

Order reduction in semiclassical cosmology

Grażyna Siemieniec-Oziębło and Andrzej Woszczyzna*

Astronomical Observatory, Jagellonian University, ul. Orła 171, 30-244 Kraków, Poland

(Received 11 May 1998; published 18 March 1999)

We investigate Robertson-Walker cosmology with the Lagrangian $R + \alpha_1 \hbar R^2 + \alpha_2 \hbar R^{\mu\nu} R_{\mu\nu} + L_{rad}$ where L_{rad} denotes a classical source with a traceless energy-momentum tensor. We weaken the self-consistence condition [L. Parker and J. Z. Simon, Phys. Rev. D **47**, 1339 (1993)]. Quantum corrections are expressed as contributions to the effective equation of state. We show that the empty space-time is stable within the class of radiation-filled expanding universes with no order reduction of the field equations. [S0556-2821(99)02906-9]

PACS number(s): 98.80.Hw

I. INTRODUCTION

Controversial fourth order differential equations, which govern the semiclassical cosmology, can be reduced to second order [1,2] and, in this way, avoid quantum-originated instabilities [3,4]. The reduction is based on the *self-consistence* condition, i.e., the assumption that both equations and solutions are perturbatively expandable in \hbar . Under this condition the universe becomes an ordinary mechanical system with a two-dimensional phase-space corresponding to the single degree of mechanical freedom—the scale factor $a(t, \hbar)$. Self-consistent theory is still renormalizable [5]; Minkowski space-time regains stability in the class of homogeneous and isotropic models; quasi-inflationary phenomena disappear [6]. Similar reduction techniques are being applied to gravity with higher than fourth-order derivatives [7] and also in other branches of physics [8].

However, imposing the self-consistency condition on the cosmological scale $a(t)$ encounters some difficulties. In a universe with vanishing spatial curvature, there still remains the freedom to multiply metrics by an arbitrary constant factor; therefore the scale $a(t)$ is not a measurable quantity. The requirement for $a(t, \hbar)$ to be \hbar expandable is physically unclear. In open or closed universes this freedom is reduced by the choice $k = \pm 1$, and the scale factors are uniquely determined by cosmological observables: the Hubble parameter H and the energy density ϵ . Yet arbitrarily small changes in any of these observables in the vicinity of critical density $\epsilon - \Lambda = 3H^2$ may result in indefinite changes of $a(t)$. The perturbative character of the energy-momentum tensor (and consequently the equations) with respect to an arbitrary chosen parameter, in general, does not imply the same property of the metrics. Finally, expanding $a(t)$ in the equations, which contain fixed curvature index k [2], limits quantum corrections to only those which preserve the same sign of the space curvature. This limitation is particularly severe for a flat universe, where generic quantum corrections would contribute to the space curvature unless the $k=0$ condition prevents that. This limitation cannot be derived directly from the Lagrangian and, in fact, it forms an additional constraint

imposed on the theory (which is not even true in classical gravity¹).

Not arguing with the very idea of self-consistency, we draw attention to some circumstances which are important for semiclassical cosmology.

(1) Without harm to the reduction procedure, one can release the consistency condition for the scale factor, demanding instead the same property for cosmological observables (the Hubble parameter, etc.)

(2) For a radiation filled universe with vanishing cosmological constant $\Lambda=0$ the self-consistency condition is superfluous, since the original equation is of second order. Terms with higher derivatives cited by classical papers [9] contain an additional hidden factor \hbar and are eventually eliminated in the first order expansion.

(3) We show that quantum corrections form the equation of state of a barotropic fluid, and discuss the stability of the Minkowski space-time on the ground of dynamic systems theory.

II. CONDITION OF SELF-CONSISTENCY FOR HUBBLE'S EXPANSION RATE

We consider semiclassical gravity theory with the Lagrangian $R + \alpha_1 \hbar R^2 + \alpha_2 \hbar R^{\mu\nu} R_{\mu\nu} + L_{rad}$, where L_{rad} represents classical radiation or another thermalized field of massless particles. Typically, cosmologies containing the R^2 and the $R^{\mu\nu} R_{\mu\nu}$ terms lead to four order equations [9] and violate the stability of empty space [3].

We write quantum terms on the right-hand side of the field equations and treat them as corrections to the energy-momentum tensor. We think of \hbar as the theory parameter, which can take arbitrary values, so the limit transition $\hbar \rightarrow 0$ defines the classical limit of the theory. The field equations we write in the Einsteinian form $R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = T_{\mu\nu}$, but with the modified, effective energy-momentum tensor

¹Note that the Lemaitre universes, which are of positive space curvature, are obtained from flat universes (not closed) when Λ diverges from zero.

*Email address: woszcz@oa.uj.edu.pl

$$T_{\mu\nu} = T_{\mu\nu}^{(\text{rad})} - \hbar \alpha_1 {}^{(1)}H_{\mu\nu} - \hbar \alpha_3 {}^{(3)}H_{\mu\nu}, \quad (1)$$

where

$${}^{(1)}H_{\mu\nu} = \frac{1}{2}R^2 g_{\mu\nu} - 2RR_{\mu\nu} - 2\Box R g_{\mu\nu} + 2\nabla_\mu \nabla_\nu R, \quad (2)$$

$${}^{(3)}H_{\mu\nu} = -R_\mu^\sigma R_{\sigma\nu} + \frac{2}{3}RR_{\mu\nu} + \frac{1}{2}R_{\sigma\rho}R^{\sigma\rho}g_{\mu\nu} - \frac{1}{4}R^2 g_{\mu\nu}, \quad (3)$$

and the constant α_3 is some combination of α_1 and α_2 [Robertson-Walker symmetry have been partially exploited to derive formula (1). For a more precise explanation see Ref [2].] The (0,0) equation derived in this way

$$0 = -\Lambda - \frac{\kappa\mu}{a^4} + \frac{1}{a^2} \left[3k + 3 \left(\frac{da}{dt} \right)^2 \right] + \frac{\alpha_1 \hbar}{a^4} \left[-18k^2 + 36k \left(\frac{da}{dt} \right)^2 + 54 \left(\frac{da}{dt} \right)^4 - 36a \left(\frac{da}{dt} \right)^2 \frac{d^2 a}{dt^2} + 18a^2 \left(\frac{d^2 a}{dt^2} \right)^2 - 36a^2 \frac{da}{dt} \frac{d^3 a}{dt^3} \right] + \frac{\alpha_3 \hbar}{a^4} \left[3 \left(\frac{da}{dt} \right)^4 + 6k \left(\frac{da}{dt} \right)^2 + 3k^2 \right] \quad (4)$$

contains four fundamental constants, two of them classical — the gravitation constant κ (further on we put $8\pi\kappa=1$), the cosmological constant Λ , and two quantum ones α_1 and α_3 . There are also two quantities which define a particular solution: the constant of motion $\mu = \epsilon_0 a_0^4$, and the index of space curvature k . Therefore, the transition from classical to quantum theory with the self-consistency $a = a_0 + \hbar a_1$ imposed on Eq. (4) preserves the type of space curvature, including the strong $k=0$ limitation for the flat universe.

One can get rid of the last two constants, and consequently, of the constraints they bring, by introducing the Hubble expansion parameter $H = (1/a)(da/dt)$. Differentiating Eq. (4) twice, we obtain the fourth order equation for H , which contains only fundamental constants²

$$0 = -3 \frac{d^2 H}{dt^2} - 18H \frac{dH}{dt} - 4H(3H^2 - \Lambda) + 18\hbar \alpha_1 \frac{d^4 H}{dt^4} + 162\hbar \alpha_1 H \frac{d^3 H}{dt^3} + \frac{d^2 H}{dt^2} \left[6(51\hbar \alpha_1 + \hbar \alpha_3) \frac{dH}{dt} + 6(90\hbar \alpha_1 + \hbar \alpha_3)H^2 - 4\Lambda(6\hbar \alpha_1 + \hbar \alpha_3) \right] + 4H \left[162\hbar \alpha_1 \left(\frac{dH}{dt} \right)^2 + \frac{dH}{dt} [3(48\hbar \alpha_1 - \hbar \alpha_3)H^2 - 2\Lambda(6\hbar \alpha_1 + \hbar \alpha_3)] - 3\hbar \alpha_3 H^4 \right]. \quad (5)$$

This equation describes the dynamics of Robertson-Walker models with arbitrary space curvature, and what is equally important, it is expressed in terms of observable quantities. A self-consistency condition imposed on measurable quantities has well defined physical meaning. We adopt Simon's ansatz to H , namely, we state that $H(t) = H_{\text{class}}(t) + \hbar H_{\text{quant}}(t)$ is perturbative in \hbar . Now, the procedure of the order reduction can be done in two ways.

(1) One can differentiate twice the zeroth-order expansion [Eq. (5) with $\alpha_1 = \alpha_3 = 0$] to find the third and fourth derivatives and eliminate them from the full Eq. (5) — this is equivalent to what is done in Ref. [2].

(2) Substitute the expansion $H(t) = H_{\text{class}}(t) + \hbar H_{\text{quant}}(t)$ directly into Eq. (5) and abandon terms second order in \hbar or higher.

²The reverse procedure would give the equation with two parameters of continuous values. Consequently, Eq. (5) formally has a broader class of solutions than Eq. (4). However, the freedom to choose k as different from $0, \pm 1$ is a trivial one, and resolves itself to rescaling the metrics by a constant factor.

In both cases we obtain the second order equation

$$\begin{aligned}
0 = & -3 \frac{d^2 H}{dt^2} - 18H \frac{dH}{dt} - 4H(3H^2 - \Lambda) \\
& + 2 \frac{d^2 H}{dt^2} \left[3(51\hbar\alpha_1 + \hbar\alpha_3) \frac{dH}{dt} + 3(90\hbar\alpha_1 + \hbar\alpha_3)H^2 - 2\Lambda(6\hbar\alpha_1 + \hbar\alpha_3) \right] \\
& + 4H \left[459\hbar\alpha_1 \left(\frac{dH}{dt} \right)^2 + \frac{dH}{dt} [3(372\hbar\alpha_1 - \hbar\alpha_3)H^2 - 2\Lambda(69\hbar\alpha_1 + \hbar\alpha_3)] \right] \\
& + 4[3(180\hbar\alpha_1 - \hbar\alpha_3)H^4 - 204\Lambda\hbar\alpha_1 H^2 + 8\Lambda^2\hbar\alpha_1], \tag{6}
\end{aligned}$$

which is nonlinear both in H and its derivatives. So strong nonlinearity allows one to find exact solutions only in some particular situations. This is not the case in Eq. (6). However, this equation becomes much more transparent after one rewrites the quantum corrections as contributions to energy density and pressure. Qualitative analysis is then enabled.

Let ϵ and P denote, respectively, the effective energy density and effective pressure, i.e., each of these quantities is supplemented by quantum corrections. The universe dynamics is determined by the system of the Raychaudhuri (7) and the continuity (8) equations

$$\frac{dH}{dt} = -H^2 - \frac{1}{6}(3P + \epsilon - 2\Lambda), \tag{7}$$

$$\frac{d\epsilon}{dt} = -3H(P + \epsilon). \tag{8}$$

We differentiate Eq. (7), substitute into Eq. (6), and apply the continuity equation (8) to get the relation between pressure, energy, and the cosmological constant in differential form

$$\frac{dP}{d\epsilon} = \frac{P + \epsilon/9}{P + \epsilon} - \frac{2}{9}\alpha_3\hbar \frac{(3P + \epsilon)^2}{P + \epsilon} - \frac{\alpha_3\hbar}{27} \frac{8\Lambda^2}{P + \epsilon}. \tag{9}$$

As a matter of fact, one can solve Eq. (9) analytically, however, the solution takes an unclear implicit form. This is much simpler to follow the other way. The solution of Eq. (9) must be a function of the energy density and cosmological constant solely, hence $P(\epsilon, \Lambda)$ is independent of the expansion rate H . Therefore the limit transition $H^2 \rightarrow 3(\epsilon + \Lambda)$ does not affect its values, and the general solution is identical with the integral found for the flat universe. In the last case the equation

$$\begin{aligned}
& -\frac{1}{18} \left[3 \frac{dH}{dt} + 2(3H^2 - \Lambda) \right] + \hbar\alpha_1 \frac{d^3 H}{dt^3} + 7\hbar\alpha_1 H \frac{d^2 H}{dt^2} \\
& + \frac{\hbar}{3} \left[12\alpha_1 \left[\frac{dH}{dt} \right]^2 + (36\alpha_1 - \alpha_3)H^2 \frac{dH}{dt} - \alpha_3 H^4 \right] = 0 \tag{10}
\end{aligned}$$

is analogous to Eq. (5). Its order reduces by 2, and finally the equation takes a particularly simple form

$$\frac{dH}{dt} = -\frac{2}{3}\epsilon + \frac{2\alpha_3\hbar}{9}(\epsilon^2 - \Lambda^2). \tag{11}$$

Now, comparing Eq. (11) with the Raychaudhuri equation (7) we obtain the equation of state of cosmological substratum in the form of the algebraic relation

$$P = \frac{1}{3}\epsilon - \frac{4\alpha_3\hbar}{9}(\epsilon^2 - \Lambda^2). \tag{12}$$

Function $P(\epsilon, \Lambda)$, defined by Eq. (12), fulfills the differential equation (9) with an accuracy to terms $o(\hbar)$. By simple calculation [10], one can confirm that the exact solutions found by Parker and Simon also obey Eq. (12).

As we have already mentioned, the equation of state (12) is barotropic, i.e., effective pressure is solely the function of the effective energy density (including the energy of vacuum Λ). While reducing the equations order we eliminate contributions to the energy-momentum tensor coming from the expansion rate [9]; therefore the universe evolution becomes a reversible process [Eqs. (7),(8) are invariant under the time reflection $t \rightarrow -t$].

Quantum corrections contained in Eq. (12), and consequently the dynamical system (7),(8) are free of the α_1 constant. The only term multiplied by α_1 which survives the reduction procedure [2], has been assimilated here by the effective energy density.³

III. THE $\Lambda = 0$ CASE

Its worth noticing that in some physically interesting situations the reduction procedure eliminating higher order derivatives is redundant. In the radiation filled universe with null cosmological constant the correction $\hbar\alpha_1 H_{\mu\nu}^{(1)}$, which formally appears as linear in \hbar , actually is quadratic, and consequently should be abandoned as the $o(\hbar)$ term. To

³In this approach the quantum corrections modify effective energy density and pressure, not the fundamental constants as in Ref. [2].

show this let us express the traceless tensor ${}^{(1)}H_\nu^\mu$ in terms of the Ricci scalar and the effective energy density

$${}^{(1)}H = \frac{R}{2}(4\epsilon - R) \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1/3 & 0 & 0 \\ 0 & 0 & -1/3 & 0 \\ 0 & 0 & 0 & -1/3 \end{bmatrix}.$$

The field equations with the energy-momentum tensor (1) show that the scalar R involves the trace of the tensor ${}^{(3)}H_\nu^\mu$, namely, $R = \alpha_3 \hbar {}^{(3)}H_\mu^\mu$, so it is a quantity linearly dependent on \hbar .

Writing ${}^{(3)}H_\nu^\mu$ in terms of the effective energy density ϵ with the accuracy to terms $o(\hbar)$ we get $R = \frac{4}{3} \alpha_3 \hbar \epsilon^2$. Tensors ${}^{(1)}H_\nu^\mu$ and ${}^{(3)}H_\nu^\mu$ can be rewritten as

$${}^{(1)}H_\nu^\mu = \frac{8}{3} \alpha_3 \hbar \epsilon^3 \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1/3 & 0 & 0 \\ 0 & 0 & -1/3 & 0 \\ 0 & 0 & 0 & -1/3 \end{bmatrix},$$

$${}^{(3)}H_\nu^\mu = \frac{1}{3} \epsilon^2 \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 5/3 & 0 & 0 \\ 0 & 0 & 5/3 & 0 \\ 0 & 0 & 0 & 5/3 \end{bmatrix}$$

$$- \frac{8}{27} \alpha_3 \hbar \epsilon^3 \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Now it is clear that only the second of the expressions $\alpha_1 \hbar {}^{(1)}H_\nu^\mu$ and $\alpha_3 \hbar {}^{(3)}H_\nu^\mu$ is essentially linear in \hbar and forms the first-order quantum contribution to the energy-momentum tensor. The first one $\alpha_1 \hbar {}^{(1)}H_\nu^\mu$, which carries all higher derivatives is actually square in \hbar . This is closely related to the absence of particle creation in the radiation-filled Robertson-Walker universe (see Ref. [11], and papers cited there.) The theory with the energy-momentum tensor $T_{\mu\nu} = T_{\mu\nu}^{(\text{rad})} - \hbar \alpha_3 H_{\mu\nu}^{(3)}$ leads to the effective equation of state $P = \frac{1}{3} \epsilon - (4\alpha_3 \hbar / 9) \epsilon^2$, which is perfectly consistent with Eq. (12).

IV. STABILITY OF THE EMPTY SPACE — DYNAMICAL SYSTEMS APPROACH

The equation of state of the form $P = P(\epsilon, \Lambda)$ [or more generally $P = P(\epsilon, \Lambda, H)$, see Ref. [12]] uniquely determines cosmological evolution. The system (7),(8), which defines the universe dynamics in the $\{H, \epsilon\}$ -phase space is autonomous. Choosing a point in the $\{H, \epsilon\}$ -phase space, one determines uniquely the metrics in the initial moment as

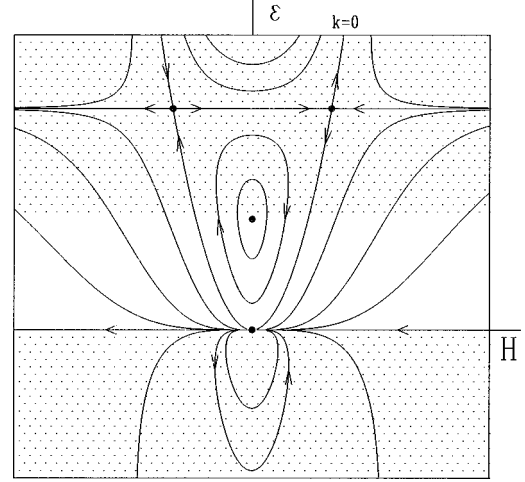


FIG. 1. Dynamics of semiclassical radiation filled universes.

well as the metrics' evolution in time.⁴ The stability of the Minkowski space-time is defined by the stability of the $(H, \epsilon) = (0, 0)$ point in the $\{H, \epsilon\}$ -phase space under the condition $\Lambda = 0$. For the equation of state (12) discussed in the preceding section the autonomous system (7),(8) reads

$$\frac{dH}{dt} = -H^2 - \frac{1}{3} \left[\epsilon - \frac{2}{3} \alpha_3 \hbar \epsilon^2 \right], \quad (13)$$

$$\frac{d\epsilon}{dt} = -4H \left[\epsilon - \frac{1}{3} \alpha_3 \hbar \epsilon^2 \right], \quad (14)$$

and its trajectories form levels of the integral

$$H^2 = \frac{\epsilon}{3} - K \sqrt{\frac{\epsilon}{G \epsilon_0 a_0^4}} - \frac{\alpha_3 \hbar}{6} K \sqrt{\frac{\epsilon^3}{G \epsilon_0 a_0^4}}. \quad (15)$$

The phase portrait of the system (13),(14) is shown in Fig. 1. For completeness and also for readers convenience, we attach Fig. 2 showing classical Friedmannian dynamics in the same representation. The phase structure of classical radiation-filled universes and the phase structure defined by Eqs. (13),(14) are topologically equivalent in the low-energy limit. This is so because one cannot enrich the structure of the $\{H, \epsilon\}$ -phase plane without violating the standard energy

⁴We abandon here a trivial freedom to multiply the flat universe metrics by a factor constant in time.

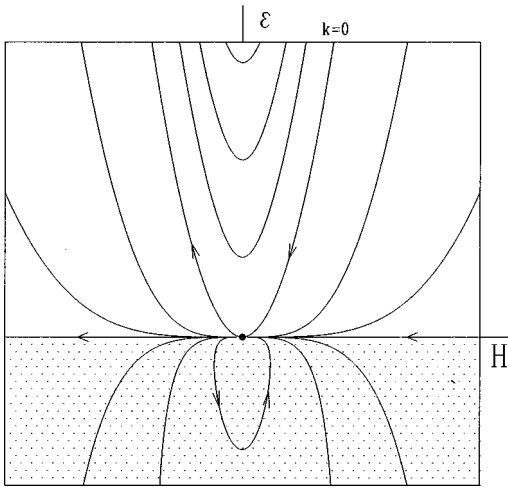


FIG. 2. Dynamics of Friedmanian (classical) radiation filled universes.

conditions. On the other hand, according to Eq. (12) these conditions are well fulfilled for low and positive energy densities. The equation of state (12) formally admits violation of the energy conditions but these states appear already in the Planckian regime and hence, far beyond the region where semiclassical approximation is valid. (The dotted region in the upper part of Fig. 1, which contains three additional critical points must be recognized as nonphysical.)

An essential property of the system (13),(14) is the absence of solutions that change the energy density from positive to negative or the reverse. (Such a behavior was possible in the original semiclassical theory and disqualified the empty space as a ground state.) Indeed, on the strength of Eq. (12), the initial condition $\epsilon=0$ results in $\epsilon+P=0$, and consequently the right-hand side of Eq. (14) vanishes. Both conditions $\epsilon=0$ and $d\epsilon/dt=0$ ensure that the state of the zero energy density is “persistent.” This is consistent with the results based on the functional integral formalism [13], where all higher derivative terms responsible for instability are eliminated by regularization of the energy-momentum tensor.

The stability of Minkowski spacetime is the same as in classical theory. In both cases, classical or the quantum, the $(H, \epsilon)=(0,0)$ point is a threefold point with elliptical sector [14] and its type does not depend on the value of \hbar . This means that the phase space is structurally stable against quantum corrections in the low-energy density limit. This

nontrivial property does not follow from the solutions analytical in \hbar , but from the form of the energy density tensor (1).⁵

V. SUMMARY AND CONCLUSIONS

In the reduced Simon-Parker theory the energy-momentum tensor is renormalized to take the hydrodynamic form with a simple, barotropic equation of state. The self-consistency conditions for semiclassical cosmology can be imposed on observable quantities and weakened. By demanding the Hubble expansion rate to be perturbative in \hbar we allow the space curvature to alter from 0 while quantum corrections to the flat universes occur.

In the particular case of the radiation-filled universe and vanishing cosmological constant, the dynamics of the Robertson-Walker universe in the (original) semiclassical theory is described by a second order equation, therefore it does not need either the reduction or additional conditions of self-consistence. The reason lies in the absence of particle creation in the radiation filled universe, which manifests itself as an additional factor \hbar “hidden” in the tensor ${}^{(1)}H_{\mu\nu}$. This eventually eliminates all the higher derivative terms.

Minkowski spacetime has the same stability character as Einsteinian gravity, which is consistent with results based on the functional integral formalism [13]. The stability of Minkowski spacetime is independent of the numerical value of the Planck constant. In the language of dynamical systems theory, this property is called the structural stability of the $\{H, \epsilon\}$ -phase space against changes of \hbar .

It’s worth noticing that the Liapunov stability of the environment with equation of state (12) with respect to position-dependent perturbations is also the same as for the classical radiation-filled universe [16], in contrast to the original semiclassical theory, where quantum corrections let inhomogeneities grow. This suggests an insignificant role for semiclassical corrections in the processes of structure formation in the early universe.

ACKNOWLEDGMENTS

We would like to acknowledge Professor Marek Demiański and Professor Lech Sokołowski for helpful discussions. This work was partially supported by Polish research Project No. KBN Nr 2 P03D 02210.

⁵In general, a threefold point may bifurcate into simple critical points under smooth changes of the equation coefficients. This is what occurs when cosmological constant appears. Solutions are analytical in Λ , though the critical point corresponding to empty space bifurcates into three simple points. Two of them represent de Sitter space time, the third one, the Einstein static universe [15]. However, no bifurcation, results from quantum corrections.

- [1] J. Z. Simon, Phys. Rev. D **43**, 3308 (1991).
 [2] L. Parker and J. Z. Simon, Phys. Rev. D **47**, 1339 (1993).
 [3] G. T. Horowitz and R. M. Wald, Phys. Rev. D **17**, 414 (1978); J. B. Hartle and G. T. Horowitz, *ibid.* **24**, 257 (1981); M. W. Suen, *ibid.* **40**, 315 (1989).

- [4] A. Starobinski, Phys. Lett. **91B**, 99 (1980).
 [5] F. D. Mazzitelli, Phys. Rev. D **45**, 2814 (1992).
 [6] J. Z. Simon, Phys. Rev. D **45**, 1953 (1992).
 [7] H. J. Schmidt, Phys. Rev. D **49**, 6354 (1994); A. B. Mayer and H. J. Schmidt, Class. Quantum Grav. **10**, 2441 (1993).

- [8] I. M. Aguirregabiria, A. Hernandez, and M. Rivas, *J. Phys. A* **19**, L651 (1997).
- [9] M. V. Fischetti, J. B. Hartle, and B. L. Hu, *Phys. Rev. D* **20**, 1757 (1979); L. H. Ford, *ibid.* **35**, 2955 (1987); B. L. Hu, *Phys. Lett.* **90A**, 375 (1982).
- [10] G. Siemienieć-Oziębło and A. Woszczyńska, *Acta Cosmologica*, **XXIII**, 1 (1997).
- [11] L. Parker, *Phys. Rev. Lett.* **50**, 1009 (1983).
- [12] V. A. Belinsky and I. M. Khalatnikov, *Sov. Phys. JETP* **42**, 205 (1975); **45**, 1 (1977).
- [13] P. O. Mazur and E. Mottola, *Nucl. Phys.* **B341**, 187 (1990).
- [14] A. A. Andronov, E. A. Leontovich, I. I. Gordon, and A.G. Maier, *Qualitative Theory of Second Order Dynamic Systems* (Wiley, New York, 1973).
- [15] A. Woszczyńska, *Gen. Relativ. Gravit.* **19**, 233 (1987).
- [16] G. Siemienieć-Oziębło and A. Woszczyńska, in *Proceedings of The Eight Marcel Grossmann Meeting on General Relativity*, Jerusalem, 1997.