to the case when V_a is replaced by a rank-2 PSTF tensor W_{ab} (see [5], p. 470), with $W_{ab}=\sigma_{ab}$. In the second term, when the round brackets are expanded, only those terms with at most one u^{a_r} survive, and

$$u^a\dot{F}_{a\cdots}=-A^a F_{a\cdots},$$

$$u^b\nabla^a F_{b\cdots}=-\left(\frac{1}{3}\Theta h^{ab}+\sigma^{ab}-\varepsilon^{abc}\omega_c\right)F_{b\cdots}.$$

Thus the covariant multipoles b_{A_l} of df/dv are

$$E^{-1}b_{A_l}=\dot{F}_{\langle A_l\rangle}-\frac{1}{3}\Theta EF'_{A_l}+D_{\langle a_l}F_{A_{l-1}\rangle}+\frac{(l+1)}{(2l+3)}D^a F_{aA_l}$$

$$-\frac{(l+1)}{(2l+3)}E^{-(l+1)}[E^{l+2}F_{aA_l}]'A^a$$

$$-E^l[E^{1-l}F_{\langle A_{l-1}}]'A_{a_l\rangle}-l\omega^b\varepsilon_{bc\langle a_l}F_{A_{l-1}\rangle}{}^c$$

$$-\frac{(l+1)(l+2)}{(2l+3)(2l+5)}E^{-(l+2)}[E^{l+3}F_{abA_l}]'\sigma^{ab}$$

$$-\frac{2l}{(2l+3)}E^{-1/2}[E^{3/2}F_{b\langle A_{l-1}}]'\sigma_{a_l\rangle}{}^b$$

$$-E^{l-1}[E^{2-l}F_{\langle A_{l-2}}]'\sigma_{a_{l-1}a_l\rangle}.\tag{70}$$

This regains the result of [5] [Eq. (4.12)] in the massless case, with minor corrections. The form given here benefits from the streamlined version of the $1+3$ covariant formalism. We reiterate that this result is exact and holds for any photon or (massless) neutrino distribution in any spacetime. We now multiply Eq. (70) by E^3 and integrate over all energies, using integration by parts and the fact that $E^n F_{a\cdots}\to0$ as $E\to\infty$ for any positive n. We obtain the multipole equations that determine the brightness multipoles Π_{A_l}:

$$K_{A_l} = \dot{\Pi}_{\langle A_l \rangle} + \frac{4}{3}\Theta\Pi_{A_l} + D_{\langle a_l}\Pi_{A_{l-1}\rangle} + \frac{(l+1)}{(2l+3)}D^b\Pi_{bA_l}$$

$$- \frac{(l+1)(l-2)}{(2l+3)}A^b\Pi_{bA_l} + (l+3)A_{\langle a_l}\Pi_{A_{l-1}\rangle}$$

$$- l\omega^b\varepsilon_{bc\langle a_l}\Pi_{A_{l-1}\rangle}{}^c - \frac{(l-1)(l+1)(l+2)}{(2l+3)(2l+5)}\sigma^{bc}\Pi_{bcA_l}$$

$$+ \frac{5l}{(2l+3)}\sigma^b{}_{\langle a_l}\Pi_{A_{l-1}\rangle b} - (l+2)\sigma_{\langle a_l a_{l-1}}\Pi_{A_{l-2}\rangle}. \tag{71}$$

Once again, this is an exact result, and it holds also for any collision term, i.e. any K_{A_l}. For decoupled neutrinos, we have $K_N^{A_l} = 0$ in this equation. For photons undergoing Thomson scattering, the left hand side of Eq. (71) is given by Eq. (68), which is exact in the kinematic and dynamic quantities, but first order in the relative baryonic velocity. Eqs. (68) and (71) thus constitute a nonlinear generalization of the FLRW-linearized case given by Challinor and Lasenby [36].

These equations describe evolution along the timelike world lines of fundamental observers, not along the lightlike geodesics of photon motion. The timelike integration is related to light cone integrations by making homogeneity assumptions about the distribution of matter in (spacelike) surfaces of constant time, as is discussed in [84].

The monopole and dipole of Eq. (71) give the evolution equations of energy and momentum density:

$$K = \dot{\Pi} + \frac{4}{3}\Theta\Pi + \frac{1}{3}D^a\Pi_a + \frac{2}{3}A^a\Pi_a + \frac{2}{15}\sigma^{ab}\Pi_{ab}, \tag{72}$$

$$K^a = \dot{\Pi}^{\langle a \rangle} + \frac{4}{3}\Theta\Pi^a + D^a\Pi + \frac{2}{5}D_b\Pi^{ab} + \frac{2}{5}A_b\Pi^{ab}$$

$$+ 4\Pi A^a - [\omega,\Pi]^a + \sigma^{ab}\Pi_b. \tag{73}$$

In the case of neutrinos, $K_N = 0 = K_N^a$, these express the conservation of energy and momentum:[14]

$$\dot{\rho}_N + \frac{4}{3}\Theta\rho_N + D_a q_N^a = -2A_a q_N^a - \sigma_{ab}\pi_N^{ab}, \tag{74}$$

$$\dot{q}_N^{\langle a \rangle} + \frac{4}{3}\Theta q_N^a + \frac{4}{3}\rho_N A^a + \frac{1}{3}D^a\rho_N + D_b\pi_N^{ab}$$

$$= -[\omega,q_N]^a - \sigma^a{}_b q_N^b - A_b\pi_N^{ab}. \tag{75}$$

FLRW linearization reduces the right hand sides to zero. For photons, K and K^a are given by Eqs. (64) and (65), and determine the Thomson rates of transfer in Eqs. (39) and (40):

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[14]As in the photon case, we omit the asterisks on the neutrino dynamic quantities, since we do not require their values in the neutrino frame.

$$U_T = 4\pi K, \quad M_T^a = \frac{4\pi}{3}K^a. \tag{76}$$

Finally, we return to the definition of temperature anisotropies. As noted above, these are determined by the Π_{A_l}. Generalizing the linearized 1+3 covariant approach in [10], we define the temperature fluctuation $\tau(x,e)$ via the directional bolometric brightness:

$$T(x)[1 + \tau(x,e)] = \left[\frac{4\pi}{r}\int E^3 f(x,E,e)dE\right]^{1/4}. \tag{77}$$

This is a 1+3 covariant and gauge-invariant definition which is also exact. We can rewrite it explicitly in terms of the Π_{A_l}:

$$\tau(x,e) = \left[1 + \left(\frac{4\pi}{\rho_R}\right)\sum_{l\geq 1}\Pi_{A_l}e^{A_l}\right]^{1/4} - 1$$

$$= \tau_a e^a + \tau_{ab}e^a e^b + \cdots. \tag{78}$$

In principle, we can extract the irreducible PSTF temperature fluctuation multipoles by using the inversion in Eq. (47):

$$\tau_{A_l}(x) = \Delta_l^{-1}\int \tau(x,e)e_{\langle A_l\rangle}d\Omega. \tag{79}$$

In the almost-FLRW case, when τ is $O(\epsilon)$, we regain from Eq. (78) the linearized definition given in [10]:

$$\tau_{A_l} \approx \left(\frac{\pi}{\rho_R}\right)\Pi_{A_l}, \tag{80}$$

where $l\geq 1$. In particular, the dipole and quadrupole are

$$\tau^a \approx \frac{3q_R^a}{4\rho_R} \quad \text{and} \quad \tau^{ab} \approx \frac{15\pi_R^{ab}}{2\rho_R}. \tag{81}$$

VI. QUALITATIVE IMPLICATIONS OF THE NONLINEAR DYNAMICAL EFFECTS

In Sec. II, we gave the nonlinear evolution and constraint equations governing the kinematic, total dynamic and gravito-electric–magnetic quantities—see Eqs. (6)–(17). In these equations, the total dynamic quantities are, using the results of Sec. III:

$$\rho = \rho_R + \rho_N + (1 + v_C^2)\rho_C + [1 + (1+w_B)v_B^2]\rho_B + \Lambda$$

$$+ \mathcal{O}[3], \tag{82}$$

$$p = \frac{1}{3}\rho_R + \frac{1}{3}\rho_N + \frac{1}{3}v_C^2\rho_C + \left[w_B + \frac{1}{3}(1+w_B)v_B^2\right]\rho_B$$

$$- \Lambda + \mathcal{O}[3], \tag{83}$$

$$q^a = q_R^a + q_N^a + \rho_C^a v_C^a + (1+w_B)\rho_B v_B^a + \mathcal{O}[3], \tag{84}$$

$$\pi^{ab} = \pi_R^{ab} + \pi_N^{ab} + \rho_C v_C^{\langle a}v_C^{b\rangle} + (1+w_B)\rho_B v_B^{\langle a}v_B^{b\rangle} + \mathcal{O}[3]. \tag{85}$$

The conservation equations for matter were given in Sec. III—see Eqs. (41)–(44). For neutrinos, the equations were given in Sec. V—see Eqs. (74) and (75). For photons, the equations follow from the results of Sec. V as

$$\dot{\rho}_R + \frac{4}{3}\Theta\rho_R + D_a q_R^a + 2A_a q_R^a + \sigma_{ab}\pi_R^{ab} = n_E\sigma_T\left(\frac{4}{3}\rho_R v_B^2 - q_R^a v_{Ba}\right) + \mathcal{O}[3],$$ (86)

$$\dot{q}_R^{\langle a\rangle} + \frac{4}{3}\Theta q_R^a + \frac{4}{3}\rho_R A^a + \frac{1}{3}D^a\rho_R + D_b\pi_R^{ab} + \sigma^a{}_b q_R^b - [\omega, q_R]^a + A_b\pi_R^{ab} = n_E\sigma_T\left(\frac{4}{3}\rho_R v_B^a - q_R^a + \pi_R^{ab}v_{Bb}\right) + \mathcal{O}[3].$$ (87)

The nonlinear dynamical equations are completed by the integrated Boltzmann multipole equations given in Sec. V—see Eq. (71). For neutrinos ($l \geqslant 2$):

$$0 = \dot{\Pi}_N^{\langle A_l\rangle} + \frac{4}{3}\Theta\Pi_N^{A_l} + D^{\langle a_l}\Pi_N^{A_{l-1}\rangle} + \frac{(l+1)}{(2l+3)}D_b\Pi_N^{bA_l} - \frac{(l+1)(l-2)}{(2l+3)}A_b\Pi_N^{bA_l} + (l+3)A^{\langle a_l}\Pi_N^{A_{l-1}\rangle} - l\omega_b\varepsilon^{bc(a_l}\Pi_N^{A_{l-1})c}$$

$$- \frac{(l-1)(l+1)(l+2)}{(2l+3)(2l+5)}\sigma_{bc}\Pi_N^{bcA_l} + \frac{5l}{(2l+3)}\sigma_b{}^{\langle a_l}\Pi_N^{A_{l-1}\rangle b} - (l+2)\sigma^{\langle a_l a_{l-1}}\Pi_N^{A_{l-2}\rangle}.$$ (88)

For photons, the quadrupole evolution equation is

$$\dot{\pi}_R^{\langle ab\rangle} + \frac{4}{3}\Theta\pi_R^{ab} + \frac{8}{15}\rho_R\sigma^{ab} + \frac{2}{5}D^{\langle a}q_R^{b\rangle} + \frac{8\pi}{35}D_c\Pi^{abc} + 2A^{\langle a}q_R^{b\rangle} - 2\omega^c\varepsilon_{cd}{}^{(a}\pi_R^{b)d} + \frac{2}{7}\sigma_c{}^{\langle a}\pi_R^{b\rangle c} - \frac{32\pi}{315}\sigma_{cd}\Pi^{abcd}$$

$$= -n_E\sigma_T\left[\frac{9}{10}\pi_R^{ab} - \frac{1}{5}q_R^{\langle a}v_B^{b\rangle} - \frac{8\pi}{35}\Pi^{abc}v_{Bc} - \frac{2}{5}\rho_R v_B^{\langle a}v_B^{b\rangle}\right] + \mathcal{O}[3].$$ (89)

In the free-streaming case $n_E = 0$, Eq. (89) reduces to the result first given in [30]. This quadrupole evolution equation is central to the proof that almost-isotropy of the CMB after last scattering implies almost-homogeneity of the universe [30].

The higher multipoles ($l > 3$) evolve according to

$$\dot{\Pi}^{\langle A_l\rangle} + \frac{4}{3}\Theta\Pi^{A_l} + D^{\langle a_l}\Pi^{A_{l-1}\rangle} + \frac{(l+1)}{(2l+3)}D_b\Pi^{bA_l} - \frac{(l+1)(l-2)}{(2l+3)}A_b\Pi^{bA_l} + (l+3)A^{\langle a_l}\Pi^{A_{l-1}\rangle} - l\omega^b\varepsilon_{bc}{}^{(a_l}\Pi^{A_{l-1})c}$$

$$- \frac{(l-1)(l+1)(l+2)}{(2l+3)(2l+5)}\sigma_{bc}\Pi^{bcA_l} + \frac{5l}{(2l+3)}\sigma_b{}^{\langle a_l}\Pi^{A_{l-1}\rangle b} - (l+2)\sigma^{\langle a_l a_{l-1}}\Pi^{A_{l-2}\rangle}$$

$$= -n_E\sigma_T\left[\Pi^{A_l} - \Pi^{\langle A_{l-1}}v_B^{a_l\rangle} - \left(\frac{l+1}{2l+3}\right)\Pi^{A_l a}v_{Ba}\right] + \mathcal{O}[3].$$ (90)

For $l = 3$, the second term in square brackets on the right of Eq. (90) must be multiplied by $\frac{3}{2}$. The temperature fluctuation multipoles τ_{A_l} are determined in principle from the radiation dynamic multipoles Π_{A_l} via Eqs. (78) and (79).

These equations show in a transparent and explicitly 1+3 covariant and gauge-invariant form precisely which physical effects are directly responsible for the evolution of CMB anisotropies in an inhomogeneous universe. They show how the matter content of the universe generates anisotropies. This happens directly through direct interaction of matter with the radiation, as encoded in the Thomson scattering terms on the right of Eqs. (86), (87), (89) and (90). And it happens indirectly, as matter generates inhomogeneities in the gravitational field via the field equations (6)–(17) and the evolution equation (44) for the baryonic velocity v_B^a. This in turn feeds back into the multipole equations via the kinematic quantities, the baryonic velocity v_B^a, and the spatial gradient $D_a\rho_R$ in the dipole equation (87). The coupling of the multipole equations themselves provides an up and down cascade of effects, shown in general by Eq. (90). Power is transmitted to the l-multipole by lower multipoles through the dominant (linear) distortion term $D^{\langle a_l}\Pi^{A_{l-1}\rangle}$, as well as through nonlinear terms coupled to the 4-acceleration ($A^{\langle a_l}\Pi^{A_{l-1}\rangle}$), baryonic velocity ($v_B^{\langle a_l}\Pi^{A_{l-1}\rangle}$), and shear ($\sigma^{\langle a_l a_{l-1}}\Pi^{A_{l-2}\rangle}$). Simultaneously, power cascades down from higher multipoles through the linear divergence term $(\mathrm{div}\,\Pi)^{A_l}$, and the nonlinear terms coupled to A^a, v_B^a and σ^{ab}. (Note that the vorticity coupling does not transmit across multipole levels.)

The equations for the radiation (and neutrino) multipoles generalize the equations given by Challinor and Lasenby [36], to which they reduce when we remove all terms $O(\epsilon v_B)$ and $O(\epsilon^2)$. In this case, i.e. FLRW linearization, there is major simplification of the equations:

$$\dot{\rho}_R + \frac{4}{3}\Theta\rho_R + \operatorname{div} q_R \approx 0, \tag{91}$$

$$\dot{q}_R^a + 4Hq_R^a + \frac{4}{3}\rho_R A^a + \frac{1}{3}D^a\rho_R + (\operatorname{div}\pi_R)^a$$

$$\approx n_E\sigma_T\left(\frac{4}{3}\rho_R v_B^a - q_R^a\right), \tag{92}$$

$$\dot{\pi}_R^{ab} + 4H\pi_R^{ab} + \frac{8}{15}\rho_R\sigma^{ab} + \frac{2}{5}D^{\langle a}q_R^{b\rangle} + \frac{8\pi}{35}(\operatorname{div}\Pi)^{ab}$$

$$\approx -\frac{9}{10}n_E\sigma_T\pi_R^{ab}, \tag{93}$$

and for $l \geq 3$

$$\dot{\Pi}^{A_l} + 4H\Pi^{A_l} + D^{\langle a_l}\Pi^{A_{l-1}\rangle} + \frac{(l+1)}{(2l+3)}(\operatorname{div}\Pi)^{A_l}$$

$$\approx -n_E\sigma_T\Pi^{A_l}. \tag{94}$$

These linearized equations, together with the linearized equations governing the kinematic and free gravitational quantities, given by Eqs. (6)–(17) with zero right hand sides, may be covariantly split into scalar, vector and tensor modes, as described in [43,35,36]. The modes can then be expanded in covariant eigentensors of the comoving Laplacian, and the Fourier coefficients obey ordinary differential equations, facilitating numerical integration. Such integrations are performed for scalar modes by Challinor and Lasenby [36], with further analytical results given in [35,36,66,38].

However, in the nonlinear case, it is no longer possible to split into scalar, vector and tensor modes [63,64,73]. A simple illustration of this arises in dust spacetimes, which may be considered as a simplified model after last scattering if we neglect the dynamical effects of baryons, radiation and neutrinos. If one attempts to carry over the linearized scalar-mode conditions [43,36]

$$\omega_a = 0 = H_{ab}$$

into the nonlinear regime, it turns out that a non-terminating chain of integrability conditions must be satisfied, so that the models are in general inconsistent unless they have high symmetry [85,86]. Thus, even in this simple case, it is not possible to isolate scalar modes. In particular, gravitational radiation, with curl $H_{ab} \neq 0$ (see [55–57]), must in general be present.

The generalized equations given above can form the basis for investigating the implications of nonlinear dynamical effects in general and second-order effects against an FLRW background in particular. More quantitative and detailed investigations along these lines are taken up in further papers. Here we will confine ourselves to a qualitative analysis.

A. Nonlinear effects on kinematic, gravitational and dynamic quantities

Evolution of the expansion of the universe Θ, given by Eq. (7), is retarded by the nonlinear shear term $-\sigma_{ab}\sigma^{ab}$, and accelerated by the nonlinear vector terms $+A_aA^a$ and $+2\omega_a\omega^a$ (see also [2]). The vorticity evolution equation (9) has a nonlinear coupling $\sigma_{ab}\omega^b$ of vorticity to shear, whose effect will depend on the alignment of vorticity relative to the shear eigendirections. The shear evolution equation (10) has tensor-tensor and vector-vector type couplings, which are the tensor counterpart of similar terms in the expansion evolution. But in addition, relative velocity effects enter via the total anisotropic stress term. From Eq. (85), we see that baryonic and cold dark matter contributions of the form $\rho v^{\langle a}v^{b\rangle}$ to the shear evolution arise at the nonlinear level. The constraint equations (13) and (14) show that acceleration and vorticity provide scalar $(A^a\omega_a)$ and vector $([\omega,A]_a)$ nonlinear source terms for respectively the vorticity and shear.

The free gravitational fields, which $1+3$ covariantly describe tidal forces and gravitational radiation (see [2,3,55–57]), and therefore in particular control the tensor contribution to CMB anisotropies, are governed by the Maxwell-like equations (11), (12), (16) and (17). This is the foundation for the electromagnetic analogy. The role of nonlinear coupling terms in these equations is more complicated—see [58] for a full discussion. Here we note that nonlinear couplings of the shear and vorticity to the energy flux and gravito-magnetic field act as source terms for the gravito-electric field—see Eq. (16)—while nonlinear couplings of the shear and vorticity to the anisotropic stress and gravito-electric field act as source terms for the gravito-magnetic field—see Eq. (17).

From Eqs. (41)–(44), we see that for baryonic and cold dark matter, nonlinear relative velocity terms act as a source for the linear parts of the evolution equations for energy density and relative velocity. While the 4-acceleration A_a is involved in correction terms in all these equations, the vorticity ω_a and shear σ_{ab} only enter nonlinear corrections of the velocity equations, and not the energy density equations. This reflects the fact that vorticity and shear are volume-preserving. The kinematic corrections to the evolution of matter relative velocity are of the form $A_av^a, [\omega,v]_a$ and $\sigma_{ab}v^b$. For the massless species, as shown by Eqs. (74), (75) and (86), (87), the same form of corrections arises in the energy flux evolution, since energy flux is of the form $\frac{4}{3}\rho v^a$ when the photon and neutrino frames are chosen as the energy frame. Vorticity also does not affect energy density, but shear does, owing to the intrinsic anisotropic stress of photons and neutrinos, which couples with the shear.

Baryonic and radiation conservation equations are both affected by nonlinear Thomson correction terms, which involve a coupling of the baryonic relative velocity v_B^a to the radiation energy density, momentum density and anisotropic stress. In particular, we note that there is a *nonzero energy density transfer* due to Thomson scattering at second order.

B. Nonlinear effects on radiation multipoles

Nonlinear Thomson scattering corrections also affect the evolution of the radiation quadrupole π_R^{ab}, as shown by Eq.

(89). In this case, the baryonic relative velocity couples to the radiation dipole q_R^a and octopole Π^{abc}. Note also the $\frac{1}{10}$ correction to the linear Thomson term $n_E \sigma_T \pi_R^{ab}$, in agreement with [19,36]. This correction arises from incorporating anisotropic effects in the scattering integral (while neglecting polarization effects, as noted earlier).

The general evolution equation (90) for the radiation dynamic multipoles Π^{A_l} shows that *five successive multipoles,* i.e. for $l-2,\ldots,l+2$, are linked together in the nonlinear case. Furthermore, the 4-acceleration A_a couples to the $l \pm 1$ multipoles, the vorticity ω_a couples to the l multipole, and the shear σ_{ab} couples to the $l \pm 2$ and l multipoles. All of these couplings are nonlinear, except for $l=1$ in the case of A_a and $l=2$ in the case of σ_{ab}. These latter couplings that survive linearization are shown in the dipole equation (87) (i.e. $\rho_R A^a$) and the quadrupole equation (89) (i.e. $\rho_R \sigma^{ab}$). The latter term drives Silk damping during the decoupling process [50]. Nonlinear corrections introduce additional acceleration and shear terms. *Vorticity corrections are purely nonlinear,* i.e. vorticity has no direct effect at the linear level, and a linear approach could produce the false impression that vorticity has no direct effect at all on the evolution of CMB anisotropies. However, for very high l, i.e. on very small angular scales, the nonlinear vorticity term could in principle be non-negligible.

The disappearance of most of the kinematic terms upon linearization is further reflected in the fact that the linearized equations link only *three* successive moments, i.e. l, $l \pm 1$. This is clearly seen in Eq. (94).

In addition to A_a and ω_a, there is a further vector coupling at the nonlinear level, i.e. the coupling of the baryonic velocity v_B^a to the $l \pm 1$ multipoles in the Thomson scattering source term of the evolution equation (90). In fact these nonlinear velocity corrections are of precisely the same tensorial form as the acceleration corrections on the left hand side, only with different weighting factors. Linearization, by removing these terms, also has the effect of removing the nonlinear contribution of the radiation multipoles $\Pi^{A_{l\pm1}}$ to the collision multipole K^{A_l}.

One notable feature of the nonlinear terms is that some of them scale like l for large l, as already noted in the case of vorticity. There are no purely linear terms with this property, which has an important consequence, i.e. that *for very high l multipoles (corresponding to very small angular scales in CMB observations), certain nonlinear terms can reach the same order of magnitude as the linear contributions.* (Note

that the same effect applies to the neutrino background.) The relevant nonlinear terms in Eq. (90) are (for $l \gg 1$)

$$-l \left(\frac{1}{4} \sigma_{bc} \Pi^{bcA_l} + \sigma^{\langle a_l a_{l-1}} \Pi^{A_{l-2} \rangle} - A^{\langle a_l} \Pi^{A_{l-1} \rangle} + \frac{1}{2} A_b \Pi^{bA_l} \right.$$

$$\left. + \omega^b \varepsilon_{bc}{}^{\langle a_l} \Pi^{A_{l-1} \rangle c} \right).$$

The observable imprint of this effect will be made after last scattering. In the free-streaming era, it is reasonable to neglect the vorticity relative to the shear. We can remove the acceleration term by choosing u^a as the dynamically dominant cold dark matter frame (i.e. choosing $v_C^a = 0$), as in [36]. It follows from Eqs. (80) and (90) that the nonlinear correction to the rate of change of the linearized temperature fluctuation multipoles is

$$\delta(\dot{\tau}^{A_l}) \sim l \left(\frac{1}{4} \sigma_{bc} \tau^{bcA_l} + \sigma^{\langle a_l a_{l-1}} \tau^{A_{l-2} \rangle} \right) \quad \text{for} \quad l \gg 1.$$

$$(95)$$

The linear solutions for τ_{A_l} and σ_{ab} can be used on the right hand side to estimate the correction to second order. Its effect on observed anisotropies will be estimated by integrating $\delta(\dot{\tau}^{A_l})$ from last scattering to now. (See [37] for the relation between the τ_{A_l} and the angular correlations.) In the case of scalar perturbations, these solutions are given by Challinor and Lasenby [36] (see also [66]).

Finally, we note that the well-known Vishniac and Rees-Sciama second-order effects also become significant at high l, and can eventually dominate the linear contributions to CMB anisotropies on small enough angular scales (typically $l > 10^3$ or more) [19].

C. Temperature fluctuation multipoles

We can normalize the radiation dynamic multipoles Π^{A_l} to define the dimensionless multipoles ($l \geq 1$)

$$\mathcal{T}^{A_l} = \left(\frac{\pi}{rT^4} \right) \Pi^{A_l} \approx \tau^{A_l}.$$

Thus the $\mathcal{T}^{A_l}$ are equal to the temperature fluctuation multipoles plus nonlinear corrections. In terms of these quantities, the hierarchy of radiation multipoles becomes

$$\frac{\dot{T}}{T} = -\frac{1}{3}\Theta - \frac{1}{3}D_a \mathcal{T}^a - \frac{4}{3}\mathcal{T}^a \frac{D_a T}{T} - \frac{2}{3} A_a \mathcal{T}^a - \frac{2}{15}\sigma_{ab}\mathcal{T}^{ab} + \frac{1}{3}n_E \sigma_T v_{Ba}(v_B^a - \mathcal{T}^a) + \mathcal{O}[3], \tag{96}$$

$$\dot{\mathcal{T}}^a = -4 \left(\frac{\dot{T}}{T} + \frac{1}{3}\Theta \right) \mathcal{T}^a - \frac{D^a T}{T} - A^a - \frac{2}{5}D_b \mathcal{T}^{ab} + n_E \sigma_T (v_B^a - \mathcal{T}^a) + \frac{2}{5}n_E \sigma_T \mathcal{T}^{ab}v_{Bb} - \sigma^a{}_b \mathcal{T}^b - \frac{2}{5}A_b \mathcal{T}^{ab}$$

$$+ [\omega, \mathcal{T}]^a - \frac{8}{5}\mathcal{T}^{ab}\frac{D_b T}{T} + \mathcal{O}[3], \tag{97}$$

$$\dot{T}^{ab} = -4\left(\frac{\dot{T}}{T} + \frac{1}{3}\Theta\right)T^{ab} - \sigma^{ab} - D^{\langle a}T^{b\rangle} - \frac{3}{7}D_c T^{abc} - \frac{9}{10}n_E\sigma_T T^{ab} + n_E\sigma_T\left(\frac{1}{2}T^{\langle a}v_B^{b\rangle} + \frac{3}{7}T^{abc}v_{Bc}\right)$$

$$+ \frac{3}{4}v_B^{\langle a}v_B^{b\rangle}\right) - 5A^{\langle a}T^{b\rangle} - \frac{4}{21}\sigma_{cd}T^{abcd} + 2\omega^c\varepsilon_{cd}^{\langle a}T^{b\rangle d} - \frac{10}{7}\sigma_c^{\langle a}T^{b\rangle c} - \frac{12}{7}T^{abc}\frac{D_c T}{T} + \mathcal{O}[3], \quad (98)$$

and, for $l > 3$,

$$\dot{T}^{A_l} = -4\left(\frac{\dot{T}}{T} + \frac{1}{3}\Theta\right)T^{A_l} - D^{\langle a_l}T^{A_{l-1}\rangle} - \frac{(l+1)}{(2l+3)}D_b T^{bA_l} - n_E\sigma_T T^{A_l} + n_E\sigma_T\left[T^{\langle A_{l-1}}v_B^{a_l\rangle} + \left(\frac{l+1}{2l+3}\right)T^{A_l b}v_{Bb}\right]$$

$$+ \frac{(l+1)(l-2)}{(2l+3)}A_b T^{bA_l} - (l+3)A^{\langle a_l}T^{A_{l-1}\rangle} + l\omega^b\varepsilon_{bc}^{(a_l}T^{A_{l-1})c} + (l+2)\sigma^{\langle a_l a_{l-1}}T^{A_{l-2}\rangle} + \frac{(l-1)(l+1)(l+2)}{(2l+3)(2l+5)}\sigma_{bc}T^{bcA_l}$$

$$- \frac{5l}{(2l+3)}\sigma_b^{\langle a_l}T^{A_{l-1}\rangle b} - 4\frac{(l+1)}{(2l+3)}T^{A_l b}\frac{D_b T}{T} + \mathcal{O}[3]. \quad (99)$$

For $l = 3$, the Thomson term $T^{\langle A_{l-1}}v_B^{a_l\rangle}$ must be multiplied by $\frac{3}{2}$.

The nonlinear multipole equations given in this form show more clearly the evolution of temperature anisotropies (including the monopole, i.e. the average temperature T). Although the T^{A_l} only determine the actual temperature fluctuations τ^{A_l} to linear order, they are a useful dimensionless measure of anisotropy. Furthermore, Eqs. (96)–(99) apply as the evolution equations for temperature fluctuation multipoles when the radiation anisotropy is small (i.e. $T^{A_l} = \tau^{A_l}$), but the spacetime inhomogeneity and anisotropy are not restricted. This includes the particular case of small CMB anisotropies in general Bianchi universes or in perturbed Bianchi universes.

FLRW linearization, i.e. the case when only first order effects relative to the FLRW limit are considered, reduces the above equations to

$$\frac{\dot{T}}{T} \approx -\frac{1}{3}\Theta - \frac{1}{3}D_a\tau^a, \quad (100)$$

$$\dot{\tau}^a \approx -\frac{D^a T}{T} - A^a - \frac{2}{5}D_b\tau^{ab}$$

$$+ n_E\sigma_T(v_B^a - \tau^a), \quad (101)$$

$$\dot{\tau}^{ab} \approx -\sigma^{ab} - D^{\langle a}\tau^{b\rangle} - \frac{3}{7}D_c\tau^{abc}$$

$$- \frac{9}{10}n_E\sigma_T\tau^{ab}, \quad (102)$$

and, for $l \geq 3$

$$\dot{\tau}^{A_l} \approx -D^{\langle a_l}\tau^{A_{l-1}\rangle} - \frac{(l+1)}{(2l+3)}D_b\tau^{bA_l} - n_E\sigma_T\tau^{A_l}. \quad (103)$$

These are the $1+3$ covariant and gauge-invariant multipole generalizations of the Fourier mode formulation of the integrated Boltzmann equations used in the standard literature (see e.g. [19] and the references therein). Equations (100)–(102) were given in [10] in the free-streaming case $n_E = 0$.

As noted before, there is still a gauge freedom here associated with the choice of 4-velocity u^a. Given any physical choice for this 4-velocity which tends to the preferred 4-velocity in the FLRW limit, the $l \geq 1$ equations are gauge invariant.

VII. CONCLUSIONS

We have used a covariant Lagrangian approach, in which all the relevant physical and geometric quantities occur directly and transparently, as PSTF tensors measured in the comoving rest space. There is no restriction on the deviation of geometric and physical quantities from FLRW limiting values, so that arbitrary nonlinear behavior may in principle be treated. We have derived the corresponding equations governing the generation and evolution of inhomogeneities and CMB anisotropies in nonlinear generality, without *a priori* restrictions on spacetime geometry or specific assumptions about early-universe particle physics, structure formation history, etc. Thus we have developed a useful approach to the analysis of local nonlinear effects in CMB anisotropies, with the clarity and transparency arising from $1+3$ covariance. The equations are readily linearized in a gauge-invariant way, and then the methods of [37] may be used to expand in scalar modes and regain well-known first-order results [66] (see also [35,36,38]).

This approach allowed us to identify and qualitatively describe some of the key local nonlinear effects, and more quantitative results will be considered in further papers. We calculated the nonlinear form of Thomson scattering multipoles (given the initial simplifying assumption of no polarization), revealing the new effect of coupling between the baryonic bulk velocity and radiation brightness multipoles of order $l \pm 1$. We also found the nonlinear effects of relative velocities of particle species on the dynamic quantities that source the gravitational field. These effects also operate on

the conservation equations, including evolution equations for the relative velocities of baryonic and cold dark matter.

Nonlinear effects come together in the hierarchy of evolution equations for the radiation dynamic (brightness) multipoles, which determine the CMB temperature anisotropies. In addition to the nonlinear Thomson contribution, we identified nonlinear couplings of the kinematic quantities to the multipoles of order $l\pm 2$, $l\pm 1$, and l. These quantities themselves are governed by nonlinear evolution equations, which provides part of the link between CMB anisotropies and inhomogeneities in the gravitational field and sources. The link is also carried by the spatial gradient of radiation energy density (equivalently, average radiation all-sky temperature), and the baryonic relative velocity. Furthermore, there is internal up- and down-transmission of power within the multipole hierarchy, supported by the kinematic couplings as well as by distortion and divergence derivatives of the multipoles.

We used our analysis of the radiation multipoles to identify new effects that operate at high l. In particular, we showed that there is a nonlinear shear correction effect on small angular scales, whose impact on the angular power spectrum was qualitatively described. The quantitative analysis of this and other nonlinear effects is a subject of further research.

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APPENDIX: EXACT NONLINEAR RELATIVE VELOCITY EQUATIONS

Change in 4-velocity:

$$\tilde{u}_a = \gamma(u_a + v_a) \quad \text{where} \quad \gamma = (1-v^2)^{-1/2}, \quad v_a u^a = 0. \tag{A1}$$

Change in fundamental algebraic tensors:

$$\tilde{h}_{ab} = h_{ab} + \gamma^2[v^2 u_a u_b + 2u_{(a}v_{b)} + v_a v_b], \tag{A2}$$

$$\tilde{\varepsilon}_{abc} = \gamma \varepsilon_{abc} + \gamma\{2u_{[a}\varepsilon_{b]cd} + u_c \varepsilon_{abd}\}v^d. \tag{A3}$$

Transformed kinematic quantities are defined by

$$\nabla_b \tilde{u}_a = \frac{1}{3}\tilde{\Theta}\tilde{h}_{ab} + \tilde{\sigma}_{ab} + \tilde{\varepsilon}_{abc}\tilde{\omega}^c - \tilde{A}_a \tilde{u}_b,$$

which implies, using $\nabla_a \gamma = \gamma^3 v^b \nabla_a v_b$, and Eq. (2), the following kinematic transformations [52]:

$$\tilde{\Theta} = \gamma\Theta + \gamma(\mathrm{div}\, v + A^a v_a) + \gamma^3 W, \tag{A4}$$

$$\tilde{A}_a = \gamma^2 A_a + \gamma^2\left\{\dot{v}_{\langle a\rangle} + \frac{1}{3}\Theta v_a + \sigma_{ab}v^b - [\omega,v]_a + \left(\frac{1}{3}\Theta v^2 + A^b v_b + \sigma_{bc}v^b v^c\right)u_a + \frac{1}{3}(\mathrm{div}\, v)v_a + \frac{1}{2}[v,\mathrm{curl}\, v]_a \right.$$
$$\left. + v^b \mathrm{D}_{\langle b}v_{a\rangle}\right\} + \gamma^4 W(u_a + v_a), \tag{A5}$$

$$\tilde{\omega}_a = \gamma^2\left\{\left(1 - \frac{1}{2}v^2\right)\omega_a - \frac{1}{2}\mathrm{curl}\, v_a + \frac{1}{2}v_b(2\omega^b - \mathrm{curl}\, v^b)u_a + \frac{1}{2}v_b\omega^b v_a + \frac{1}{2}[A,v]_a + \frac{1}{2}[\dot{v},v]_a + \frac{1}{2}\varepsilon_{abc}\sigma^b{}_d v^c v^d\right\}, \tag{A6}$$

$$\tilde{\sigma}_{ab} = \gamma\sigma_{ab} + \gamma(1+\gamma^2)u_{(a}\sigma_{b)c}v^c + \gamma^2 A_{(a}[v_{b)} + v^2 u_{b)}] + \gamma \mathrm{D}_{\langle a}v_{b\rangle} - \frac{1}{3}h_{ab}[A_c v^c + \gamma^2(W - \dot{v}_c v^c)] + \gamma^3 u_a u_b\left[\sigma_{cd}v^c v^d\right.$$
$$+ \frac{2}{3}v^2 A_c v^c - v^c v^d \mathrm{D}_{\langle c}v_{d\rangle} + \left(\gamma^4 - \frac{1}{3}v^2\gamma^2 - 1\right)W\right] + \gamma^3 u_{(a}v_{b)}\left[A_c v^c + \sigma_{cd}v^c v^d - \dot{v}_c v^c + 2\gamma^2\left(\gamma^2 - \frac{1}{3}\right)W\right]$$
$$+ \frac{1}{3}\gamma^3 v_a v_b[\mathrm{div}\, v - A_c v^c + \gamma^2(3\gamma^2 - 1)W] + \gamma^3 v_{\langle a}\dot{v}_{b\rangle} + v^2\gamma^3 u_{(a}\dot{v}_{\langle b\rangle)} + \gamma^3 v_{(a}\sigma_{b)c}v^c - \gamma^3[\omega,v]_{(a}\{v_{b)} + v^2 u_{b)}\}$$
$$+ 2\gamma^3 v^c \mathrm{D}_{\langle c}v_{(a\rangle}\{v_{b)} + u_{b)}\}, \tag{A7}$$

where

$$W \equiv \dot{v}_c v^c + \frac{1}{3} v^2 \operatorname{div} v + v^c v^d D_{\langle c} v_{d \rangle}.$$

Transformed dynamic quantities are [52]

$$\widetilde{\rho} = \rho + \gamma^2 [v^2(\rho+p) - 2q_a v^a + \pi_{ab} v^a v^b], \tag{A8}$$

$$\widetilde{p} = p + \frac{1}{3} \gamma^2 [v^2(\rho+p) - 2q_a v^a + \pi_{ab} v^a v^b], \tag{A9}$$

$$\widetilde{q}_a = \gamma q_a - \gamma \pi_{ab} v^b - \gamma^3 [(\rho+p) - 2q_b v^b + \pi_{bc} v^b v^c] v_a - \gamma^3 [v^2(\rho+p)$$
$$- (1+v^2) q_b v^b + \pi_{bc} v^b v^c] u_a, \tag{A10}$$

$$\widetilde{\pi}_{ab} = \pi_{ab} + 2\gamma^2 v^c \pi_{c(a} \{u_{b)} + v_{b)} \} - 2v^2 \gamma^2 q_{(a} u_{b)} - 2\gamma^2 q_{\langle a} v_{b \rangle} - \frac{1}{3} \gamma^2 [v^2(\rho+p)$$

$$+ \pi_{cd} v^c v^d] h_{ab} + \frac{1}{3} \gamma^4 [2v^4(\rho+p) - 4v^2 q_c v^c + (3-v^2) \pi_{cd} v^c v^d] u_a u_b$$

$$+ \frac{2}{3} \gamma^4 [2v^2(\rho+p) - (1+3v^2) q_c v^c + 2\pi_{cd} v^c v^d] u_{(a} v_{b)} + \frac{1}{3} \gamma^4 [(3-v^2)(\rho+p)$$

$$- 4q_c v^c + 2\pi_{cd} v^c v^d] v_a v_b. \tag{A11}$$

Gravito-electric–magnetic field: using [67]

$$C_{ab}{}^{cd} = 4\{u_{[a} u^{[c} + h_{[a}{}^{[c}\} E_{b]}{}^{d]} + 2\varepsilon_{abe} u^{[c} H^{d]e} + 2u_{[a} H_{b]e} \varepsilon^{cde} = 4\{\widetilde{u}_{[a} \widetilde{u}^{[c} + \widetilde{h}_{[a}{}^{[c}\} \widetilde{E}_{b]}{}^{d]} + 2\widetilde{\varepsilon}_{abe} \widetilde{u}^{[c} \widetilde{H}^{d]e} + 2\widetilde{u}_{[a} \widetilde{H}_{b]e} \widetilde{\varepsilon}^{cde},$$

we find the transformation [52]

$$\widetilde{E}_{ab} = \gamma^2 \{(1+v^2) E_{ab} + v^c [2\varepsilon_{cd(a} H_{b)}{}^d + 2E_{c(a} u_{b)} + (u_a u_b + h_{ab}) E_{cd} v^d - 2E_{c(a} v_{b)} + 2u_{(a} \varepsilon_{b)cd} H^{de} v_e]\}, \tag{A12}$$

$$\widetilde{H}_{ab} = \gamma^2 \{(1+v^2) H_{ab} + v^c [-2\varepsilon_{cd(a} E_{b)}{}^d + 2H_{c(a} u_{b)} + (u_a u_b + h_{ab}) H_{cd} v^d - 2H_{c(a} v_{b)} - 2u_{(a} \varepsilon_{b)cd} E^{de} v_e]\}. \tag{A13}$$

This may be compared with the electromagnetic transformation

$$\widetilde{E}_a = \gamma \{E_a + [v,H]_a + v^b E_b u_a\},$$

$$\widetilde{H}_a = \gamma \{H_a - [v,E]_a + v^b H_b u_a\},$$

where

$$F_{ab} = 2u_{[a} E_{b]} + \varepsilon_{abc} H^c = 2\widetilde{u}_{[a} \widetilde{E}_{b]} + \widetilde{\varepsilon}_{abc} \widetilde{H}^c.$$

Note that all the transformations above are given explicitly in terms of irreducible quantities (i.e. irreducible in the original u^a-frame).

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