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Entropy of 2D black holes from counting microstates

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We present a microscopical derivation of the entropy of the black hole solutions of the Jackiw-Teitelboim theory. We show that the asymptotic symmetry of two-dimensional (2D) anti–de Sitter space is generated by a central extension of the Virasoro algebra. Using a canonical realization of this symmetry and Cardy's formula we calculate the statistical entropy of 2D black holes, which turns out to agree, up to a factor $\sqrt{2}$, with the thermodynamical result. [S0556-2821(99)50406-2]

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The idea of asymptotic symmetry plays an important role in the recent developments in string theory and black hole physics. The anti–de Sitter– (ADS-)conformal field theory (CFT) correspondence [1] is just one example of how asymptotic symmetries can be used to bring in touch different theories in spacetimes of different dimensions. The conjectured equivalence between supergravity on D-dimensional ADS space and conformal field theory on the $(D-1)$-dimensional boundary is a very useful tool to gain information about the nonperturbative regime of gauge theories and to solve the problem of the microscopic interpretation of black hole entropy.

The previous ideas have found a nice application for $D=3$. It is well known since the work of Brown and Henneaux [2] that the asymptotic symmetry group of ADS_3 is the conformal group in two dimensions. Using this result Strominger has calculated the entropy of the three-dimensional (3D) Bañados-Teitelboim-Zanelli (BTZ) black hole by counting excitations of ADS_3 [3]. A nice feature of this microscopical derivation of the black hole entropy is that it does not use string theory or supersymmetry, but just general properties of 3D gravity. This fact makes the Strominger calculation of Ref. [3] more similar to that of Carlip [4] than to statistical derivations of black hole entropy that rely both on supersymmetry and string theory [5].

It looks very natural to try to apply the microstate counting procedure of Strominger to two-dimensional (2D) black hole solutions in ADS spacetime. The simplest 2D gravity theory that admits ADS space as solution is the Jackiw-Teitelboim (JT) model [6]. The JT model admits solutions that can be interpreted as 2D black holes in ADS space and that behave very similarly to their four- and three-dimensional cousins. One can associate with them a Hawking temperature and a thermodynamical entropy [7]. Moreover, at the semiclassical level takes place the evaporation process, whose Hawking radiation flux has been already calculated [7].

In this Rapid Communication we present a microscopical derivation of the entropy of the black hole solutions of the JT model. The approach used in Ref. [3] for the 3D case cannot be immediately extended to the 2D one. The obstruction is mainly due to the dimensionality of the $x\rightarrow\infty$ boundary of ADS_2, which makes both the mathematical treatment and the physical interpretation of the results highly nontrivial. For this reason we will present here only the main outcomes of our investigation. The details of the calculations and a thorough discussion of the physical meaning of our results will be published elsewhere.

We compute the entropy of the JT black hole by counting states on the one-dimensional, timelike, $x\rightarrow\infty$, boundary of ADS_2. To this end we first show how the $SL(2,R)$ isometry group of ADS_2 can be promoted to an asymptotic symmetry group on the boundary. This asymptotic symmetry group turns out to be generated by a central extension of the Virasoro algebra. Using a canonical realization of the asymptotic symmetry, we calculate the central charge c of the algebra. Applying Cardy's formula [8] for the asymptotic density of states, we calculate the statistical entropy of the JT black hole reproducing, up to a factor $\sqrt{2}$, the thermodynamical result.

The JT model is described by the action

$$A=\frac{1}{2}\int \sqrt{-g}\, d^2x\, \eta(R+2\lambda^2), \tag{1}$$

where λ is the 2D cosmological constant and η is a scalar field related to the usual definition of the dilaton ϕ by $\eta=\exp(-2\phi)$. The theory admits solutions that can be interpreted as 2D black holes in ADS space, which in a Schwarzschild gauge take the form [7]:

$$ds^2=-(\lambda^2x^2-a^2)dt^2+(\lambda^2x^2-a^2)^{-1}dx^2, \quad \eta=\eta_0\lambda x, \tag{2}$$

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where η_0 is an integration constant and a^2 is related to mass M of the black hole by

$$M=\frac{1}{2}\eta_0 a^2\lambda. \tag{3}$$

Two-dimensional dilaton gravity does not allow for a dimensionful analog of the Newton constant. However, it is evident from the action (1) that the inverse of the scalar field η represents the (coordinate-dependent) coupling constant of the theory, whereas the inverse of the integration constant η_0 plays the role of a dimensionless 2D Newton constant.

All the solutions (2) are locally anti–de Sitter, but have different global properties. In particular, we consider the $a=0$ solutions (which following the notation of Ref. [7] will be denoted by ADS^0) as the ground state of the model. ADS^0 is not geodesically complete and differs globally from full 2D ADS space [the $a^2=-1$ solution in Eq. (2)] [7]. A similar phenomenon occurs also for the 3D BTZ black hole solutions.

Using standard arguments one can easily calculate the thermodynamical parameters associated to the black hole (2). For the entropy S we have [7]

$$S=4\pi\sqrt{\frac{\eta_0 M}{2\lambda}}=2\pi\eta_h, \tag{4}$$

where η_h is the value of the scalar field at the horizon. In two spacetime dimensions we do not have an area law for the black hole entropy. However, the second equality in Eq. (4) can be interpreted as a generalization to 2D of the Bekenstein-Hawking entropy. This follows simply from the fact that according to Eq. (2), η is nothing but the ''radial'' coordinate of the 2D space.

The anti–de Sitter space is invariant under the $SO(1,2)\sim SL(2,R)$ group of isometries which, in the case of ADS^0, are generated by the three Killing vectors

$$^{(1)}\chi=\frac{1}{\lambda}\frac{\partial}{\partial t},$$

$$^{(2)}\chi=t\frac{\partial}{\partial t}-x\frac{\partial}{\partial x},$$

$$^{(3)}\chi=\lambda\left(t^2+\frac{1}{\lambda^4x^2}\right)\frac{\partial}{\partial t}-2\lambda tx\frac{\partial}{\partial x}. \tag{5}$$

The asymptotic symmetries are best investigated in the Hamiltonian formalism. With the parametrization

$$ds^2=-N^2dt^2+\sigma^2(dx+N^xdt)^2, \tag{6}$$

the Hamiltonian of the JT theory reads [9]

$$H=\int dx(N\mathcal{H}+N^x\mathcal{H}_x). \tag{7}$$

N and N^x act as usual as Lagrange multipliers enforcing the constraints,

$$\mathcal{H}=-\Pi_\eta\Pi_\sigma+\sigma^{-1}\eta''-\sigma^{-2}\sigma'\eta'-\lambda^2\sigma\eta=0, \tag{8}$$

$$\mathcal{H}_x=\Pi_\eta\eta'-\sigma\Pi'_\sigma=0,$$

where

$$\Pi_\eta=N^{-1}(-\dot{\sigma}+(N^x\sigma)'),\qquad \Pi_\sigma=N^{-1}(-\dot{\eta}+N^x\eta'), \tag{9}$$

are the momenta conjugate to η and σ, respectively. A dot denotes derivative with respect to t and a prime with respect to x.

In case of non-compact spacelike surfaces, however, it is well known that, in order to have well defined variational derivatives, one must add to the Hamiltonian a surface term δJ, which in general depends on the boundary conditions imposed on the fields [10]. In our case, the boundary reduces to a point and the variation δJ must be given by

$$\delta J=-\lim_{x\to\infty}[N(\sigma^{-1}\delta\eta'-\sigma^{-2}\eta'\delta\sigma)-N'(\sigma^{-1}\delta\eta)+N^x(\Pi_\eta\delta\eta-\sigma\delta\Pi_\sigma)]. \tag{10}$$

Using suitable boundary conditions, this can be written as a total variation at infinity of a functional J.

We have now to fix the boundary conditions at spatial infinity such that the metric behaves asymptotically as that of ADS_0 and to study under which transformations they are preserved. We require that, for $x\to\infty$

$$g_{tt}\sim-\lambda^2x^2+o(1),$$

$$g_{tx}\sim o\left(\frac{1}{x^3}\right),$$

$$g_{xx}\sim\frac{1}{\lambda^2x^2}+o\left(\frac{1}{x^4}\right). \tag{11}$$

Actually, in order to enforce anti–de Sitter behavior at infinity, one could choose milder asymptotic conditions. However, our stronger conditions are needed in order to have well-defined charges J. The asymptotic conditions (11) imply

$$\sigma\sim\frac{1}{\lambda x}+o\left(\frac{1}{x^3}\right),\quad N\sim\lambda x+o\left(\frac{1}{x}\right),\quad N^x\sim o\left(\frac{1}{x}\right). \tag{12}$$

Imposing that the asymptotic form (11) of the metric is conserved under the action of the Killing vectors χ^μ, one obtains that these must have the form

$$\chi^t = T(t) + \frac{1}{2\lambda^4}\frac{d^2T(t)}{dt^2}\frac{1}{x^2} + o\left(\frac{1}{x^4}\right),$$
$$\chi^x = -\frac{dT(t)}{dt}x + o\left(\frac{1}{x}\right), \tag{13}$$

where T is an arbitrary function of t. Diffeomorphisms with $T=0$ fall off rapidly as $x\to\infty$. They represent ''pure'' gauge transformations.

One still has to consider how the transformations (13) affect the dilaton. The variation of a scalar field η is given by $\mathcal{L}_\chi \eta = \chi^\mu \partial_\mu \eta$, which is asymptotically $o(x)$ for η of the form (2), and hence of the same order as the field itself. This is quite disturbing, but is an inescapable consequence of the scalar nature of the dilaton, and is also in accordance with the fact that η is defined up to the scale factor η_0 by the field equations. The previous considerations together with Eq. (9) permit us to fix the asymptotic behavior of the remaining canonical variables:

$$\eta \sim o(x), \qquad \Pi_\sigma \sim o(1), \qquad \Pi_\eta \sim o(x^{-4}). \tag{14}$$

We can now write down the algebra generated by the asymptotic symmetries (13). Since the anti–de Sitter space has a natural periodicity in t, it is convenient to expand the function $T(t)$ in a Fourier series in the interval $0<t<2\pi/\lambda$. The generators of the asymptotic symmetries read then

$$A_k = \frac{1}{\lambda}\left(1 - \frac{k^2}{2\lambda^2 x^2}\right)\cos(k\lambda t)\frac{\partial}{\partial t} + kx\sin(k\lambda t)\frac{\partial}{\partial x},$$
$$B_k = \frac{1}{\lambda}\left(1 - \frac{k^2}{2\lambda^2 x^2}\right)\sin(k\lambda t)\frac{\partial}{\partial t} - kx\cos(k\lambda t)\frac{\partial}{\partial x}, \tag{15}$$

where k is an integer. The generators satisfy the commutation relations

$$[A_k, A_l] = \frac{1}{2}(k-l)B_{k+l} + \frac{1}{2}(k+l)B_{k-l},$$
$$[B_k, B_l] = -\frac{1}{2}(k-l)B_{k+l} + \frac{1}{2}(k+l)B_{k-l}, \tag{16}$$
$$[A_k, B_l] = -\frac{1}{2}(k-l)A_{k+l} + \frac{1}{2}(k+l)A_{k-l}.$$

In the Hamiltonian formalism, the symmetries associated with the Killing vectors χ^μ are generated by the phase space functionals $H[\chi]$, defined as

$$H[\chi] = \int dx(\chi^\perp \mathcal{H} + \chi^\parallel \mathcal{H}_x) + J[\chi], \tag{17}$$

where $\chi^\perp = N\chi^t$, $\chi^\parallel = \chi^x + N^x\chi^t$, and the surface term $J[\chi]$ can be interpreted as the charge associated with the symmetry generator χ^μ. In view of the boundary conditions discussed above and adjusting the arbitrary constant so that J vanishes for ADS^0, the functional $J[\chi]$ can be written in finite form as

$$J[\chi] = \lim_{x\to\infty} \eta_0\Bigg[-(\lambda x)\chi^\perp(\eta' - \lambda) + (\lambda x)\frac{\partial \chi^\perp}{\partial r}(\eta - \lambda x) + \frac{\lambda^4 x^3}{2}\chi^\perp\left(g_{xx} - \frac{1}{\lambda^2 x^2}\right) + \frac{1}{\lambda x}\chi^\parallel \Pi_\sigma\Bigg]. \tag{18}$$

In general, the Poisson bracket algebra of $H[\chi]$ yields a projective representation of the asymptotic symmetry group [2]:

$$\{H[\chi], H[\omega]\} = H[[\chi,\omega]] + c(\chi,\omega), \tag{19}$$

where c is the central charge of the algebra. By enforcing the constraints $\mathcal{H}_\nu = 0$ the charges $J[\chi]$ give themselves a realization of the asymptotic symmetry group through the Dirac bracket, so that

$$\{J[\chi], J[\omega]\}_{DB} = J[[\chi,\omega]] + c(\chi,\omega). \tag{20}$$

In the case of three-dimensional anti–de Sitter space, the previous arguments give a simple way to calculate the central charge of the algebra [2]. One just needs to observe that the surface deformation algebra $[\chi,\omega]_{SD}$ is isomorphic to the algebra of the asymptotic symmetries and that the variation of $J[\chi]$ under surface deformations is given by the Dirac bracket,

$$\delta_\omega J[\chi] = J[[\chi,\omega]] + c(\chi,\omega). \tag{21}$$

By evaluating the previous equation for ADS^0, one finds that the central charge $c(\chi,\omega)$ is just given by the charge $J[\chi]$ evaluated on the surface deformed by ω.

In the case of 2D anti–de Sitter space, however, the previous calculation method cannot work, at least in the form described above. In fact, the boundary being a point, the functional derivatives appearing in the Poisson bracket (19) can be defined only for pure gauge transformations, for which the charge $J[\chi]$ vanishes. Moreover, the Dirac brackets (20) have no meaning as long as the $x\to\infty$ boundary is a point. As a consequence, the surface deformation algebra has no definite action on the charges $J[\chi]$, and Eq. (21) cannot be used to calculate the central charge.

The simplest way to cure the disease is to define the time-independent charges

$$\hat{J}[\chi] = \frac{\lambda}{2\pi}\int_0^{2\pi/\lambda} dt\, J[\chi]. \tag{22}$$

The functional derivatives of $\hat{J}[\chi]$ can be easily defined, so that the Dirac bracket algebra $\{\hat{J}[\chi], \hat{J}[\omega]\}_{DB}$ has now a meaning. One can also verify that the action of the surface deformation on the charges $\hat{J}[\chi]$ gives a realization of the algebra (16). Let us comment briefly on the physical meaning of the charges $\hat{J}$. Apart from $J[A_0]$, which gives the mass M of the solution, the other charges $J[A_k]$ are in general time-dependent. This means that besides the mass there are no conserved quantities. This fact is strongly related to the presence of the dilaton and its behavior under the transformations (13). On the other hand the charges $\hat{J}$ represent a sort of averaged charges that can be used to give a canonical representation of the algebra (16).

We can now easily calculate the central charges c. We just need to use in Eq. (21) the charges $\hat{J}$ instead of J. One gets

$$c(A_k,A_l)=c(B_k,B_l)=0, \quad c(A_k,B_l)=\eta_0 k^3 \delta_{|k|\,|l|}. \tag{23}$$

Defining new generators $L_k=-(B_k-iA_k)$, and shifting L_0 by a constant, one obtains the Virasoro algebra,

$$[L_k,L_l]=(k-l)L_{k+l}+\frac{c}{12}(k^3-k)\delta_{k+l}, \quad c=24\eta_0. \tag{24}$$

To calculate the entropy of a generic black hole solution of mass M in terms of states living on the boundary, we just need to use Cardy's formula for the asymptotic density of states:

$$S=2\pi\sqrt{\frac{c\, l_0}{6}}, \tag{25}$$

where l_0 is the eigenvalue of the Virasoro generator L_0, which for a black hole of mass M is given by

$$l_0=\frac{M}{\lambda}. \tag{26}$$

Inserting Eq. (26) and the value of the central charge c given by Eq. (24) into Eq. (25), we find, for the statistical entropy,

$$S=4\pi\sqrt{\frac{\eta_0 M}{\lambda}}, \tag{27}$$

which agrees, up to a factor $\sqrt{2}$, with the thermodynamical result (4). The lack of knowledge about the theory on the boundary renders difficult explaining this discrepancy between the statistical and the thermodynamical result. Nevertheless, a simple explanation of the factor $\sqrt{2}$ can be found if one considers the model (1) as a circular symmetric dimensional reduction of three-dimensional gravity, with the field η parametrizing the radius of the circle. Using the notation of Ref. [3], the 2D dilaton gravity action can be obtained from the 3D one by the ansatz

$$ds^2_{(3)}=ds^2_{(2)}+16G\eta^2 d\varphi^2, \tag{28}$$

where G is the 3D Newton constant and $0\leqslant\varphi\leqslant 2\pi$. In this context the 2D black hole (2) can be considered as the dimensional reduction of the $J=0$ (zero angular momentum) BTZ black hole. Simple calculations show that both the mass and the thermodynamical entropy of the BTZ black hole agree with our 2D results. The same is not true for the statistical entropy. From the 3D point of view we have contributions to the mass of the black hole coming from both the right- and left-movers oscillators of the 2D conformal field theory living on the boundary of ADS_3. Because $J=0$ implies that the number of right-movers equals that of left-movers, we have $l_0=M/2\lambda$, which inserted in the Cardy's formula reproduces the thermodynamical entropy (4). From the 2D point of view only oscillators of one sector contribute to the mass of the black hole giving $l_0=M/\lambda$ and the statistical entropy (27). These results are in accordance with those obtained by Strominger in a recent paper [12], where ADS_2 is generated as the near-horizon, near-extremal limit of ADS_3. At first sight this seems to imply that there is no intrinsically 2D explanation of the statistical entropy of 2D black holes. This is certainly true as long as the field η is interpreted as the radius of the internal circle, because the $x\to\infty$ boundary of ADS_2 corresponds to the region $\eta\to\infty$, where the space decompactifies and the 2D theory becomes intrinsically 3D.

The previous considerations do not apply when ADS_2 arises as near-horizon geometry of higher dimensional black holes with no intermediate ADS_3 geometry involved. We do not have a complete explanation of the factor $\sqrt{2}$ in this case. In our opinion what is needed in order to find an explanation of this discrepancy is a complete understanding of the role played in our derivation by the global topology of ADS_2. Full ADS_2 has a cylindrical topology with two disconnected timelike boundaries. This fact plays a crucial role in Ref. [12] because it makes the string theory on ADS_2 a theory of open strings. By studying the black hole solutions of the JT theory we are forced to cut the spacetime on the $x=0$ "singularity," so that only one timelike boundary of full ADS_2 is available. It seems to us that a thorough understanding of the statistical entropy of 2D black holes will be at hand only when this point will be fully clarified.

Our derivation of the statistical entropy of 2D black holes, though very simple and elegant, has the same drawbacks as the derivation of Strominger [3] (for a critical review see Ref. [11]). In particular the question remains open about the origin and the location of the relevant degrees of freedom on the boundary, whose number of excitations account for the entropy of the black hole. In our case, the nature of these degrees of freedom is even more mysterious than in the 3D case. Even though one has no explicit description of the degrees of freedom that are responsible for the entropy of the BTZ black hole, the underlying field theory is well known, being 2D conformal field theory with given central charge. For 2D black holes, instead, we know very little about the theory that should describe the excitations on the boundary. The one-dimensional nature of the latter implies that we are dealing with some kind of particle quantum mechanics, rather than quantum field theory. The quantum mechanical system, whose states span a representation of the Virasoro algebra (24), is most likely a very unconventional one. In this context the implementation of the ADS/CFT correspondence in the 2D case could help to shed light on the nature of this quantum mechanical system. On the other hand the fact that one can use particle quantum mechanics (even though in a still mysterious form) to explain the entropy of 2D black holes seems to us a very exciting possibility.

Note added. After this manuscript was completed we became aware of the existence of the paper of Ref [13], in which the asymptotic symmetries of 2D anti–de Sitter space are discussed.

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