Gravitationally induced interference of gravitational waves by a rotating massive object

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We discuss an interesting effect induced by the rotation of a massive object acting as a lens for coherent gravitational radiation. We show that the result is a concentric interference pattern which is shifted due to the effect of the angular momentum on the phase of radiation, an effect analogous to the Aharonov-Bohm effect. The possibility of detecting lensed gravitational waves is discussed in the context of upcoming gravitational wave detectors. [S0556-2821(99)03406-2]

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I. INTRODUCTION

One of the predictions of Einstein's general theory of relativity and one of its early tests concerned light deflection by the Sun. A deflecting mass represents the simplest gravitational lens configuration [1] in what has given rise to a whole research field in theoretical and observational astrophysics [2]. Another important prediction of Einstein's theory is the existence of gravitational waves which emanate from massive objects in nonspherical motion [3].

In this paper we want to discuss gravitational lensing of gravitational waves. The treatment of such a phenomenon demands the use of wave optics rather than geometric optics for the following reasons. Gravitational waves interact only very weakly with matter so that their coherence is preserved over cosmological distances, and in many cases gravitational waves are nearly monochromatic. These properties of gravitational waves are relevant in gravitational lensing for the situation when source, lens and observer are nearly collinear. Since infinitely many rays with almost equal path length intersect at the observer, diffraction and interference should invalidate geometrical optics. This is not the case for sources of electromagnetic radiation: rays originating from points not on but near the optic axis reach the observer at separate times which can exceed the coherence time. In the case of gravitational lensing of gravitational waves, however, the use of wave optics is not only justified but necessary. Along the whole focal line the intensity is amplified by the factor $8\pi^2 m/\lambda$ [4], m being the deflector's mass and λ the radiation's wavelength.

In the case of gravitational lensing by a rotating massive object, thanks to the coherence of gravitational waves, we discover a behavior which has so far been discovered only at the atomic level in two occasions. One is the interference of electron-waves passing either side of a magnetic solenoid, the Aharonov-Bohm effect [5]; the other is the interference of neutron-waves affected by the Corriolis force in the earth's gravitational field, the Colella-Overhauser-Werner (COW) experiment [6]. The phase shift in the Aharonov-Bohm effect is given by the line integral of the vector potential $\int d\vec{l}\cdot\vec{A}$. Similarly one expects that the phase shift of gravitational waves is given by the line integral of the gravitomagnetic potential $\vec{A}=2(\vec{J}\times\vec{r})/r^3$, with $\vec{J}$ being the angular momentum of a rotating massive object:

$$\int d\vec{l}\cdot\vec{A}=4(\vec{\xi}/\xi^2)\cdot(\vec{n}\times\vec{J}), \tag{1}$$

where $\vec{\xi}$ is the impact parameter vector, and $\vec{n}$ is the normalized vector pointing from the source to the lens. We show in Sec. III that this is really the case for the weak field metric.

Gravitational lensing by a Kerr black hole has been discussed previously [7–9], but none of these computations gave a description of the interference of gravitational waves. Moreover the value used for the Kerr parameter in [9] is overly extreme. In Sec. II we derive the Fresnel-Kirchhoff diffraction formula for the linearized Kerr metric. We use the Fresnel-Kirchhoff diffraction formula adapted to the linearized Kerr metric to compute the interference/diffraction function. In Sec. IV we discuss the possibility of detecting lensed gravitational waves in the context of the upcoming Laser Interferometric Gravitational Wave Observatory (LIGO), and in Sec. V we summarize our results. The treatment of the effect of the background curvature on gravitational wave propagation in the short wave approximation is relegated to the appendix. We use the convention $c=G=1$.

II. THE FRESNEL-KIRCHHOFF DIFFRACTION FORMULA

A. Preliminary considerations

Interference corresponds to the sum of two or more waves yielding a resultant intensity that deviates from the sum of the component intensities. The intensity of gravitational waves peaks along the focal line which stretches along the source-lens axis behind the gravitational lens. Similarly to Young's double slit experiment, the interference of coherent waves results from the gravitational deflection by an angle $\delta_0=4m/b$, where b is the impact parameter, of the two separate null geodesics which correspond to the double image in geometrical optics. According to the lens equation Ref. [2] Sec. 2.1 the slit width (or distance between two images) is of order of the Einstein radius defined as

$$\xi_0 \equiv \sqrt{4m\frac{ds}{d+s}}, \tag{2}$$

where d and s are the distances from the observer to the lens and the source, respectively. We assume that $d,s \gg m$, i.e. observer and source are located far away from the lens. In this situation each component of the gravitational waves can be treated as a scalar wave since the rotation of the polarization is in general too small to affect the interference pattern. In fact the rotation angle of the polarization plane of a wave lensed by a Kerr black hole is derived in Ref. [8] as $\chi = \frac{5}{4}\pi m^2 a \cos\Theta/b^3$, with Θ being the angle between the angular momentum vector of the deflector and the line of sight, and a being the Kerr parameter. Given the impact parameter $b \approx \sqrt{4md}$ and the Kerr parameter $a \leqslant m$, this angle is vanishingly small. Therefore we can neglect the polarization tensor e_{ab} in our derivation below.

B. The diffraction formula

We write the gravitational wave tensor as $h_{ab} = \mathcal{A} e_{ab} e^{iS} = \mathcal{E} e_{ab}$ (see Appendix A), and treat $\mathcal{E}$ as a scalar wave satisfying the propagation equation

$$\Box \mathcal{E} = \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\mu}\left(g^{\mu\nu}\sqrt{-g}\frac{\partial}{\partial x^\nu}\right)\mathcal{E} = 0. \tag{3}$$

We consider a lens object whose size is much smaller than the Einstein radius ξ_0, Eq. (2), so we can specialize to a linearized Kerr metric

$$ds^2 = -(1-2m/r)dt^2 + (1+2m/r)dr^2 - \frac{4am}{r}\sin^2\tilde{\theta} d\tilde{\phi} dt + r^2 d\tilde{\theta}^2 + r^2 \sin^2\tilde{\theta} d\tilde{\phi}^2. \tag{4}$$

For monochromatic waves of frequency ω, i.e., $\mathcal{E} \propto e^{-i\omega t}$, Eq. (3) becomes

$$\Delta\mathcal{E} + i\omega \frac{4am}{r^3}\frac{\partial \mathcal{E}}{\partial \tilde{\phi}} + (1+2m/r)\omega^2 \mathcal{E} = 0, \tag{5}$$

where Δ is the spatial Laplacian. Though we do not take account of the cosmological expansion in the metric, Eq. (4), the formulas derived below apply also to cosmological situations (considered in Sec. IV B) when the distances d,s and $d+s$ are replaced by their corresponding ''angular size distances'' [10], because we assume that the wavelength is much shorter than the horizon scale. We show in Appendix B that the Kirchhoff integral theorem is applicable to Eq. (5), so the wave amplitude at the observer is written as

$$\mathcal{E}_{\rm obs} = -\frac{1}{4\pi}\oint_{\Sigma_1}[U_2^* \vec{\nabla}\mathcal{E} - \mathcal{E}(\vec{\nabla}U_2)^*]\cdot d\vec{S}, \tag{6}$$

where U_2 is a solution of Eq. (5) which behaves like an incoming spherical waves $\to e^{-i\omega r_2}/r_2$ when the distance from the observer $r_2 \to 0$, and Σ_1 is a large closed surface containing the observer (Fig. 3).

The disturbance leaving the source S has the form of an outgoing spherical wave $\mathcal{E} \to A_0 e^{i\omega r_1}/r_1$ as $r_1 \to 0$, where A_0 is a constant and r_1 is the distance from the source. The Huygens-Fresnel principle is directly traceable to this integral resulting from the scalar wave equation. The gravitational wave originating at S propagates to the surface Σ_1 and from there the superposition of secondary waves sums up to an amplitude at the observer. The surface Σ_1 can be chosen arbitrarily, but to suit our purposes we use a finite plane perpendicular to the line of sight. To close the surface, we attach a sphere from which the contribution to the integral will be negligible because we can blow this spherical portion up to infinity, so that at the time when the disturbance at the observer is considered no contributions from this spherical surface could have reached O. We therefore have to take into account only contributions from the (lens) plane Σ (see [11]).

We shall use the eikonal approximation for the solutions $\mathcal{E}$ and U_2 (Appendix A) assuming that the wavelength is much shorter than the curvature radius $(m/\xi_0^3)^{-1/2}$ around the lens object,

$$\mathcal{E} = A_0\, e^{iS_1}/r_1 \quad \text{on the source side of the plane } \Sigma, \tag{7}$$

$$\text{and } U_2 = e^{-iS_2}/r_2 \quad \text{on the observer side of the plane } \Sigma, \tag{8}$$

where S_1 and S_2 are, using $\omega r \gg 1$, regular solutions of the eikonal equation, Eq. (A9), in their respective domains. From Eqs. (6)–(8) we obtain the Fresnel-Kirchhoff diffraction formula:

$$\mathcal{E}_{\rm obs} = \frac{A_0}{4\pi i}\oint_{\Sigma}\frac{e^{i(S_1+S_2)}}{r_1 r_2}(\vec{\nabla}S_1 - \vec{\nabla}S_2)\cdot \vec{n}\, dS, \tag{9}$$

where $d\vec{S} = \vec{n} dS$, $\vec{n}$ being the unit normal vector of Σ pointing towards the observer.

In astronomical situations the distances r_1 and r_2 are much larger than the effective scale of the plane of integration which is of the order of the Einstein radius because of the stationary phase approximation (Sec. III B). We can therefore replace them by constant distances, i.e. r_1 by the distance from the source to the lens, s, and r_2 by the distance from the lens to the observer, d, and consequently write them in front of the integrand. In the lowest order approximation, $S_1 = \omega r_1$ and $S_2 = \omega r_2$, so that $\vec{n}\cdot\vec{\nabla}S_1 = (\partial S_1/\partial r_1)\cos(\vec{n},\vec{r_1}) \approx \omega$ and $\vec{n}\cdot\vec{\nabla}S_2 = (\partial S_2/\partial r_2)\cos(\vec{n},\vec{r_2}) \approx -\omega$, because the angles involved are small. Therefore, from Eq. (9) we obtain

$$\mathcal{E}_{\rm obs} = \frac{\omega A_0}{2\pi i s d}\int_\Sigma d^2\xi\, e^{i(S_1+S_2)}, \tag{10}$$

where ξ is the two dimensional coordinates in the lens plane Σ and dS is approximated by $d^2\xi$. In the following section we will evaluate the eikonals S_1 and S_2.

III. INTERFERENCE AND DIFFRACTION OF GRAVITATIONALLY LENSED GRAVITATIONAL WAVES

A. Evaluation of the eikonals S_1 and S_2

With the metric Eq. (4), the eikonal equation, Eq. (A9), is written as

$$\omega^2(1+2m/r)-\omega\frac{4am}{r^3}\frac{\partial S}{\partial\tilde{\phi}}-(1-2m/r)\left(\frac{\partial S}{\partial r}\right)^2 -\frac{1}{r^2}\left(\frac{\partial S}{\partial\tilde{\theta}}\right)^2-\frac{1}{r^2\sin^2\tilde{\theta}}\left(\frac{\partial S}{\partial\tilde{\phi}}\right)^2=0, \qquad (11)$$

(S in Appendix A corresponds to $S-\omega t$ here). We solve Eq. (11) perturbatively by setting $S=S_0+\epsilon$, where S_0 is the eikonal in the Euclidean space satisfying $\delta_{ij}(\partial S_0/\partial x_i)(\partial S_0/\partial x_j)=\omega^2$, while the correction term ϵ is linear in the mass m and the angular momentum, $\vec{J}=\vec{a}m$, of the lens object.

Let us first consider the case of perfect collinearity among source, lens and observer. Assuming regularity of S_1 and S_2 in their domain, the solutions of Eq. (11) are approximately, i.e., up to linear order in m and a, given respectively by

$$S_1=\omega\left[s+\frac{r^2}{2s}-r\cos\theta+m\left\{2\ln\left(\frac{1}{r}-\frac{\cos\theta}{s}\right) -2\ln(1+\cos\theta)+\cos\theta-\frac{2\vec{r}}{r^2}\cdot\frac{\vec{n}\times\vec{a}}{\vec{n}\cdot\hat{r}-1}\right\}\right], \qquad (12)$$

where $\theta\in[0,\pi/2]$, and

$$S_2=\omega\left[d+\frac{r^2}{2d}+r\cos\theta+m\left\{2\ln\left(\frac{1}{r}+\frac{\cos\theta}{d}\right) -2\ln(1-\cos\theta)-\cos\theta+\frac{2\vec{r}}{r^2}\cdot\frac{\vec{n}\times\vec{a}}{\vec{n}\cdot\hat{r}+1}\right\}\right], \qquad (13)$$

where $\theta\in[\pi/2,\pi],\theta$ being the polar angle measured from the z-axis which points from the lens to the source. In the eikonal approximation, e^{iS_1}/r_1 and e^{-iS_2}/r_2 give respectively an outgoing wave from the source, which is valid in the source side, and an incoming wave to the observer, which is valid in the observer side.

In general, we expect a displacement $\vec{\eta}$ of the source parallel to the lens plane from the alignment observer-deflector (see Fig. 1). We have

$$S_1'=\omega\left[s+\frac{|\vec{r}-\vec{\eta}|^2}{2s}-\left|\vec{r}-\vec{\eta}\right|\cos\theta +m\left\{2\ln\left(\frac{1}{r}-\frac{\cos\theta}{s}\right)-2\ln(1+\cos\theta) +\cos\theta-\frac{2\vec{r}}{r^2}\cdot\frac{\vec{n}\times\vec{a}}{\vec{n}\cdot\hat{r}-1}\right\}\right]. \qquad (14)$$

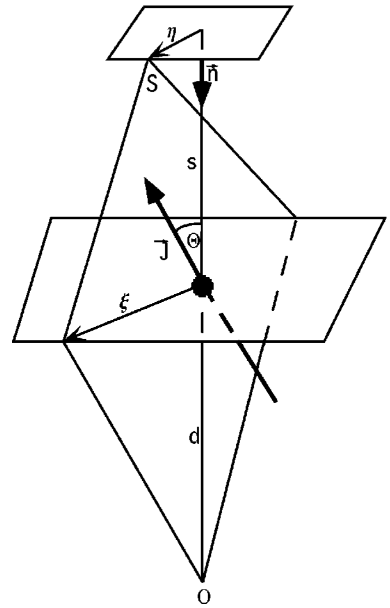

FIG. 1. Geometry in case of nonalignment.

We use the eikonal relative to the unlensed case

$$S=S_1'+S_2-\omega|(d+s)\vec{n}-\vec{\eta}|, \qquad (15)$$

to evaluate the Fresnel-Kirchhoff diffraction formula, Eq. (10), since we are not interested in the constant phase factor of $\mathcal{E}$. Furthermore, we do not need the full information of the waves, Eq. (12) and Eq. (13), but only the value at the lens plane where $\theta=\pi/2$ and $\vec{n}\cdot\hat{r}=0$. The value of S at the lens plane is

$$S=\omega\left[\frac{|(d+s)\boldsymbol{\xi}-d\,\boldsymbol{\eta}|^2}{2sd(d+s)}+4m\ln\left(\frac{1}{\xi}\right)+4m\frac{\boldsymbol{\xi}}{\xi^2}\cdot(\vec{n}\times\vec{a})\right]. \qquad (16)$$

Here we have replaced $\vec{r}$ by $\boldsymbol{\xi}$ defined in the lens plane, and hereafter 2-dimensional vectors in the plane perpendicular to the line of sight are expressed by boldface symbols. The first term is determined by the (Euclidean) geometry of the lens configuration, the second term is due to the influence of the Newton potential, while the third term is a result of the Kerr rotation.

Next, we rescale the coordinates to make them dimensionless according to [2]:

$$\boldsymbol{x}=\frac{\boldsymbol{\xi}}{\xi_0},\ \ \boldsymbol{y}=\frac{d}{d+s}\frac{\boldsymbol{\eta}}{\xi_0}, \qquad (17)$$

where ξ_0 is the Einstein radius defined in Eq. (2). The eikonal in terms of the dimensionless variables $\boldsymbol{x}$ and $\boldsymbol{y}$ is

$$S=4m\omega\left[\frac{1}{2}|\boldsymbol{x}-\boldsymbol{y}|^2-\ln x+\frac{\boldsymbol{x}}{x^2}\cdot\left(\vec{n}\times\frac{\vec{a}}{\xi_0}\right)\right]. \qquad (18)$$

The eikonal S is sometimes called a Fermat potential, the stationarity of S with respect to $\boldsymbol{x}$ yielding classical paths of geometrical optics.

B. Wave amplitude and image positions as stationary points of the eikonal

We evaluate the Fresnel-Kirchhoff formula Eq. (10) using the eikonal obtained in the previous subsection. We define

$$f=4m\omega,\quad \boldsymbol{\alpha}=\vec{n}\times\frac{\vec{a}}{\xi_0}=\vec{n}\times\frac{\vec{J}}{m\xi_0},\tag{19}$$

$$T(\boldsymbol{x},\boldsymbol{y})=\frac{1}{2}|\boldsymbol{x}-\boldsymbol{y}|^2-\ln x+\frac{\boldsymbol{x}}{x^2}\cdot\boldsymbol{\alpha}.\tag{20}$$

Note that f represents the ratio of the gravitational radius of the lens to the wavelength. The vector $\boldsymbol{\alpha}$ points in the direction perpendicular to the projection of the angular momentum vector onto the lens plane, and its magnitude $|\boldsymbol{\alpha}|$ is of order $\sim\sqrt{m/d}$ which is much smaller than unity. From Eqs. (10) and (18) we see that the wave amplitude at the observer is

$$\mathcal{E}_{\rm obs}(\boldsymbol{y})=\frac{\mathcal{E}_0 f}{2\pi i}\int d^2\boldsymbol{x}\ \exp[ifT(\boldsymbol{x},\boldsymbol{y})],\tag{21}$$

where $\mathcal{E}_0=A_0/(d+s)$ is the wave amplitude for the case of no lensing. We note that Eq. (20) can be rewritten as

$$T(\boldsymbol{x},\boldsymbol{y})=\frac{1}{2}|\boldsymbol{x}-\boldsymbol{y}|^2-\ln|\boldsymbol{x}-\boldsymbol{\alpha}|,\tag{22}$$

to first order in α. Equation (22) suggests that we introduce new coordinates

$$\tilde{\boldsymbol{x}}:=\boldsymbol{x}-\boldsymbol{\alpha},\quad \tilde{\boldsymbol{y}}:=\boldsymbol{y}-\boldsymbol{\alpha}.\tag{23}$$

This coordinate transformations are just parallel translations by the constant vector $\boldsymbol{\alpha}$, and we can change the integral variable of Eq. (21) from $\boldsymbol{x}$ to $\tilde{\boldsymbol{x}}$:

$$\mathcal{E}_{\rm obs}(\tilde{\boldsymbol{y}})=\frac{\mathcal{E}_0 f}{2\pi i}\int d^2\tilde{\boldsymbol{x}}\ \exp[ifT(\tilde{\boldsymbol{x}},\tilde{\boldsymbol{y}})],\tag{24}$$

where

$$T(\tilde{\boldsymbol{x}},\tilde{\boldsymbol{y}})=\frac{1}{2}|\tilde{\boldsymbol{x}}-\tilde{\boldsymbol{y}}|^2-\ln\tilde{x}.\tag{25}$$

Therefore it is clear that the whole pattern is just shifted by the amount $\boldsymbol{\alpha}$.

As we show in Sec. IV.A, we consider the case where $f\gg 1$, i.e., when the wavelength is much shorter than the gravitational radius of the lens. In this case we can evaluate Eq. (24) using the stationary phase approximation. The phase function $T(\tilde{\boldsymbol{x}})$ in Eq. (25) has two stationary points $\tilde{\boldsymbol{x}}_\pm$ as solutions of $\tilde{\nabla}T(\tilde{\boldsymbol{x}})=0,(\tilde{\nabla}:=\partial/\partial\tilde{\boldsymbol{x}})$. $\tilde{\boldsymbol{x}}_+$ is a minimum, and $\tilde{\boldsymbol{x}}_-$ is a saddle point. Actually these stationary points correspond to the images in the geometric optics approximation, and the condition $\tilde{\nabla}T(\tilde{\boldsymbol{x}})=0$ describes Fermat's principle of least time [2]. Expanding $T(\tilde{\boldsymbol{x}})$ around the stationary points and performing the Gauss integrals, the resulting wave amplitude is given as the superposition of contributions from the two stationary points:

$$\mathcal{E}_{\rm obs}=\mathcal{E}_0\sum_{j=\pm}|\mu(\tilde{\boldsymbol{x}}_j)|^{1/2}\ \exp[ifT(\tilde{\boldsymbol{x}}_j)-i\pi n_j/2],\tag{26}$$

where $n_+=0$ and $n_-=1$ correspond respectively to the minimum and to the saddle point, and

$$\mu(\tilde{\boldsymbol{x}}):=\det[\tilde{\nabla}\otimes\tilde{\nabla}T(\tilde{\boldsymbol{x}})]^{-1}.\tag{27}$$

When $\tilde{y}\lesssim 1/f$ the stationary phase approximation is not valid anymore. This case is treated separately below by evaluating the diffraction integral Eq. (24) exactly.

Since the phase function $T(\tilde{\boldsymbol{x}})$ in Eq. (25) has exactly the same form as that for the non-rotating lens, we can use the familiar formulas for the Schwarzschild lens [2] with $\boldsymbol{x}$ and $\boldsymbol{y}$ replaced by $\tilde{\boldsymbol{x}}$ and $\tilde{\boldsymbol{y}}$. The two stationary points $\tilde{\boldsymbol{x}}_\pm$ are

$$\tilde{\boldsymbol{x}}_\pm=\frac{\tilde{\boldsymbol{y}}}{2\tilde{y}}(\tilde{y}\pm\sqrt{\tilde{y}^2+4}).\tag{28}$$

The curvatures Eq. (27) at these stationary points are

$$\mu_\pm(\tilde{y})=\frac{1}{2}\pm\frac{\tilde{y}^2+2}{2\tilde{y}\sqrt{\tilde{y}^2+4}}.\tag{29}$$

The phase difference between them is

$$\Delta T(\tilde{y})=T(\tilde{\boldsymbol{x}}_-)-T(\tilde{\boldsymbol{x}}_+)\tag{30}$$

$$=\frac{\tilde{y}}{2}\sqrt{\tilde{y}^2+4}+\ln\frac{\sqrt{\tilde{y}^2+4}+\tilde{y}}{\sqrt{\tilde{y}^2+4}-\tilde{y}}.\tag{31}$$

Thus, the total wave intensity is magnified by the factor

$$|\mathcal{E}_{\rm obs}/\mathcal{E}_0|^2=|\mu_+|+|\mu_-|+2|\mu_+\mu_-|^{1/2}\ \sin(f\Delta T)\tag{32}$$

$$=\frac{\tilde{y}^2+2+2\sin(f\Delta T)}{\tilde{y}\sqrt{\tilde{y}^2+4}}.\tag{33}$$

The third term of this equation expresses the interference of waves reaching from the two images. By exchanging the roles of source and observer, i.e., d and s, we expect by reciprocity an interference pattern to appear on the observer plane. This interference pattern is circular and has the same shape as for the case of a non-rotating lens. The effect of the lens spin is just to shift the whole pattern translationally by the small vector $\boldsymbol{\alpha}$. In physical units, this shift on the observer plane is

$$\frac{d+s}{s}a\sin\Theta, \tag{34}$$

where Θ is the angle between the angular momentum vector of the lens and the line of sight.

Let us estimate the width of these interference fringes near the caustic where $1/f \lesssim y \ll 1$. From Eq. (33) the condition for constructive interference is $f\,\Delta T(\tilde{y}) = \pi(2n+\frac{1}{2})$ with the integer n. Since $\Delta T(\tilde{y}) \simeq 2\tilde{y}$ for $\tilde{y} \ll 1$, the distance between adjacent fringes near the center $\tilde{y}=0$ of the interference pattern is given by $\Delta\tilde{y} = \pi/f$, or by

$$\frac{\pi}{\omega}\sqrt{\frac{d(d+s)}{4ms}}, \tag{35}$$

in physical units on the observer plane.

Since the stationary phase approximation breaks down for $\tilde{y} \lesssim 1/f$, diffraction effects become important at the center of the interference pattern, where the wave amplitude has its maximum. Performing the diffraction integral Eq. (24) with $\tilde{y}=0$ yields

$$\mathcal{E}_{\rm obs} = -i\mathcal{E}_0 f\int_0^\infty d\tilde{x}\tilde{x}^{1-if}e^{if\tilde{x}^2/2} \tag{36}$$

$$= \mathcal{E}_0(f/2)^{if/2}e^{\pi f/4}\Gamma(1-\tfrac{1}{2}if), \tag{37}$$

and the wave intensity at the center of the interference pattern is

$$|\mathcal{E}_{\rm obs}/\mathcal{E}_0|^2 = \frac{\pi f}{1-e^{-\pi f}} \simeq 4\pi m\omega, \tag{38}$$

since $f \gg 1$. This is exactly the same as the maximum magnification for the case of a Schwarzschild lens [4].

We conclude that (1) the whole interference pattern is shifted by $\boldsymbol{\alpha}$, which can be understood as a dragging of the gravitational wave by the rotation of the massive object, and (2) all the other features of the circular diffraction/interference pattern are the same as in the case of a nonrotating lens. The analogy to the Aharonov-Bohm effect should now be apparent, the quantity $\boldsymbol{\alpha}$ proportional to the angular momentum of the lens object having a similar effect on the interference of gravitational waves as the magnetic flux has on the interference of electron waves.

IV. NUMERICS

A. Orders of magnitude — the case of our galactic center

In the case of large distances of the source from the lens, we can set $s\to\infty$. In this case the shift will be almost exactly $a\sin\Theta$, and the distance between fringes near the focal line (caustic) will depend only on the wavelength of the gravitational radiation λ and the distance to the lens d. Let us take as a typical wavelength of gravitational radiation $\lambda = 3\times10^8$ cm, which corresponds to a frequency of 1 Hz, and for $d = 8$ kpc$=2.4\times10^{23}$ cm, i.e., the distance to our center of galaxy, taking $m = 10^8 M_\odot$, which corresponds to a geometrical mass of 3×10^{13} cm, i.e., the mass of the black hole located at the center of our galaxy. For these values the Einstein radius (impact parameter) is $\xi_0 = \sqrt{4md} \approx 5\times10^{18}$ cm; we can easily check that $\alpha = (a/\xi_0)\sin\,\Theta$ is less than 6×10^{-6}, i.e., a very small number, the fact we used in the approximation which has lead us to Eq. (22). In this case we have the following distance between fringes located near the caustic:

$$\frac{\lambda}{4}\sqrt{\frac{d}{m}} \approx 0.4 \text{ AU}. \tag{39}$$

The interference pattern we expect is therefore huge. If we assume $a = 0.1m$ and $\Theta = \pi/2$, the pattern will be shifted by

$$a = 3\times10^{12} \text{ cm} \approx 0.2 \text{ AU}. \tag{40}$$

The maximum magnification is given by

$$\mu = 8\pi^2\frac{m}{\lambda} \approx 8\times10^6. \tag{41}$$

This gigantic magnification motivates a brief discussion, given in the next section, on the possibility to detect gravitational waves lensed not only by the center of our own galaxy but by the centers of other galaxies as well.

B. Detection rate estimate — lenses dispersed in the universe

In this section we estimate quantitatively the detection rate of lensed gravitational waves following the method in Ref. [13]. We consider a situation in which waves from coalescing neutron star binaries are lensed by intervening massive objects (e.g., galaxies) whose gravitational radius is larger than wavelength so that the geometric optic approximation is valid [12]. Lensing would increase the detection rate since faint signals from some distant sources become detectable because of the amplification of the wave intensity. Figure 2 plots the number of detection events per year for the LIGO-type detector versus the minimum detector noise $h_{\rm min}$ (defined to be $\sqrt{f_0 S_0}$ in the notation of Ref. [14]). Solid, dashed and dotted curves show respectively the event rate without gravitational lensing, the increase of event rate due to the lensing amplification and the rate of those lensed events in which the two images are both detectable. The three curves correspond to cases in which the cosmological parameters are $(\Omega_0,\lambda_0)=(1,0),(0.2,0)$ and $(0.2,0.8)$ from bottom to top. In the figure the threshold of detection is set to be S/N>5, the density parameter contributed from lens objects is 0.004 (this is the observed value of the density parameter of galaxies [10]), the coalescence rate density at present epoch is 7.4×10^{-9} Mpc^{-3}yr^{-1} [15], the maximum redshift of sources is 5, and $H_0 = 70$ km s^{-1}Mpc^{-1}. Since the value of $h_{\rm min}$ is 4.2×10^{-24} for the advanced LIGO superinterferometer [16], we conclude from the figure that the detection of gravitationally lensed gravitational waves (more than once per year) may be possible by lowering the noise by

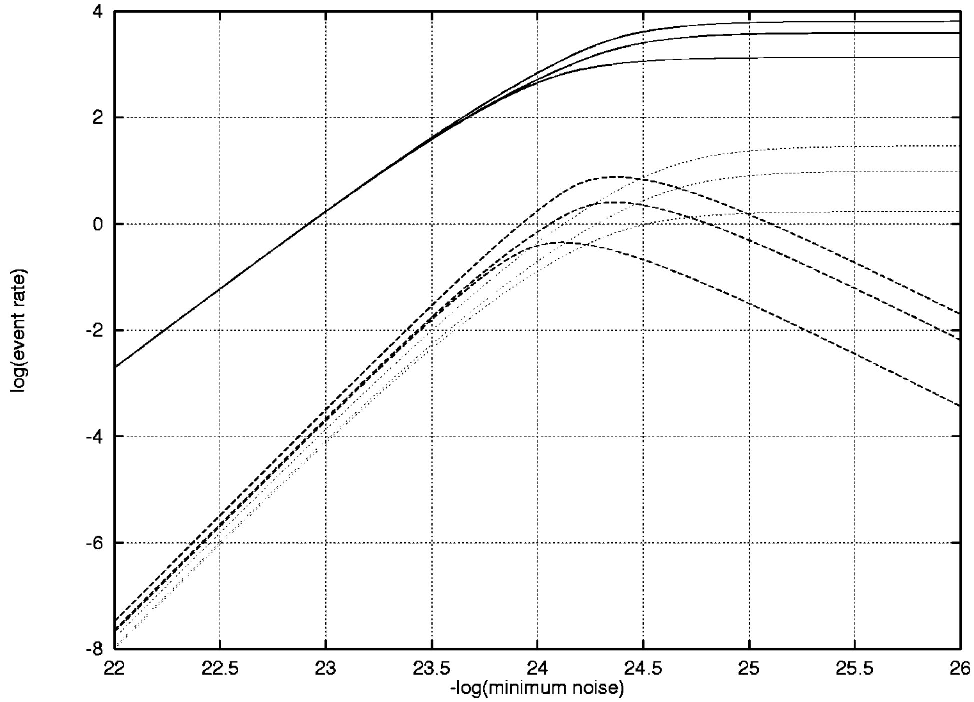

FIG. 2. The number of detection events per year for the LIGO-type detector versus the minimum detector noise $h_{\rm min}$. Solid: event rate without lensing magnification; dashed: increase of event rate due to lensing magnification; dotted: lensed event rate with both two images detectable. Three curves correspond to different cosmological models $(\Omega_0,\lambda_0)=(1,0),(0.2,0)$ and $(0.2,0.8)$ from bottom to top. See text for the assumed values of parameters.

a factor of several beyond the level of advanced interferometers if our universe is dominated by the cosmological constant.

V. SUMMARY AND DISCUSSION

We pointed out that in astrophysical situation there exists a parallel phenomenon with the Aharonov-Bohm effect or the COW experiment in quantum mechanics, namely, gravitational lensing of gravitational waves by a spinning massive object. The main result of our paper is summarized as follows. Due to the high coherence of gravitational waves, an interference pattern is formed near the caustic on the observer plane. This pattern is circular whether the lens is spinning or not, and the effect of lens spin is to shift the pattern translationally by the amount $a\sin\Theta$ in the observer plane, where a is the Kerr parameter and Θ is the angle between the lens angular momentum vector and the line of sight. This shift occurs in the direction perpendicular to the angular momentum vector projected onto the lens plane, i.e. the direction of dragging, leaving all the other properties of the diffraction-interference pattern unchanged. We conclude that the gravitomagnetic field around a Kerr black hole affects not only the path of the gravitational wave [7,9] but causes also a phase shift in parallel with the magnetic field or the Corriolis force field in the above mentioned quantum mechanical effects, resulting in a shift of the interference pattern formed near the caustic.

It should be pointed out, however, that in the Aharonov-Bohm effect no force acts upon the electron waves and we have a pure shift of the interference pattern based on the phase shift resulting from a change in magnetic flux inside the solenoid. On the other hand, the COW experiment is essentially the same as the effect we described, the Corriolis force acting upon the neutron-waves, causing the interference pattern to shift, just as the gravitational dragging acts upon the gravitational waves.

In real astrophysical situations it is unlikely that we can measure the Kerr parameter of a lensing object from the above shift of the interference pattern for the following reasons. (1) As we found above, except for this shift the overall interference pattern does not depend on the Kerr parameter. Unlike for the case in atomic physics, we cannot prepare a large screen on the observer plane. In other words the sole observable effect would be the misalignment of the lens and the two images, i.e., the stationary points. These are on one line with the lens when the lens is not spinning. But the angle of this misalignment ~ (Kerr parameter)/(distance to lens) is tiny and currently irresolvable. (2) This tiny misalignment may well be due to other reasons such as the external shear from nearby objects surrounding the lens. (3) As shown in Sec. IVB detection of lensed waves would also be rare for the laser interferometer detectors under construction such as the advanced LIGO [17].

Einstein wrote in his seminal paper on the gravitational lensing effect of electromagnetic radiation [1] that ''there is no hope of observing this phenomenon directly.'' Gravitational lensing of electromagnetic radiation, however, has been observed on several occasions in the meantime. Due to the weak interaction and therefore high coherence of gravitational waves, interference of gravitational waves by lensing certainly occurs in our universe even if difficult to detect.

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APPENDIX A: GRAVITATIONAL WAVE PROPAGATION IN THE EIKONAL APPROXIMATION

Let us recall the effect of the background on gravitational wave propagation [3]. As long as the amplitude of the waves $|h_{ab}|\ll 1$, where h_{ab} obeys the linearized field equations and represents a weak gravitational wave, the vacuum propagation equation is

$$h_{ab;c}{}^{;c}+2R^{B}_{cadb}h^{cd}=0, \tag{A1}$$

subject to the transverse traceless Lorentz gauge

$$h^{c}{}_{c}=0=h_{ab}{}^{;b}, \tag{A2}$$

where the covariant derivative is taken using the background metric form, and R^{B}_{cadb} is the background Riemann tensor.

If the wavelength λ is much smaller than the typical radius of curvature of the background than one expects wave behavior to go over to particle motion. Such waves appear, relative to the observers of interest, as nearly plane and monochromatic on a scale large compared with a typical wavelength, but very small compared with the typical radius of curvature of spacetime. Following the usual eikonal expansion of the phase, we have

$$h_{ab}=\mathcal{R}e\left\{e^{iS/\epsilon}\left(\mathcal{A}_{ab}+\frac{\epsilon}{i}\mathcal{B}_{ab}\right)+0(\epsilon^{2})\right\}, \tag{A3}$$

where ϵ is a dummy expansion parameter with eventual value unity which serves to identify orders of magnitude.

For an observer with proper time τ, world line $x^{a}(\tau)$ and 4-velocity $u^{a}=dx^{a}/d\tau$, the circular frequency ω and the wave vector k_{a} of the wave are defined as

$$\omega=-\frac{dS}{d\tau}=-S_{,a}u^{a}=k_{a}u^{a}. \tag{A4}$$

Furthermore the scalar amplitude is defined as

$$\mathcal{A}\equiv\left(\frac{1}{2}\mathcal{A}^{*}_{ab}\mathcal{A}^{ab}\right)^{1/2}, \tag{A5}$$

and the polarization tensor as

$$e_{ab}\equiv\frac{1}{\mathcal{A}}\mathcal{A}_{ab}. \tag{A6}$$

Then the equation of motion is

$$\begin{aligned}0=&-\frac{1}{\epsilon^{2}}\mathcal{A}_{ab}S_{;c}S^{;c}\\&+\frac{i}{\epsilon}(2\mathcal{A}_{ab;c}S^{;c}+\mathcal{A}_{ab}S_{;c}{}^{;c}+\mathcal{B}_{ab}S_{;c}S^{;c})\\&+2\mathcal{B}_{ab;c}S^{;c}+\mathcal{A}_{ab;c}{}^{;c}+\mathcal{B}_{ab}S_{;c}{}^{;c}+2R^{B}_{cadb}\mathcal{A}^{cd}\\&+\frac{\epsilon}{i}(\mathcal{B}_{ab;c}{}^{;c}+2R^{B}_{cadb}\mathcal{B}^{cd}).\end{aligned} \tag{A7}$$

If $\mathcal{A}_{ab}$ is assumed to vanish at most on hypersurfaces, the leading term says

$$k_{a}k^{a}=0, \tag{A8}$$

i.e., the wave vector $k^{a}=-g^{ab}S_{,b}$ is a null vector, and the phase must obey the eikonal equation

$$g^{ab}S_{,a}S_{,b}=0. \tag{A9}$$

This equation which determines the characteristic wavefronts is a generalization of the time-dependent eikonal equation of classical optics [11]. Moreover, since k_{a} is a gradient, i.e., $k_{a;b}=k_{b;a}$, it is straightforward to show that

$$k_{a;b}k^{b}=0, \tag{A10}$$

which means that k_{a} is tangent to an affinely parametrized geodesic.

The gauge condition reads

$$\frac{i}{\epsilon}\mathcal{A}_{ab}S^{;b}+\mathcal{A}_{ab}{}^{;b}+\mathcal{B}_{ab}S^{;b}+\frac{\epsilon}{i}\mathcal{B}_{ab}{}^{;b}=0. \tag{A11}$$

The leading term of the gauge condition says

$$e_{ab}k^{b}=0, \tag{A12}$$

which means that the polarization is orthogonal to the rays.

The ϵ^{-1} term of the equation of motion implies

$$2\mathcal{A}_{ab;c}k^{c}+\mathcal{A}_{ab}k_{c}{}^{;c}=0, \tag{A13}$$

which by contraction with $\mathcal{A}^{*ab}$ and using the definitions of the scalar amplitude and polarization tensor gives

$$2\mathcal{A}_{;c}k^{c}+\mathcal{A}k_{c}{}^{;c}=0, \tag{A14}$$

or equivalently

$$(\mathcal{A}^{2}k^{a})_{;a}=0, \tag{A15}$$

which means that the scalar amplitude decreases as the rays diverge from each other expressing the conservation of gravitons. Moreover, by definition of the polarization tensor, Eq. (A13) together with Eq. (A14) implies

$$e_{ab;c}k^{c}=0, \tag{A16}$$

meaning the polarization tensor is parallelly transported along these null geodesics.

APPENDIX B: THE KIRCHHOFF INTEGRAL THEOREM

We rewrite the propagation equation, Eq. (5), for monochromatic scalar waves U of frequency ω, in the case of a linearized Kerr metric as

$$\tilde{\Delta}U=-(1+2m/r)\omega^2 U, \tag{B1}$$

where $\tilde{\Delta}=\Delta+i\omega(4am/r^3)(\partial/\partial\tilde{\phi})$ and Δ is the spatial Laplacian. For two waves U_1 and U_2 which both satisfy Eq. (B1), the following volume integral vanishes:

$$0=\int_V[U_2^*\tilde{\Delta}U_1-U_1(\tilde{\Delta}U_2)^*]dV \tag{B2}$$

$$=\int_V[U_2^*\Delta U_1-U_1(\Delta U_2)^*]dV + 4iam\omega\int_V\frac{dV}{r^3}\frac{\partial}{\partial\tilde{\phi}}(U_1U_2^*). \tag{B3}$$

We take the integral volume V to be the one sandwiched between the two closed surfaces Σ_1 and Σ_2, both containing the observer (Fig. 3). For this V, the second term of Eq. (B3) becomes a two-dimensional integral on Σ_1 after performing the $d\tilde{\phi}$ integral. Then, this term is smaller than the first term by an order $\sim am/r_0^2$, where r_0 is the typical distance from the lens to the stationary points of the phase of $U_1U_2^*$ on Σ_1. For our choice of Σ_1 in Eq. (9), r_0 is of order of the Einstein radius ξ_0 [Eq. (2)]. Therefore the second term of Eq. (B3), of the order of $am/\xi_0^2 \ll 1$, can be safely neglected compared with the first term.

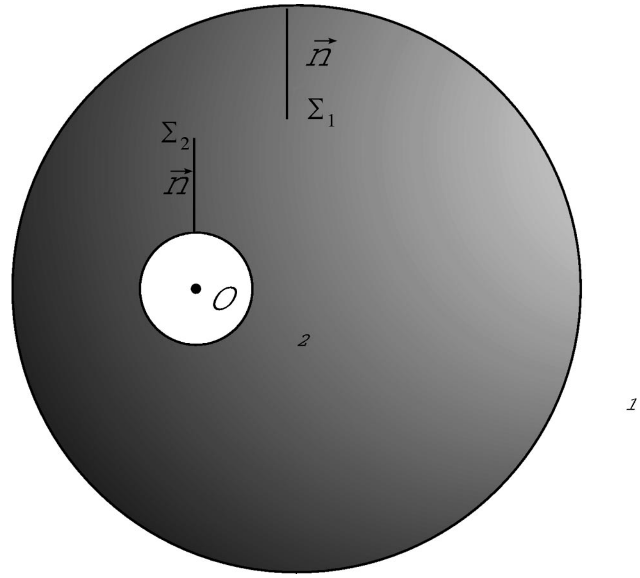

FIG. 3. A doubly connected region surrounding the observer O.

The rest of the derivation can be found in standard textbooks discussing wave optics [11]. Using Green's theorem, Eq. (B3) is rewritten as

$$\left(\oint_{\Sigma_1}+\oint_{\Sigma_2}\right)(U_2^*\vec{\nabla}U_1-U_1\vec{\nabla}U_2^*)\cdot d\vec{S}=0, \tag{B4}$$

where $d\vec{S}$ is surface-element normal vector pointing inward to the volume V. We take Σ_2 to be a sphere whose center is at the observer, and let its radius shrink to zero. Setting $U_2 = e^{-i\omega r_2}/r_2$ to be incoming spherical waves in flat space in the vicinity of the observer, the second integral of Eq. (B4) yields 4π times the value of U_1 at the observer. Thus Eq. (6), where we replaced U_1 with $\mathcal{E}$, has been justified.

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