Generalized Bertlmann-Martin inequalities for confining potentials

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The generalized Bertlmann-Martin inequalities for the moments of the ground state wave function of a two-body system are shown to be saturated within a few percent for confining potentials. Two kinds of potentials are investigated: the superposition of $1/r$ and r^p, and the anharmonic oscillator potentials. The application to $q\bar{q}$ hadrons is discussed. [S0556-2821(99)00409-9]

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I. INTRODUCTION

Studying the spectroscopy of heavy quark systems, about 20 years ago, Bertlmann and Martin proposed a number of inequalities valid for the case of a particle moving in a central potential, or a two-body system with a scalar interaction [1,2]. Among various inequalities, the one which retained our attention was derived from the Thomas-Reiche-Kuhn sum rule [3], and is linking the rms radius of the ground state wave function to the lowest dipole transition energy. In a previous paper [4], devoted to the spectroscopy of hypernuclei, we have generalized this result to higher multipole transitions in three dimensions. This provides us with a series of recurrent inequalities, which will be denoted generalized Bertlmann-Martin (GBM) inequalities. Each sum rule of multipolarity λ gives access to the 2λ-th moment of the ground state wave function $\langle r^{2\lambda}\rangle$.

It is interesting to note that an alternative derivation of the original Bertlmann-Martin inequalities has been proposed by Common, Martin and Stubbe [5]. It is directly based on properties of the Schrödinger equation proved by Common [6], whereas we are using sum rules. The goal of the present work being merely focused on practical applications, we simply refer the reader to the book by Grosse and Martin [7] for deeper studies.

For the harmonic oscillator potential, these relationships turn to equalities. We have further found, by analytical as well as numerical investigations, that for finite range potentials the degree of saturation is increasing with the binding energy of the ground state. As soon as the recurrent inequalities are saturated within 1–2 %, they can be used to construct the ground state wave function from its moments. In such a case the method could be applied to the inverse problem. The example of a Λ hyperon bound to a nuclear core was displayed in Ref. [4].

From these results, it is expected that for confining potentials the generalized recurrent inequalities should approach equalities to the desired level of accuracy, namely within few %. In fact, the answer is known for the infinite square well potential, for which $\langle r^2\rangle$ is given by the Bertlmann-Martin inequality to better than 1%. Consequently, physical systems described by confining potentials should constitute a privileged domain of application of this method.

The purpose of the present work is to check the degree of saturation of the GBM inequalities for a sample of confining potentials. An interesting feature is that the degree of saturation is increasing with the order of the moment $n=2\lambda$. Consequently, only the lowest order has to be calculated; in practice $n=10$ is a sufficient limit. The results are displayed in Sec. II after recalling shortly the inequalities. Considering physical situations, we discuss their application to the $q\bar{q}$ systems in Sec. III. Conclusions are drawn in Sec. IV.

II. RESULTS FOR CONFINING POTENTIALS

To generalize the Bertlmann-Martin inequality to higher multipoles, we consider the operator

$$Q_{\lambda,\nu}(\vec{r})=r^{\lambda}Y_{\lambda,\nu}(\theta,\phi)\quad . \tag{1}$$

For a particle of mass m moving in a central potential, the commutation relations give

$$[H,Q_{\lambda,\nu}]=-\frac{\hbar^2}{2m}(\vec{\nabla}Q_{\lambda,\nu})\cdot\vec{\nabla}-\frac{\hbar^2}{2m}\triangle Q_{\lambda,\nu}. \tag{2}$$

For the particular choice of $Q_{\lambda,\nu}$ (1), the second term of the right hand side vanishes. Assuming spherical symmetry, setting $\nu=0$, it is easy to get from the double commutator $[Q_{\lambda,0},[H,Q_{\lambda,0}]]$ the sum rule linear in energy ($\lambda\geqslant 1$)

$$\sum_j\,(E_j(\lambda)-E_0)|\langle 0|Q_{\lambda,0}|j,\lambda\rangle|^2=\frac{\hbar^2}{2m}\lambda(2\lambda+1)\langle r^{2\lambda-2}\rangle. \tag{3}$$

Here E_0 is the ground state energy and $E_j(\lambda)$ is the energy of the jth level of multipolarity λ. The sum is running over all states of the same multipolarity. Remember that the $Q_{\lambda,\mu}$ operators connect the ground state to natural parity states $\pi=(-)^{\lambda}$, only. Following the technique proposed by Bertlmann and Martin [1], the inequalities are obtained by minoring the sum. This is done by replacing all energy differences $E_j(\lambda)-E_0$ by the lowest value $E_1(\lambda)-E_0$, for each multipolarity. Then, by using the closure approximation, we get

$$\langle r^{2\lambda}\rangle \leq \frac{\hbar^2}{2m}\lambda(2\lambda+1)\frac{\langle r^{2\lambda-2}\rangle}{(E_1(\lambda)-E_0)} \equiv \langle r^{2\lambda}\rangle_{GBM},$$

$$\text{for } \lambda \geq 1. \tag{4}$$

These recurrent relationships emphasize the role played by the levels of the yrast line, i.e., the line joining the lowest energy states as a function of the multipolarity. The original Bertlmann-Martin inequality corresponds to $\lambda=1$. In a very similar way, from the monopole transition operator r^2 (which does not belong to the class of the $Q_{\lambda,\nu}$), we get the following relationship [4]:

$$\langle r^4\rangle \leq \frac{2\hbar^2}{m}\frac{\langle r^2\rangle}{(E_{2s}-E_0)} + \langle r^2\rangle^2 \equiv \langle r^4\rangle_M. \tag{5}$$

Consequently, we get the estimate of $\langle r^4\rangle$ from both Eqs. (4) and (5). Their ratio can be used as a consistency check.

Note that in the case of a two-body problem, the particle mass m in Eqs. (2)–(5) has to be replaced by the reduced mass μ.

It has to be noted that the above inequalities are particularly useful when they are close to become equalities, i.e., when they are saturated within a couple of percents. This situation occurs as soon as the lowest excited state $|1,\lambda\rangle$ carries a large fraction of the sum rule. The limiting case in this respect is given by the harmonic oscillator, for which the 1 phonon excitation exhausts the sum rule. On the other hand, the situation is much less promising for potentials with singularity at the origin. For instance, in the case of the Coulomb potential the expectation value $\langle r^2\rangle$ equals to only 3/4 of the GBM value [RHS of relation (4) for $\lambda=1$]. For such potentials, Bertlmann and Martin have modified the RHS of the inequalities by introducing an additional multiplicative factor. Modified relation (4) for $\lambda=1$ acquires the form

$$\langle r^2\rangle \leq \frac{\hbar^2}{2m(E_{1p}-E_0)}\left(1-\frac{1}{4}C\right), \tag{6}$$

where

$$C=\left(\frac{E_{2s}+E_{1s}-2E_{1p}}{E_{2s}-E_{1s}}\right)^2 . \tag{7}$$

Factor C has been constructed by fitting the Coulomb case while simultaneously respecting the extreme situation of the harmonic oscillator (HO), i.e., C=1 and 0 for Coulomb and HO, respectively. Numerous tests with power potentials of the form r^α for $-1<\alpha<+\infty$, and superpositions of the form $-A(1/r)+Br, A,B>0$ revealed that estimate (6) is valid to within a few per mille. We therefore believe it is justified to use this correction factor also in our calculations of generalized GBM inequalities. We will refer to corrected GBM inequalities whenever the factor C is applied.

It is clear that this approach is applicable to the two-body problem, replacing m by the reduced mass. The extension to two fermion systems requires the handling of the spin degrees of freedom. They enter the Hamiltonian either in the spin-orbit coupling or in the hyperfine ($\vec{\sigma}_1\cdot\vec{\sigma}_2$) interaction, with different consequences.

Suppose the spin-orbit interaction small enough to leave the sum rules unchanged, the strength will be distributed among the two partners according to their degeneracy. Consequently the lowest energy difference $E_1(\lambda)-E_0$ has to be replaced by the weighted average

$$\frac{1}{2(2\lambda+1)}\sum_j (2j+1)(E_1(j)-E_0)$$

with $j=\lambda\pm1/2$. Note that the same kind of average has to be introduced in the case when the degeneracy of the λ states is removed by nonspherical interaction.

The situation is somehow different with the hyperfine term. First of all the singlet and triplet states behave like two independent sets of eigenvalues. Thus the generalized GBM inequalities can be used separately to get the moments of both ground states (S = 0 and S = 1). Whereas the structure of the singlet states is simple, the excited triplet states occur also as triplets having $\vec{J}=\vec{L}+\vec{S}$. In this case it is again necessary to introduce a weighted average for the energy differences.

In order to get a quantitative estimate of the virtues of the relationships (4) and (5) for confining potentials, we have investigated two classes. The first class corresponds to potentials constructed for $q\bar{q}$ systems that can be expressed in a form

$$V(r)=-\frac{\alpha}{r}+\beta r^p+\Lambda+\frac{\gamma}{m_i m_j}\frac{\exp(-r/r_0)}{r r_0^2}\vec{\sigma}_1\cdot\vec{\sigma}_2. \tag{8}$$

A characteristic example of this class is the linear potential ($p=1$) of Bhaduri [8] with $\alpha=0.52$, $\beta=0.186$ GeV^2, $\gamma=\alpha$, $\Lambda=-0.9135$ GeV, $r_0=2.305$ GeV^{-1}, $m_u=m_d=337$ MeV, $m_s=600$ MeV, $m_c=1.870$ GeV, and $m_b=5.259$ GeV. First, we studied potentials with $\gamma=0$ (the hyperfine interaction was switched off) and kept the other parameters unchanged. For the sake of comparison we took $p=1/2$, 1 and 2. The results of calculations represented by the equivalent radii $R(n)=\langle r^n\rangle^{1/n}$ divided by the GBM values from the RHS of Eq. (4), $(\langle r^n\rangle/\langle r^n\rangle_{GBM})^{1/n}$, (denoted by a) and corrected GBM values (b) are presented in Table I. The corrected GBM values are clearly closer to the equivalent radii than the original ones. Moreover, the GBM predictions are improving with increasing p; for $p=2$ the relations (4) are becoming practically equalities. For completeness, the estimates of $\langle r^4\rangle_{GBM}$ from Eq. (4) divided by $\langle r^4\rangle_M$ from relation (5) are listed in the last row of the table. The results for the Bhaduri potential ($p=1$ and $\gamma=\alpha$) for singlet (S) and triplet (T) cases are presented in Table II.

As the next example we have chosen the quartic anharmonic oscillator

$$V(r)=\tfrac{1}{2}ar^2+br^4. \tag{9}$$

TABLE I. The equivalent radius R(n) divided by the GBM value $[(\langle r^n\rangle/\langle r^n\rangle_{GBM})^{1/n}]$ is listed up to n=10 in columns a. The confining potential is of the form $V(r)=V_0+\alpha/r+\beta r^p$ where $p=1/2,1,2$ (see text). In columns b, the corrected GBM sum rule values are used (see text). The ratios $\langle r^4\rangle_{GBM}/\langle r^4\rangle_M$ [see relations (4) and (5)] are listed in the last row for completeness.

	$p=1/2$		$p=1$		$p=2$	
	a	b	a	b	a	b
n=2	0.982	1.003	0.993	1.001	0.999	1.001
n=4	0.986	0.997	0.995	0.999	0.990	1.000
n=6	0.991	0.998	0.996	0.999	1.000	1.000
n=8	0.991	0.997	0.997	0.999	1.000	1.000
n=10	0.992	0.997	0.997	0.999	1.000	1.000
$\langle r^4\rangle_{GBM}/\langle r^4\rangle_M$	0.999		0.999		0.999	

This potential has been subject to numerous studies in the past thanks to its possible applications in the description of molecular vibrations and in modelling of nonlinear quantum field theories. In order to investigate the sensitivity of the GBM predictions to the strength of the quadratic and quartic terms in potential (9) we varied parameters a and b between −10 and 1, and 1 and 10, respectively. Four typical potential examples are drawn in Fig. 1. Their corresponding GBM values are displayed in Table III. The worst case is the one denoted AO3, which develops a pronounced minimum at finite distance. Note in particular that in this case the ratio of the two $\langle r^4\rangle$ estimates differ by 22%. This is exceptional among the investigated potentials.

III. APPLICATION TO $q\bar{q}$ SYSTEMS

It is very tempting to apply the considerations of the preceding section to meson spectra. Since the earlier days of the quark model, non-relativistic (or semi-relativistic) calculations of the hadron spectra have successfully reproduced the experimental levels (see for instance Ref. [9] for recent fits). Unfortunately, the electromagnetic form factors, which give access to the particle sizes, have been measured only for the lightest mesons, the π^+ and the K^+. This is limiting the scope of our analysis and requires some caveats.

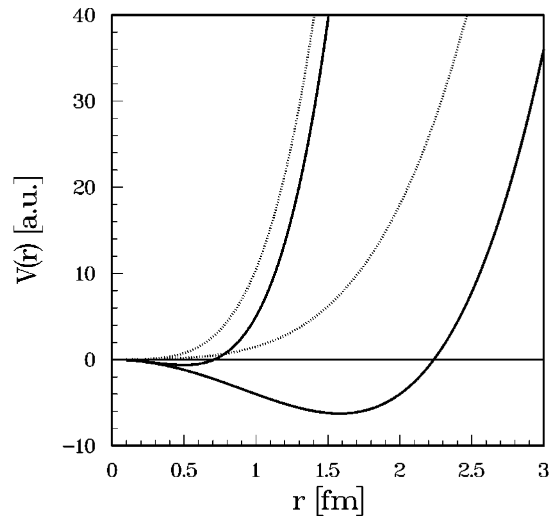

FIG. 1. Shape of the anharmonic potentials used to check the degree of saturation of the GBM inequalities (Table III). From the left to the right the 4 lines correspond to potentials AO2, AO4, AO1 and AO3, respectively.

The generalized inequalities (4) and (5) have been derived in the framework of non-relativistic quantum mechanics. They remain valid in the case of the Dirac equation. At the ''no-pair'' approximation, relativistic corrections leave the sum rules (3) unchanged, as recently shown by Romero and Aucar [10]. We have also verified that, up to minor corrections arising mainly from its non-locality, the solutions of the Dirac equation saturate the GBM inequalities equally well as the Schrödinger ones [11].

However, it is not clear at all that the ''no-pair'' approximation is sufficient to describe both the spectra and the form factors of mesons, a task which requires to go beyond relativistic quantum mechanics. It is precisely one of our goals to show explicitly the limits of potential models.

As far as the data are concerned, we rely on the last issue of the Review of Particle Physics [12]. The levels we are taking into account for the S=0 and S=1 states of I=1 $n\bar{n}$, I=1/2 $n\bar{s}$ and $n\bar{c}$ and I=0 $c\bar{c}$ are summarized in Table IV. Our analysis does not exhaust all $q\bar{q}$ systems; we concentrate

TABLE II. The equivalent radius R(n) divided by the GBM value $[(\langle r^n\rangle/\langle r^n\rangle_{GBM})^{1/n}]$ is listed up to n=10 for Bhaduri [8] singlet (S) and triplet (T) potential (see text). In columns a the original GBM sum rule was used whereas in columns b we applied the corrected sum rule values (see text). The ratios $\langle r^4\rangle_{\rm GBM}/\langle r^4\rangle_M$ [see relations (4) and (5)] are listed in the last row for completeness.

	S		T	
	a	b	a	b
n=2	0.947	1.003	0.997	0.999
n=4	0.973	1.001	0.998	0.999
n=6	0.983	1.003	0.998	0.999
n=8	0.991	1.005	0.998	0.998
n=10	0.990	1.001	0.998	0.998
$\langle r^4\rangle_{GBM}/\langle r^4\rangle_M$	0.929		0.992	

TABLE III. The generalized GBM inequalities for the anharmonic oscillator $V(r)=1/2\alpha r^2+\beta r^4$ and parameters $\alpha=1$, $\beta=1$ (AO1), $\alpha=1$, $\beta=10$ (AO2), $\alpha=-10$, $\beta=1$ (AO3), and $\alpha=-10$, $\beta=10$ (AO4). The ratios $\langle r^4\rangle_{GBM}/\langle r^4\rangle_M$ [see relations (4) and (5)] are listed in the last row for completeness.

	AO1	AO2	AO3	AO4
n=2	0.998	0.997	0.861	0.990
n=4	0.997	0.997	0.921	0.992
n=6	0.997	1.000	0.948	0.994
n=8	0.998	0.999	0.962	0.995
n=10	0.999	0.999	0.972	0.996
$\langle r^4\rangle_{GBM}/\langle r^4\rangle_M$	1.002	0.996	0.778	0.986

TABLE IV. Mesonic states used in the present analysis.

	$n\bar{n}$ I=1	$n\bar{s}$ I=1/2	$c\bar{n}$ I=1/2	$c\bar{c}$ I=0
$^1S_0$	π 139.	K 494	D 1869	η_c 2979
$^1P_1$	b_1 1231.	K_1 1273.	D_1 2423	h_c 3526
$^1D_2$	π_2 1670.	K_2 1580., 1773.		
$^1F_3$	a_3 2080.	K_3 2324.		
$^1G_4$		K_4 2490.		
2^1S_0	π 1300.±100.	K 1460.		η_c 3594.
$^3S_1$	ρ 770.	K^* 892.	D^* 2010.	J/Ψ 3097.
$^3P_0$	a_0 984.	K_0^* 1429.		χ_{c0} 3415.
$^3P_1$	a_1 1230.	K_1 1402.		χ_{c1} 3511.
$^3P_2$	a_2 1318.	K_2^* 1425.	D_2^* 2456.	χ_{c2} 3556.
$^3D_1$	ρ 1720.	K^* 1714.		Ψ 3770.
$^3D_2$		K_2 1816.		
$^3D_3$	ρ_3 1690.	K_3^* 1770.		
$^3F_2$		K_2^* 1975.		
$^3F_4$	a_4 2040.	K_4^* 2045.		
$^3G_5$	ρ_5 2330.	K_5^* 2382.		
2^3S_1	ρ 1460.	K^* 1412.		$\Psi(2s)$ 3686

our effort on cases having a sufficient number of clear spin assignations. Even so some uncertainties remain, and they are indicated in Table IV.

The GBM inequalities yield the moments of the ground state wave function in terms of the average of the relative distance between the two quarks $\langle r^n \rangle$. Charge form factors are measured from the center of mass. Taking into account the charges of the quarks (e_1, e_2), the moments of the charge radius $\langle r_c^n \rangle$ are given by

$$\langle r_c^n \rangle = \left[e_1 \frac{1}{(1+\varepsilon)^n} + e_2 \frac{\varepsilon^n}{(1+\varepsilon)^n} \right] \langle r^n \rangle. \qquad (10)$$

If m_1 and m_2 refer to the mass of the quark 1 and 2, respectively, with $m_1 \leqslant m_2, \varepsilon = m_1/m_2$.

Another difficulty arises from the values of the quark masses. Using constituent quark masses of 336 MeV for the u and the d quarks, and 600 MeV for the quark s, lead to charged radii an order of magnitude smaller than experiments. This is a well known defect of non-relativistic quark model. It is due to the fact that the lowest order diagram coupling the quark to the electromagnetic field accounts only for a minor contribution. The dressing of the quark does improve the situation but it is not sufficient to match the experiments [9].

On the other hand, unless they are provided us by independent measurements, the masses appear in GBM inequalities as free parameters. Consequently, adjusting the masses to reproduce the charge rms radius, for instance, gives a model capable of describing both the spectrum (at least the so called yrast line) and the charge form factor. Unfortunately, this scheme is lacking universality. To be explicit, if we stick to the π^+ and the K^+ cases, it is not possible to find m_n $(n=u,d)$ and m_s values fitting simultaneously both spectra and charge form factors.

This is a first obvious limitation of the potential models. It is confirmed by the very fact that to our knowledge no potential model has been able to reach such a solution yet.

The best we can afford at this stage is to fit the charge rms radius of the K^+ and search for the largest rms radius of the π. This procedure yields $m_n = 111.4$ MeV and $m_s = 222.8$ MeV, which is about twice lower than the dressed quark masses used by Semay and Silvestre-Brac [9]. It would be interesting to check how far these masses are compatible with heavier mesons. The lack of data or independent predictions for $\langle r_c^2 \rangle$ does not allow us to reach further conclusions.

To illustrate the situation, we display in Table V $\langle r_c^n \rangle^{1/n}$

TABLE V. GBM equivalent radii $R(n) = \langle r_c^n \rangle_{GBM}^{1/n}$ of the π and K charge radius for three different values of the quark masses m_s and m_u, the corresponding reduced mass μ are indicated. The experimental values are presented for comparison. In the case of the $K, n \geqslant 4$, two values are listed; the first line corresponds to $K_2(1580)$, the second line to $K_2(1773)$.

	m_s (MeV)	600	434	222.8	
	m_u/m_s	0.56	0.509	0.5	Expt. [13,14]
	μ (MeV)	215	146.4	74.27	
π	n=2	0.282	0.348	0.490	0.683±0.08
	n=4	0.350	0.432	0.608	0.90±0.01
	n=6	0.410	0.505	0.711	1.12±0.01
K	n=2	0.332	0.412	0.580	0.58±0.04
	n=4	0.430	0.537	0.758	0.79±0.05
		0.366	0.456	0.644	
	n=6	0.492	0.616	0.869	0.99±0.06
		0.418	0.524	0.739	
	n=8	0.558	0.699	0.987	1.18±0.07
		0.474	0.594	0.839	

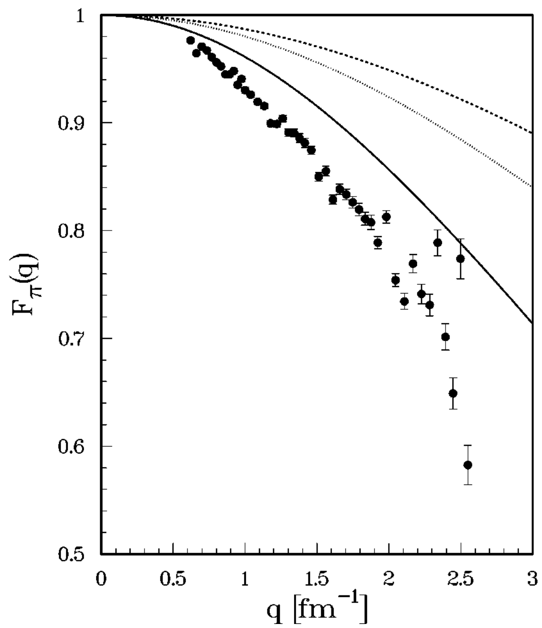

FIG. 2. Electromagnetic form factor of the π^+ calculated from the moments $\langle r_c^n \rangle^{1/n}$ listed in Table V for three different masses of the quarks (see Table V). The dashed line corresponds to usual constituent quark masses, the dotted line to dressed quarks and the solid line to adjusted masses, respectively (see text). The experimental values are taken from Ref. [13].

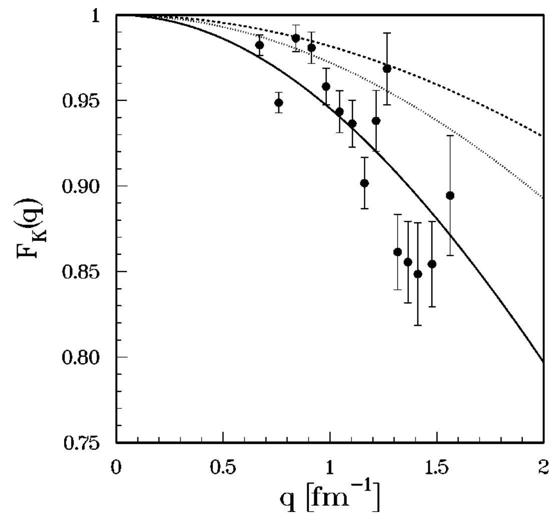

FIG. 3. Electromagnetic form factor of the K^+ calculated from the moments $\langle r_c^n \rangle^{1/n}$ listed in Table V for three different masses of the quarks (see Table V). The dashed line corresponds to usual constituent quark mases, the dotted line to dressed quarks and the solid line to adjusted masses, respectively (see text). The experimental values are taken from Ref. [14].

for the π and the K sectors for 3 sets of the masses. The first one corresponds to usual non-relativistic calculations, the second uses dressed quark masses form [9], and the third is adjusted as stated above. Comparison is made with the experimental values of Amendolia *et al.* [13,14]. Note that beyond the rms radius, experimentally the higher moments are obtained by a single pole fit, which is of limited validity. For this reason, we have plotted in Figs. 2 and 3 the form factors corresponding to the three set of masses and compared these quantities directly to the experimental form factors. In the K^+ case, we have verified that the uncertainties on the GBM moments due to the dubious assignment of the K_2 state (see Table IV and V) have practically no effect on the form factor up to $q \approx 1.6\ \mathrm{fm}^{-1}$. Similarly, the adjustment of the K^+ charge rms radius leads to a form factor (solid line in Fig. 3) which is barely distinguishable from the pole fit up to $q \approx 2.0\ \mathrm{fm}^{-1}$.

At least two quantities exist, which are independent of the quark masses, and are thus firm and critical predictions of potential models. The first one is the ratio of the $\langle r^4 \rangle$ values derived either from the quadrupole or the monopole excitation. From the results displayed in Tables I–III, this ratio is expected to be close to 1, possibly less than 1 in the case of an anharmonic potential with a minimum at a finite distance (see AO3). Values for the π, K, ρ, K^* and J/ψ are displayed in Table VI. They clearly indicate limitations of the usual potential model description of the ρ and K^*.

The next prediction independent of the quark masses is the ratio of the triplet to singlet values for $\langle r^n \rangle^{1/n}$. They are listed in Table VII for the ρ, K^*, J/ψ and D^*. As expected from any potential model these ratios are >1 and decreasing with n. It just reflects the repulsive interaction of the hyperfine potential at the origin in the triplet state, whereas at large distances the wave function is dictated by the confining part of the potential. In the absence of data allowing a direct check of these predictions, we note that for the ρ and the K^*, the mean square radii are twice the values of the π and the sum of the π and the K values, respectively. This is what can be naively expected from the fact that the ρ decays mostly into 2 π and the K^* into $K+\pi$. A contrario, from the analysis of the slope of the ρ photoproduction differential cross section, by using the vector dominance model together with the Glauber model, the size of the ρ is found roughly equal to that of the π. This is corroborated by the Pomeron exchange model [15]. In other words the question remains open since none of these reasonings can be taken with confidence.

IV. CONCLUSIONS

Investigating the generalized Bertlmann-Martin inequalities for confining potentials, we found a high degree of saturation. Consequently they constitute an efficient tool to be used in the framework of the inverse problem, reconstructing the ground state wave function and the potential from the yrast energy levels. The correction proposed for the rms radius by Bertlmann and Martin, based on the Coulomb potential, brings further improvements. In practice, however, to

TABLE VI. Ratio of $\langle r^4 \rangle$ obtained from the quadrupole and the monopole inequalities. This ratio is independent of the quark masses.

π	K	ρ	K^*	J/Ψ
$1.05\pm.05$	$1.15-.98$	$.90\pm.03$	$.87\pm.03$	1.07

TABLE VII. Ratio of the equivalent radii for triplet to singlet state $\langle r^n\rangle_{S=1}^{1/n}/\langle r^n\rangle_{S=0}^{1/n}$. These ratios are independent of the mass of the quarks.

n	ρ	K^*	J/Ψ	D^*
2	1.51	1.19	1.13	1.11
4	1.14	1.05–1.10	1.12	
6	1.07	1.08		
8		1.04		

the level of 1–2% it can be neglected.

For power-law potentials r^p, the degree of saturation is decreasing with decreasing slopes. Nevertheless for p=1/2 it is still better than 2% for the rms radius, which is the worst case. Anharmonic oscillators are close to the harmonic oscillator, unless they develop a profound well depth at finite distance (see case AO3).

As far as spin-dependent Hamiltonians are concerned, we have investigated the case of a hyperfine interaction added to a Coulomb and a linear term. Typically such a potential has been used by Bhaduri to calculate meson spectra. The hyperfine part is attractive in the singlet and repulsive in the triplet states. The two sets of states are totally independent. In fact they can be coupled only via tensor forces or nonsymmetric spin-orbit coupling. Both are expected to be negligible in the $q\bar{q}$ systems. It turns out the GBM inequalities are saturated within 5% and 2% for the singlet and triplet states, respectively. Note that in the singlet case, the correction to GBM brings the saturation of the rms radius inequality to .1%.

To our knowledge, the best confined system in nature is provided us by the hadrons, in particular by $q\bar{q}$ mesons. Describing their properties within potential models, i.e., by solving a Schrödinger or a Dirac equation, is really justified only for heavy quarks. Unfortunately, data related to the size are only available for the lightest mesons, the π^+ and the K^+. Since we do not know *a priori* if the GBM inequalities can be applied to such systems, the present study is merely a test showing the limitations of potential models. The advantage of our method is that it is independent of the shape of the potential. It does not require either complicated fitting procedure.

The study of the π^+ and K^+ form factors underlines the role played by the quark mass. Considering the masses as free parameters to be adjusted to the experimental rms charge radius, it is not possible to find a unique set reproducing π^+ and K^+ data simultaneously. This reduces considerably the predicting power of the method.

Two quantities are independent of the quark masses, and appear thus as particularly valuable tests. The ratios of $\langle r^n\rangle^{1/n}$ for S=1 and S=0 states constitute one set of them. Information on the rms radii of the ρ and K^* would be very interesting in this respect. On the other hand, the consistency check which is provided us by the ratio of the two estimates of $\langle r^4\rangle$ put question marks on the usual potential description of the ρ and the K^*.

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