# Ambiguities in the zero momentum limit of the thermal $\pi^0\gamma\gamma$ triangle diagram

François Gelis

*Laboratoire de Physique Théorique LAPTH, BP110, F-74941, Annecy le Vieux Cedex, France*
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Modifications of the $\pi^0\rightarrow 2\gamma$ decay amplitude by thermal effects have already been considered by several authors, leading to quite different results. I consider in this paper the triangle diagram connecting a neutral pion to two photons in a constituent quark model, within the real-time formulation of thermal field theory, and study the zero external momentum limit of this diagram. It appears that this limit is not unique and depends strongly on the kinematical configuration of the external particles. This nonuniqueness is shown to explain the contradiction between existing results. I end with some considerations suggesting that this decay amplitude may be significantly modified by the resummation of hard thermal loops, due to infrared singularities. [S0556-2821(99)04705-0]

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## I. INTRODUCTION

During the past two years, a lot of work has been devoted to the study of the relationship between the axial anomaly and the $\pi^0\rightarrow 2\gamma$ decay rate at finite temperature, most notably by Pisarski, Tytgat, and Trueman [1–6]. The purpose of this series of papers was to explain the following basic fact: the coefficient of the axial anomaly is independent of the temperature while the amplitude for the $\pi^0\rightarrow 2\gamma$ decay is modified. The problem was therefore to explain why the relationship that relates at zero temperature the pion decay amplitude to the axial anomaly ceased to be valid in a hot medium.

This work has been initiated by a calculation of the pion decay rate in a constituent quark model, performed by Pisarski in the imaginary time formalism [1,2]. More precisely, it consists in the calculation of the triangle diagram connecting the pseudoscalar to the two emitted photons, via a quark loop. This diagram is considered at the first nonvanishing order in the small external momenta. The result found in Ref. [1] is that this diagram is proportional to $m/T^2$ where $m$ is the mass of the quark in the loop and $T$ the temperature of the heat bath, while the result found at zero temperature for the same diagram is proportional to $1/m$. The consequence of this result is that the pion decay rate into two photons vanishes if the chiral symmetry is restored at high temperature, since $m\rightarrow 0$.

The same diagram has been calculated in the real time formalism by the authors of Refs. [7,8], and also by Gupta and Nayak (GN) in Ref. [9] who studied the zero momentum limit of this diagram. GN's result for this diagram in the zero external momentum limit is proportional to $m/mT$. The dramatic difference is the behavior of this decay amplitude as a function of the quark mass, because this behavior was crucial in Pisarski's calculation [1,2] to derive his conclusion about the pion decay rate in a hot chirally symmetric phase.

The purpose of the present paper is to reconsider the calculation of the triangle diagram already studied by Pisarski and Gupta and Nayak, in order to explain the discrepancy between the results they found. To that effect, we perform this calculation in the "retarded-advanced"[1] version of the real time formalism, but we stay at a more general level than Refs. [1,7–9] concerning the kinematical configuration of the external particles. In particular, we do not assume that external particles are on-shell. Similar to Refs. [1,9], we are interested in a small external momenta expansion for this diagram. We arrive at the conclusion that this discrepancy is due to the nonuniqueness of the zero momentum limit of the considered Green's function. It appears indeed that this limit depends on the kinematical configuration of the external legs and that Pisarski and GN's calculations correspond to very different configurations, GN's configuration being the most physical for the decay of a pion into real photons. Then, we come back to Pisarski's statement about the pion decay rate in a hot chiral phase and show that, because of infrared singularities, it may remain valid in GN's kinematical configuration (i.e., in the physical configuration) despite a different dependence in the mass $m$ if one considers the correction provided by hard thermal loops.

In Sec. II, we derive the expression for the triangle diagram in the retarded-advanced formalism, and its relationship with the pion decay rate. Then, we prove the existence of a limit of zero external momentum, in a sense to be made precise later.

In Sec. III, we first give an expression for the zero momentum limit showing clearly that this limit is not unique and depends on the kinematical configuration of the external particles. The remaining of this section is devoted to the detailed study of this limit in three particular configurations. The first configuration studied corresponds to a situation where both of the emitted photons have zero energy: the zero momentum limit reproduces in this case Pisarski's result. The second important case is obtained with real photons and a pion at rest in the frame of the plasma: this case reproduces GN's result. Finally, a third simple case corresponds to the

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[1] The choice of using the retarded-advanced formalism is just technical. The main point is to show that both Pisarski's and GN's results can be recovered from a common formalism.
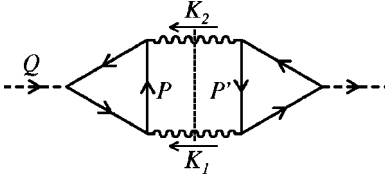
FIG. 1. Self-energy of the pseudoscalar involved in the decay in $2\gamma$.

decay of a static pion into two static virtual photons, that may subsequently decay into lepton pairs.

In Sec. IV, we study the implications of the above results for Pisarski's assertion concerning the annulation of the pion decay amplitude in a hot chiral phase. Despite the fact that this statement is incorrect in the bare theory if one considers the physical situation in which the photons are real, the interplay of infrared singularities in this calculation makes the resummation of hard thermal loops necessary. The consequence of this resummation is to change the parameter playing the role of an infrared regulator. This has the effect of making the pion decay amplitude vanish in a hot chirally symmetric phase, even when one is considering the physical decay into real photons.

Technical details are relegated to three appendixes. In Appendix A, we remind the reader of the potentially dangerous effect of changing the variables in divergent expressions since this is of some relevance for our calculation. Appendix B gives the general expression of the functions $A(K_1,K_2)$ and $B(K_1,K_2)$ that appear at intermediate stages of the calculations. Finally, Appendix C gives some details about a few integrals that appear in this paper.

## II. TRIANGLE DIAGRAM IN THE ''R/A'' FORMALISM

### A. $\pi^0$ decay rate

The decay rate of pions in a thermal bath is related to the $\pi^0\pi^0$ retarded self-energy via the relation [10,11]

$$\frac{dN}{dt\,d\boldsymbol{x}} = -\frac{dq_0\,d^3\boldsymbol{q}}{(2\pi)^4}2e^{q_0/T}n_B(q_0)\mathrm{Im}\,\Pi^{RA}(q_0,\boldsymbol{q}), \quad (1)$$

which gives the number of $\pi^0$ decays per unit time and per unit volume of the plasma, in the four momentum range $dq_0d^3\boldsymbol{q}$. This formula does not take into account the reverse process (see, for instance, Refs. [10,12]) $2\gamma\to\pi^0$ that can *a priori* occur in the medium. This is justified if one considers a quark-gluon plasma of small dimensions, like the one expected in heavy ion collisions. Indeed, if the size of the plasma is smaller than the mean free path of photons, the
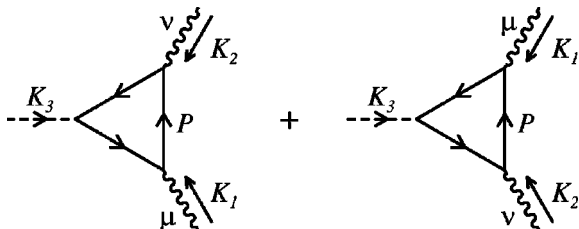
produced photons will escape from the plasma without thermalizing. The imaginary part of the $\pi^0\pi^0$ two-point function is a sum over all the possible cuts through the corresponding diagram, which means that this formula gives the total decay rate, i.e., the sum of the contribution of all the channels. In order to select a particular channel, one must look at the appropriate cut.

Similar to Refs. [1,9], I use a linear sigma model (see Ref. [13], for instance) where the fermion fields are constituent quarks, in which the mesons are coupled to quark fields as indicated by the following Lagrangian:[2]

$$\mathcal{L} = i\bar{\Psi}\slashed{D}\Psi - 2g\bar{\Psi}(\sigma t_0 + i\boldsymbol{\pi}\cdot\boldsymbol{t}\gamma^5)\Psi. \quad (2)$$

I consider two flavors of quarks and $N=3$ colors. The $t$ matrices are normalized with $t_0=1/2$ and $\mathrm{Tr}(t_at_b)=\delta_{ab}/2$. This coupling is invariant under the chiral symmetry $\mathrm{SU}(2)_L\times\mathrm{SU}(2)_R$. When this symmetry is spontaneously broken, the $\sigma$ field acquires a nonvanishing vacuum expectation value[3] $\langle\sigma\rangle$, which gives a mass $m=g\langle\sigma\rangle$ to the constituent quarks. In this model, the decay of pions in two photons appear only in the discontinuity of the three loop $\pi^0\pi^0$ self-energy. Indeed, each external pseudo-scalars must be connected to a quark loop, and these two loops must be linked by the two photons. Then, among all the possible cuts, one must consider the cut that crosses the photon propagators (see Fig. 1). Making use of the cutting rules for the ''R/A'' formalism [15], we find that the cut depicted in Fig. 1 contributes:

$$\mathrm{Im}\,\Pi^{RA}(q_0,\boldsymbol{q})$$

$$= -\frac{1}{2}\int\frac{d^4K_1}{(2\pi)^4}\int\frac{d^4K_2}{(2\pi)^4}2\pi\epsilon(k_1^0)\delta(K_1^2)2\pi\epsilon(k_2^0)\delta(K_2^2)$$

$$\times(2\pi)^4\delta(Q+K_1+K_2)\Gamma^{ARR}_{\mu\nu}(Q,K_1,K_2)$$

$$\times\Gamma^{RAA\,\mu\nu}(Q,K_1,K_2), \quad (3)$$

where $\Gamma^{ARR}_{\mu\nu}(Q,K_1,K_2)$ is the triangle diagram connecting the pseudoscalar to two photons. This object will be the subject of our study from now on ($\Gamma^{RAA}_{\mu\nu}$ is closely related to the previous one). In fact, two diagrams contribute to this one-loop three-point function because of the possibility of crossing the photons in the final state, as outlined in Fig. 2. In order to take the two configurations into account, it is suffi-

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[2] We can also include in this model an explicit chiral symmetry breaking term such as $\epsilon\sigma$, that induces a small mass $m_\pi$ for the pion. In this paper, we assume this mass to be very small. In fact, this mass is the scale of the external momenta in the small momentum expansion I will perform later.

[3] This vacuum expectation value can be identified with the pion decay constant $f_\pi$ for two flavors at zero temperature. At nonzero temperature, they differ somehow (see, for instance, Ref. [14]). Anyway, both of them vanish when the chiral symmetry is restored.



FIG. 2. One-loop triangle diagrams contributing to $\pi^0\to\gamma\gamma$.

cient to calculate in detail the first one, and then add the term obtained by interchanging the indices $(1,\mu)\leftrightarrow(2,\nu)$.

## B. Matrix element

Let us first give the value of the vertex function $\Gamma_{\mu\nu}^{ARR}$. Using the Feynman's rules established for the R/A formalism (see Refs. [16,17]), a straightforward calculation gives for one flavor of electric charge $e$

$$\Gamma_{\mu\nu}^{ARR}(K_3,K_1,K_2)$$

$$=4mNe^2g\,\epsilon_{\mu\nu\alpha\beta}k_1^\alpha k_2^\beta\int\frac{d^4P}{(2\pi)^4}\{n_F(p^0+k_2^0)$$

$$\times S^A(P-K_1)S^A(P)\mathrm{Disc}\,S^R(P+K_2)+n_F(p^0)$$

$$\times S^A(P-K_1)S^R(P+K_2)\mathrm{Disc}\,S^R(P)+n_F(p^0-k_1^0)$$

$$\times S^R(P)S^R(P+K_2)\mathrm{Disc}S^R(P-K_1)\}$$

$$+(K_1,\mu)\leftrightarrow(K_2,\nu), \tag{4}$$

where $m$ is the mass of the quark running in the loop[4] and $S^{R,A}(P)\equiv i/(P^2-m^2\pm ip^00^+)$ the scalar part of the retarded (advanced) quark propagator. $n_F$ is the Fermi-Dirac's distribution defined as $n_F(x)\equiv1/[\exp(x/T)+1]$. In the following, we can forget about the retarded or advanced labels for the denominators. Indeed, to recover the correct prescriptions, it is sufficient to perform at the very end of the calculation the substitutions

$$k_1^0\rightarrow k_1^0+i0^+,\quad k_2^0\rightarrow k_2^0+i0^+,\quad k_3^0\rightarrow k_3^0-2i0^+. \tag{5}$$

It is worth recalling that the discontinuity of the quark propagator generates a Dirac's delta function

$$\mathrm{Disc}\,S^R(P)=2\pi\epsilon(p^0)\delta(P^2-m^2) \tag{6}$$

which enables us to do easily one of the integrations. To be more definite, it is convenient to use these Dirac's functions to perform the integration over the variable $p^0$, so that we are left with a three-dimensional integration

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$$\Gamma_{\mu\nu}^{ARR}(K_3,K_1,K_2)$$

$$=4mNe^2g\,\epsilon_{\mu\nu\alpha\beta}k_1^\alpha k_2^\beta\int\frac{d^3\boldsymbol{p}}{(2\pi)^3}$$

$$\times\sum_{\epsilon=\pm}\left\{\frac{n_F(\omega_{\boldsymbol{p}+k_2})-\theta(-\epsilon)}{2\omega_{\boldsymbol{p}+k_2}}\right.$$

$$\times\frac{i}{2P\cdot K_2+K_2^2}\frac{i}{2P\cdot(K_1+K_2)+K_2^2-K_1^2}\Bigg|_{\substack{p^0+k_2^0=\\\epsilon\omega_{\boldsymbol{p}+k_2}}}$$

$$+\frac{n_F(\omega_{\boldsymbol{p}})-\theta(-\epsilon)}{2\omega_{\boldsymbol{p}}}\frac{i}{-2P\cdot K_1+K_1^2}\frac{i}{2P\cdot K_2+K_2^2}\Bigg|_{p^0=\epsilon\omega_{\boldsymbol{p}}}$$

$$+\frac{n_F(\omega_{\boldsymbol{p}-k_1})-\theta(-\epsilon)}{2\omega_{\boldsymbol{p}-k_1}}\frac{i}{2P\cdot K_1-K_1^2}$$

$$\times\frac{i}{2P\cdot(K_1+K_2)+K_2^2-K_1^2}\Bigg|_{\substack{p^0-k_1^0=\\\epsilon\omega_{\boldsymbol{p}-k_1}}}\Bigg\}+(K_1,\mu)\leftrightarrow(K_2,\nu), \tag{7}$$

where we denote $\omega_{\boldsymbol{p}}\equiv\sqrt{(\boldsymbol{p}^2+m^2)}$. This expression[5] of the vertex function will be the basis of further considerations.

## C. Existence of a zero external momenta limit

We are interested now in the zero momentum limit of this vertex function in order to understand the origin of the discrepancy between Pisarski's and GN's result. Only two of the three external momenta are independent ones due to the energy-momentum conservation: therefore we choose to consider $K_1$ and $K_2$ as independent momenta and replace everywhere[6] $K_3$ by $-K_1-K_2$. In order to take the zero momentum limit $K_1,K_2\rightarrow0$, we must assume that $K_1$ and $K_2$ are functions of some parameter $\lambda$, such that $K_1(\lambda=0)=K_2(\lambda=0)=0$. This amounts to consider the limit along a path that ends at the origin $K_{1,2}=0$ in momentum space. We will see later that this limit does not exist if we consider the most general paths. In fact, the limit exists only if the path admits a tangent at the origin. This amounts to writing in the vicinity of the origin

$$K_1\equiv\lambda\hat{K}_1+\mathcal{O}(\lambda^2),\quad K_2\equiv\lambda\hat{K}_2+\mathcal{O}(\lambda^2), \tag{8}$$

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[5]The reader who may wonder why we do not replace $P+K_2$ by $P$ in the first term and $P-K_1$ by $P$ in the third one is referred to Appendix A.

[6]From now on, we drop the explicit reference to the argument $K_3$ in $\Gamma_{\mu\nu}^{ARR}$.

where the components of $\hat{K}_{1,2}$ are fixed and of order unity.[7] The components of $\hat{K}_{1,2}$ are the components of the tangent vector of the path along which the limit is considered. Taking such a parametrization for the components of the momenta amounts to assume that this path admits a tangent at the origin, but not necessarily that it is a straight line. By this substitution, we are lead to considering the limit when $\lambda \to 0$ of a univariate function $F(\lambda)$, the $\hat{K}_{1,2}$ playing the role of constant parameters.

We now want to show that the integral appearing in Eq. (7) has a finite limit when $\lambda \to 0$. If we recall Eq. (7), we can see that this integral is the sum of six terms (three terms, plus the terms obtained in the symmetrization with respect to the external photons), each term behaving similar to $\lambda^{-2}$ in the limit $\lambda \to 0$. Therefore, in order to obtain a finite result, we must expand the integrand in Eq. (7) up to the order $\lambda^0$, and show that we have cancellations among the various terms in order to eliminate the orders $\lambda^{-2}$ and $\lambda^{-1}$.

The order $\lambda^{-2}$ is easy to obtain, since we can drop the $\lambda$ dependence in the statistical functions to extract it, which gives

$$
\Gamma_{\mu\nu}^{ARR}(K_1,K_2)\big|_{\lambda^{-2}} = \frac{4mNe^2 g}{\lambda^2} \epsilon_{\mu\nu\alpha\beta} k_1^\alpha k_2^\beta
$$

$$
\times \int \frac{d^3 p}{(2\pi)^3} \sum_{\epsilon=\pm} \frac{n_F(\omega_p) - \theta(-\epsilon)}{2\omega_p}
$$

$$
\times \left\{ \frac{i}{2P\cdot\hat{K}_2} \frac{i}{2P\cdot(\hat{K}_1+\hat{K}_2)} \bigg|_{p^0=\epsilon\omega_p} \right.
$$

$$
- \frac{i}{2P\cdot\hat{K}_1} \frac{i}{2P\cdot\hat{K}_2} \bigg|_{p^0=\epsilon\omega_p}
$$

$$
\left. + \frac{i}{2P\cdot\hat{K}_1} \frac{i}{2P\cdot(\hat{K}_1+\hat{K}_2)} \bigg|_{p^0=\epsilon\omega_p} \right\}
$$

$$
+ (\hat{K}_1,\mu) \leftrightarrow (\hat{K}_2,\nu) = 0. \tag{9}
$$

As we can see, the cancellation of the order $\lambda^{-2}$ is in fact a consequence of the energy-momentum conservation (it works because we have replaced $K_3$ by $-K_1-K_2$).

The cancellation of the order $\lambda^{-1}$ is a consequence of the parity properties in $K_1$ and $K_2$ of the vertex function. Indeed, looking at Eq. (7), it is rather straightforward to check the identity

$$
\Gamma_{\mu\nu}^{ARR}(K_1,K_2) = \Gamma_{\mu\nu}^{ARR}(-K_1,-K_2). \tag{10}
$$

Making use of the variable $\lambda$, it can be rewritten as

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[7]Therefore, $\lambda$ is the scale of the external momenta and is of the same order of magnitude as the pion mass $m_\pi$, assumed to be infinitesimal.

$$
\Gamma_{\mu\nu}^{ARR}(\lambda,\hat{K}_1,\hat{K}_2) = \Gamma_{\mu\nu}^{ARR}(-\lambda,\hat{K}_1,\hat{K}_2). \tag{11}
$$

In other words, the one-loop vertex function is an even function of $\lambda$. This implies automatically that the terms of order $\lambda^{-1}$ in the Laurent's expansion of the integral are vanishing. For this cancellation to occur, it is essential to perform the symmetrization with respect to the external photons. Therefore, if we write

$$
\Gamma_{\mu\nu}^{ARR}(\lambda,\hat{K}_1,\hat{K}_2) = 4mNe^2 g\, \epsilon_{\mu\nu\alpha\beta} k_1^\alpha k_2^\beta \tilde{\Gamma}_{\mu\nu}^{ARR}(\lambda,\hat{K}_1,\hat{K}_2),
$$

$$
\tag{12}
$$

then $\lim_{\lambda\to 0} \tilde{\Gamma}_{\mu\nu}^{ARR}(\lambda,\hat{K}_1,\hat{K}_2)$ is finite.

## III. NONUNIQUENESS OF THE LIMIT

### A. Generalities

After some tedious expansions,[8] we find

$$
\lim_{\lambda\to 0} \Gamma_{\mu\nu}^{ARR}(\lambda,\hat{K}_1,\hat{K}_2) = 4mNe^2 g\, \epsilon_{\mu\nu\alpha\beta} k_1^\alpha k_2^\beta
$$

$$
\times \int \frac{d^3 p}{(2\pi)^3} \left\{ \frac{3}{8} \frac{1-2n_F(\omega_p)}{\omega_p^5} \right.
$$

$$
- \frac{A(\hat{K}_1,\hat{K}_2)}{4} \frac{n_F'(\omega_p)}{\omega_p^4} \prod_{i=1}^3 \frac{1}{[(\mathcal{P}_+\cdot\hat{K}_i)(\mathcal{P}_-\cdot\hat{K}_i)]^2}
$$

$$
\left. - \frac{B(\hat{K}_1,\hat{K}_2)}{4} \frac{n_F''(\omega_p)}{\omega_p^3} \prod_{i=1}^3 \frac{1}{(\mathcal{P}_+\cdot\hat{K}_i)(\mathcal{P}_-\cdot\hat{K}_i)} \right\}, \tag{13}
$$

where we denote $\mathcal{P}_\pm \equiv (\omega_p, \pm p)$. The functions $A$ and $B$ are quite intricate; since their detailed expression is not really helpful here, they have been quoted in Appendix B. Moreover, it should be emphasized that, strictly speaking, the zero momentum limit does not exist since there is a residual dependence upon $\hat{K}_1$ and $\hat{K}_2$. Indeed, the fact that the above expression still depends on $\hat{K}_1$ and $\hat{K}_2$ means that the value of the zero momentum limit depends upon the path chosen to reach the point $K_1=K_2=0$ in momentum space. It also indicates that this limit exists only if this path admits a tangent at the origin (i.e., the limits exist only if the ratios $k_i^0/k_i^\mu$ of the components are constant in the vicinity of the origin). As a partial conclusion, one can say that the only limit that exists is the limit $\lambda\to 0$ along a given path that has a tangent at the origin, but not something such as the ''zero momentum limit.''[9]

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[8]At this stage we have proven the existence of the limit $\lambda\to 0$, we can speed up the calculations by making using of some computer algebra system such as MAPLE for instance.

[9]The same limitation exists when one takes the soft momentum limit of some amplitude in order to get its hard thermal loop (HTL) part, and is therefore not new in thermal field theory.

The conditions for the existence of the zero momentum limit and its nonuniqueness should not be a surprise. Examples of such a phenomenon are well known in thermal field theory. For instance, the same calculation applied to the $\Pi_{00}^{RA}$ component of the photon polarization tensor in massless QED leads to

$$\lim_{\lambda \to 0} \Pi_{00}^{RA}(\lambda,\hat{K}) = 4e^2 \int \frac{d^3\boldsymbol{p}}{(2\pi)^3} n_F'(p) \frac{(\boldsymbol{p} \cdot \hat{\boldsymbol{k}})^2}{(\mathcal{P}_+ \cdot \hat{K})(\mathcal{P}_- \cdot \hat{K})}, \tag{14}$$

which is nothing but the HTL contribution to this function. Here also, the residual dependence upon $\hat{K}$ indicates the nonuniqueness of the limit. In both cases, this remaining dependence on how the small momentum limit is reached implies that the corresponding term in an effective Lagrangian is nonlocal.

There is though an important difference between Eq. (13) and the HTL amplitudes. The hard thermal loop approximation consists in retaining only two orders in the expansion in powers of $\lambda$ (the lowest order is trivially vanishing due to momentum conservation). In the case of Eq. (13), we have combined two diagrams so that the second order is also vanishing. We therefore need to calculate the third order of this expansion, and this is why the functions $A$ and $B$ are much more involved than what is usually encountered in hard thermal loops [Eq. (14), for instance]. As a consequence, one may expect that the effective $\pi^0 \gamma\gamma$ coupling near the critical point exhibits a nonlocality of a completely different nature.[10]

Before going on with some specific kinematical configurations, a comment is worth concerning the zero-temperature limit of Eq. (13). Since for $m > 0$ we have $\lim_{T\to 0} n_F(\omega_{\boldsymbol{p}}) = \lim_{T\to 0} n_F'(\omega_{\boldsymbol{p}}) = \lim_{T\to 0} n_F''(\omega_{\boldsymbol{p}}) = 0$, the zero-temperature limit is trivial:

$$\lim_{\lambda \to 0, T \to 0} \Gamma_{\mu\nu}^{ARR}(\lambda,\hat{K}_1,\hat{K}_2) = 4mNe^2 g \,\epsilon_{\mu\nu\alpha\beta} k_1^\alpha k_2^\beta$$

$$\times \int \frac{d^3\boldsymbol{p}}{(2\pi)^3} \frac{3}{8\omega_{\boldsymbol{p}}^5}. \tag{15}$$

As one can see, the integral is now totally independent of the kinematical configuration of the external particles. Therefore, the fact that the numerical coefficient in front of the zero momentum limit of this diagram may not be uniquely defined is a purely thermal effect.

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[10]Let us recall that near $T = 0$, the nonlocality of the anomalous couplings is found to be closely related to that of hard thermal loops [4,18]. More precisely, HTL-like amplitudes are encountered in thermal corrections at the order $T^2/f_\pi^2$ in a low-temperature expansion. Near the chiral phase transition, we are in the opposite limit $T \gg f_\pi$, and it is likely that new nonlocal terms appear.

### B. Spacelike photons

A first possibility is to consider the situation where $k_{1,2}^0 = 0$ while $\boldsymbol{k}_{1,2} \neq \boldsymbol{0}$. This corresponds to external spacelike photons. In this particular case, the functions $A$ and $B$ become much simpler:

$$A(\hat{K}_1,\hat{K}_2) = -3(\boldsymbol{p} \cdot \hat{\boldsymbol{k}}_1)^4 (\boldsymbol{p} \cdot \hat{\boldsymbol{k}}_2)^4 [\boldsymbol{p} \cdot (\hat{\boldsymbol{k}}_1 + \hat{\boldsymbol{k}}_2)]^4,$$

$$B(\hat{K}_1,\hat{K}_2) = -(\boldsymbol{p} \cdot \hat{\boldsymbol{k}}_1)^2 (\boldsymbol{p} \cdot \hat{\boldsymbol{k}}_2)^2 [\boldsymbol{p} \cdot (\hat{\boldsymbol{k}}_1 + \hat{\boldsymbol{k}}_2)]^2. \tag{16}$$

Plugging these expressions into Eq. (13), we find

$$\lim_{\lambda \to 0} \Gamma_{\mu\nu}^{ARR}(\lambda,\hat{K}_1,\hat{K}_2)$$

$$= 4mNe^2 g \,\epsilon_{\mu\nu\alpha\beta} k_1^\alpha k_2^\beta \int \frac{d^3\boldsymbol{p}}{(2\pi)^3}$$

$$\times \left\{ \frac{3}{8} \frac{1 - 2n_F(\omega_{\boldsymbol{p}})}{\omega_{\boldsymbol{p}}^5} + \frac{3}{4} \frac{n_F'(\omega_{\boldsymbol{p}})}{\omega_{\boldsymbol{p}}^4} - \frac{1}{4} \frac{n_F''(\omega_{\boldsymbol{p}})}{\omega_{\boldsymbol{p}}^3} \right\}. \tag{17}$$

We can perform at this point the analytic continuation of Eq. (5). Since the functions $A(\hat{K}_1,\hat{K}_2)$ and $B(\hat{K}_1,\hat{K}_2)$ exactly cancel the denominators of Eq. (13), this analytic continuation does not introduce any imaginary part in the result. This fact is a consequence of a result proven by Evans [19], according to which all the retarded and advanced Green's functions are equal if the external energies are set to zero.

The angular integration is trivial here since it just amounts to multiplying the result by $4\pi$. It remains to perform the integral over $p = \|\boldsymbol{p}\|$. This integral cannot be performed analytically if $m \neq 0$, but we can consider performing an expansion of the result in powers of $m/T$, assuming $m \ll T$. In fact, replacing $m$ by zero in the expression inside the brackets, we can see that the integral over $p$ is infrared-safe without the need of this mass. As a consequence, the first term of the expansion in powers of $m/T$ is trivial to extract

$$\lim_{\lambda \to 0} \Gamma_{\mu\nu}^{ARR}(\lambda,\hat{K}_1,\hat{K}_2)$$

$$= 4mNe^2 g \,\epsilon_{\mu\nu\alpha\beta} k_1^\alpha k_2^\beta \int_0^{+\infty} \frac{dp}{(2\pi)^2}$$

$$\times \left\{ \frac{3}{4} \frac{1 - 2n_F(p)}{p^3} + \frac{3}{2} \frac{n_F'(p)}{p^2} - \frac{1}{2} \frac{n_F''(p)}{p} \right\}$$

$$\times \left[ 1 + \mathcal{O}\left(\frac{m}{T}\right) \right]. \tag{18}$$

Integrating by parts in order to get rid of the inverse powers of $p$, we obtain

$$\lim_{\lambda \to 0} \Gamma_{\mu\nu}^{ARR}(\lambda, \hat{K}_1, \hat{K}_2) = -\frac{mNe^2g}{4\pi^2T^2} \epsilon_{\mu\nu\alpha\beta} k_1^\alpha k_2^\beta$$

$$\times \int_0^{+\infty} dx \ln(x) \hat{n}_F'''(x) \left[ 1 + \mathcal{O}\left(\frac{m}{T}\right) \right],$$

$$(19)$$

where we denote $\hat{n}_F(x) \equiv 1/[\exp(x)+1]$ and $x \equiv p/T$. Making use of

$$\hat{n}_F'''(x) = 6\hat{n}_F^4(x) - 12\hat{n}_F^3(x) + 7\hat{n}_F^2(x) - \hat{n}_F(x), \quad (20)$$

and of Eq. (C6), we finally find[11]

$$\lim_{\lambda \to 0} \Gamma_{\mu\nu}^{ARR}(\lambda, \hat{K}_1, \hat{K}_2) = \frac{7\zeta(3)mNe^2g}{16\pi^4T^2} \epsilon_{\mu\nu\alpha\beta} k_1^\alpha k_2^\beta$$

$$\times \left[ 1 + \mathcal{O}\left(\frac{m}{T}\right) \right], \quad (21)$$

which is equivalent to formula (11) of Ref. [1]. Therefore, we have shown that Pisarski's result, obtained in the imaginary time formalism with external momenta set to zero right from the beginning, corresponds in fact to a zero-momentum limit taken with spacelike external photons.

This fact can be interpreted as follows: since in the imaginary time formalism the energy component of four vectors is a discrete quantity, the only possible way of taking the "zero momentum limit" in this formalism is to first set the external "energies" to the discrete value zero, and then consider the limit of zero three momenta. The above analysis shows that the limit is unique once the external energies are set to zero [the dependence on $\hat{k}_{1,2}$ has disappeared in Eq. (17)], which implies that the imaginary time formalism leads to a uniquely defined limit that coincides with the result obtained here with spacelike photons.

It is worth mentioning the paper by Baier, Dirks, and Kober [21], who reproduced the result of Ref. [1] in a somewhat different framework. Instead of calculating the triangle diagram in a particular model, they considered the Wess-Zumino-Witten [22,23] functional near the chiral symmetry restoration. Intermediate steps of their work involve the calculation in the imaginary time formalism of a function where the external momenta are set to zero. It seems that this technical analogy with Ref. [1] is the reason of the agreement. Since the zero momentum limit of the $\pi^0\gamma\gamma$ triangle is not uniquely defined, a complete calculation of the Wess-Zumino-Witten Lagrangian near the chirally symmetric phase should be extremely careful when using the imaginary time formalism (or avoid it), in order to get the correct non-locality for the couplings contained in this functional.

### C. Real photons

Gupta and Nayak chose to consider the decay of a massive pion at rest in the frame of the plasma into two real photons. This choice corresponds to the constraints $\mathbf{k}_1 + \mathbf{k}_2 = \mathbf{0}$ and $k_1^0 = k_2^0 = \|\mathbf{k}_{1,2}\|$. If one gives a small mass $m_\pi$ to the decaying pion (this mass is of order $\lambda$ and becomes infinitesimal when $\lambda \to 0$), then the pion must be static when its momentum goes to zero in the frame of the plasma. This remark indicates that there is a *unique* way of considering the small momentum limit of the decay of a slightly massive pion into real photons.

This implies some simplifications for the functions $A$ and $B$:

$$A(\hat{K}_1, \hat{K}_2) = 16\hat{k}_1^{0\,4} \omega_p^4 [(\omega_p \hat{k}_1^0)^2 - (\mathbf{p} \cdot \hat{\mathbf{k}}_1)^2]^4,$$

$$B(\hat{K}_1, \hat{K}_2) = 4\hat{k}_1^{0\,2} \omega_p^2 (\mathbf{p} \cdot \hat{\mathbf{k}}_1)^2 [(\omega_p \hat{k}_1^0)^2 - (\mathbf{p} \cdot \hat{\mathbf{k}}_1)^2],$$

$$(22)$$

and for the vertex function

$$\lim_{\lambda \to 0} \Gamma_{\mu\nu}^{ARR}(\lambda, \hat{K}_1, \hat{K}_2)$$

$$= 4mNe^2g \, \epsilon_{\mu\nu\alpha\beta} k_1^\alpha k_2^\beta \int \frac{d^3\mathbf{p}}{(2\pi)^3} \left\{ \frac{3}{8} \frac{1 - 2n_F(\omega_p)}{\omega_p^5} \right.$$

$$\left. - \frac{1}{4} \frac{n_F'(\omega_p)}{\omega_p^4} - \frac{1}{4} \frac{n_F''(\omega_p)}{\omega_p^3} \frac{(\mathbf{p} \cdot \hat{\mathbf{k}})^2}{\omega_p^2 - (\mathbf{p} \cdot \hat{\mathbf{k}})^2} \right\}, \quad (23)$$

where we denote $\hat{\mathbf{k}} \equiv \mathbf{k}_1 / \|\mathbf{k}_1\|$ the unit vector in the direction of the emission of the first photon. The analytic continuation of Eq. (5) generates a term $\delta[\omega_p^2 - (\mathbf{p} \cdot \hat{\mathbf{k}})^2]$. Anyway, since $\omega_p > p$, the Green's function $\Gamma_{\mu\nu}^{ARR}$ remains real. A few words are worth saying concerning the condition $m_\pi < 2m$, which is found in Refs. [8,9] to simplify the result by forbidding some processes (such as the decay of the pion into a thermalized $q\bar{q}$ pair, followed by the decay of a thermalized $q\bar{q}$ pair into two photons), since this is the place where this condition can appear in the present calculation. The reason why this condition does not appear explicitly in the present calculation is related to the zero momentum limit I have taken, which is equivalent to $m_\pi \to 0$, so that the above inequality is automatically satisfied.[12]

As one can see now, the angular integral is not defined if the quark mass is vanishing, due to a collinear singularity. This could have been expected since we are looking at the emission of real photons. The angular integration gives the expression

---

[11]The formula (C6) of Appendix C naturally leads to the quantities $\zeta(-2)$ and $\zeta'(-2)$. In order to simplify the result, we use the identities $\zeta(-2) = 0$ and $\zeta'(-2) = -\zeta(3)/4\pi^2$ (see, for instance, Ref. [20]).

[12]As said before, this is reasonable within the linear $\sigma$ model with an explicit symmetry breaking term going to zero. Whether this model is a good one near the critical temperature is definitely a relevant question, but since the results I want to compare have been derived in this model, I must use it also.

$$\lim_{\lambda \to 0} \Gamma_{\mu\nu}^{ARR}(\lambda, \hat{K}_1, \hat{K}_2)$$

$$= 4mNe^2 g\, \epsilon_{\mu\nu\alpha\beta} k_1^\alpha k_2^\beta \int_0^{+\infty} \frac{p^2\, dp}{(2\pi)^2} \left\{ \frac{3}{4} \frac{1 - 2n_F(\omega_p)}{\omega_p^5} \right.$$

$$\left. - \frac{n_F'(\omega_p)}{2\omega_p^4} + \frac{n_F''(\omega_p)}{2\omega_p^3}\left[ 1 - \frac{\omega_p}{2p}\ln\left(\frac{\omega_p+p}{\omega_p-p}\right)\right]\right\}. \quad (24)$$

In addition to the potential collinear singularity, another dramatic difference of this case with respect to the previous one lies in the infrared behavior of the integral. It is now impossible to take the limit $m \to 0$ in the expression inside the brackets because the integral over $p$ would behave as $dp/p^2$ at small $p$. This means that the expansion of the integral in powers of $m/T$ begins with a term in $1/mT$, to be compared with the $1/T^2$ of the previous situation. Integrating by parts the above equation, we can transform it into

$$\lim_{\lambda \to 0} \Gamma_{\mu\nu}^{ARR}(\lambda, \hat{K}_1, \hat{K}_2)$$

$$= 4mNe^2 g\, \epsilon_{\mu\nu\alpha\beta} k_1^\alpha k_2^\beta$$

$$\times \int_0^{+\infty} \frac{p\, dp}{(2\pi)^2} \frac{1 - 2n_F(\omega_p)}{4\omega_p^4} \ln\left(\frac{\omega_p+p}{\omega_p-p}\right), \quad (25)$$

which is equivalent to the result given by Gupta and Nayak for the decay of a static pion into two real photons [see formula (2.12) of [9]]. The first term of the expansion in powers of $m/T$ is

$$\lim_{\lambda \to 0} \Gamma_{\mu\nu}^{ARR}(\lambda, \hat{K}_1, \hat{K}_2) = \frac{mNe^2 g}{8\pi mT} \epsilon_{\mu\nu\alpha\beta} k_1^\alpha k_2^\beta \left[1 + \mathcal{O}\left(\frac{m}{T}\right)\right]. \quad (26)$$

### D. Photons at rest

Another simple case is the situation where the emitted photons are both massive and produced at rest in the frame of the plasma (they can subsequently decay into lepton pairs). Therefore, the kinematical constraints we must enforce are now $\boldsymbol{k}_{1,2} = \boldsymbol{0}$ while $k_{1,2}^0 \neq 0$. With these constraints, the functions $A$ and $B$ become trivial:

$$A(\hat{K}_1, \hat{K}_2) = 0, \quad B(\hat{K}_1, \hat{K}_2) = 0, \quad (27)$$

so that we have

$$\lim_{\lambda \to 0} \Gamma_{\mu\nu}^{ARR}(\lambda, \hat{K}_1, \hat{K}_2)$$

$$= 4mNe^2 g\, \epsilon_{\mu\nu\alpha\beta} k_1^\alpha k_2^\beta \int \frac{d^3\boldsymbol{p}}{(2\pi)^3} \frac{3}{8} \frac{1 - 2n_F(\omega_p)}{\omega_p^5}. \quad (28)$$

Again, the angular integration is trivial and for the remaining integral on the variable $p$ we can only perform an expansion in powers of $m/T$. The analytic continuation of Eq. (5) has no effect on this result. Here also, this integral is infrared divergent if we put $m = 0$ in the integrand. As a consequence, the result of the integration behaves as $1/mT$ instead of $1/T^2$. More precisely, we have

$$\lim_{\lambda \to 0} \Gamma_{\mu\nu}^{ARR}(\lambda, \hat{K}_1, \hat{K}_2) = \frac{3mNe^2 g}{32\pi mT} \epsilon_{\mu\nu\alpha\beta} k_1^\alpha k_2^\beta \left[1 + \mathcal{O}\left(\frac{m}{T}\right)\right]. \quad (29)$$

### E. Generic result

The above particular examples have demonstrated clearly the nonuniqueness of the zero momentum limit of the triangle diagram responsible for the pion decay in two photons. Moreover, the particularity of the first situation must be emphasized: when one expands the integral in powers of $m/T$, there is a cancellation of the terms of order $m/mT$ so that the first nonvanishing terms is of order $m/T^2$. This difference can be traced back in the infrared behavior of the integrand (at soft $p$). In order to study Eq. (13) in the infrared sector, we can just drop the mass $m$, and remember that the effect of $m$ would be to regularize integrals such as $\int dp/p^2 \sim m^{-1}$. If we drop the mass $m$ and study the soft $p$ behavior of the functions $A(\cdots)$ and $B(\cdots)$ of Appendix B, we find generically

$$A(\hat{K}_1, \hat{K}_2) \sim p^{12}, \quad (30)$$

$$B(\hat{K}_1, \hat{K}_2) \sim p^6. \quad (31)$$

If we plug these expressions in Eq. (13), we find that the terms proportional to $n_F$ and $n_F'$ are infrared singular (in the absence of $m$) so that they give $m/mT$, while the term proportional to $n_F''$ gives $m/T^2$. We can therefore write the generic result as

$$\lim_{\lambda \to 0} \Gamma_{\mu\nu}^{ARR}(\lambda, \hat{K}_1, \hat{K}_2)$$

$$= 4mNe^2 g\, \epsilon_{\mu\nu\alpha\beta} k_1^\alpha k_2^\beta \left[\frac{\alpha_0}{mT} + \frac{\alpha_1}{T^2} + \mathcal{O}\left(\frac{m}{T^3}\right)\right], \quad (32)$$

where the coefficients $\alpha_0, \alpha_1, \ldots$, are complicated functions of the $\hat{K}_i$. The condition $\alpha_0 = 0$ defines a zero measure subset of all the configurations, to which Pisarski's configuration belongs. In this sense, the behavior found by Pisarski is exceptional.

## IV. IR SENSITIVITY OF $\pi^0 \to 2\gamma$ AND HARD THERMAL LOOPS

### A. Preliminaries

The behavior of the decay rate of the $\pi^0$ into $2\gamma$ when the chiral symmetry is restored is closely related to the behavior of the $\Gamma_{\mu\nu}^{ARR}$ function in the limit where the mass $m$ goes to zero. The above study shows how this behavior depends on the kinematical configuration of the external photons. In particular, we observe that the imaginary time calculation performed with external momenta set to zero does not correspond to the physical situation where the emitted photons are real, but rather to a situation where the photons are both spacelike. The fact that the imaginary time calculation does

not correspond to real photons could have been expected thanks to the absence of any collinear singularity in this approach.

The problem is now that GN's situation, which seems more physical because the photons are assumed to be real, leads to a very different behavior for the triangle diagram at small $m$. Indeed, Pisarski's result behaves as $m/T^2$ and therefore vanish in the limit of chiral symmetry restoration. On the contrary, GN's result behaves as $m/mT$ and therefore tends to a nonvanishing constant when we consider the same limit. The question is therefore: is the conclusion that the $\pi^0 \to 2\gamma$ decay rate vanishes if the chiral symmetry is restored at finite temperature correct, since it has been derived using the result for spacelike photons? At first sight, it seems that this conclusion is erroneous, because it makes more sense to consider the result established for real photons in this context.

### B. Infrared sensitivity and hard thermal loops

Nevertheless, another aspect of the problem is to be considered, which may have important consequences in the limit of chiral symmetry restoration. Indeed, as seen above, the zero momentum limit in the case of real photons contains a strong infrared divergence, which gives the factor $1/mT$ (instead of $1/T^2$) once regularized by the mass $m$. This means that the integral over the loop momentum is dominated by the soft scale. More accurately, the momentum $p$, of order $m$, becomes softer and softer as one approaches closer to the chiral symmetry restoration. Therefore, there is a point when the loop momentum is soft enough to justify the resummation of hard thermal loops [24–27] on the quark propagators, as outlined in Fig. 3. Since we have two coupling constants $e$ and $g$ in our model, we can define two soft scales $eT$ and $gT$. But since the coupling constant $g$ is related to strong interaction while $e$ comes from electro-magnetic interactions of the quarks, one may expect that loop corrections involving the constant $g$ are the dominant ones. As a consequence, we will consider only loop corrections involving the $\sigma$ or $\boldsymbol{\pi}$, as shown in Fig. 4. Looking at the Lagrangian in Eq. (2), we see that the coupling of the $\sigma$ field to the quark field is $-ig$ while the coupling of the $\boldsymbol{\pi}$ to the quarks is $g\gamma^5$ (I do not write here the isospin matrices since they appear in such a way that the end result for the quark self-energy is proportional to the identity in flavor space). Moreover, in order to derive the contribution of $\sigma$ and $\boldsymbol{\pi}$ loops to the HTL correction to the quark propagator, we can neglect the mass $m$ of
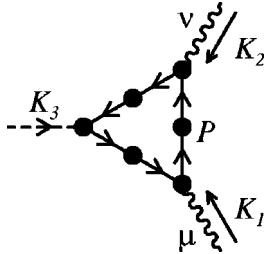


FIG. 3. $\pi^0\gamma\gamma$ Green's function at one loop in the HTL expansion.
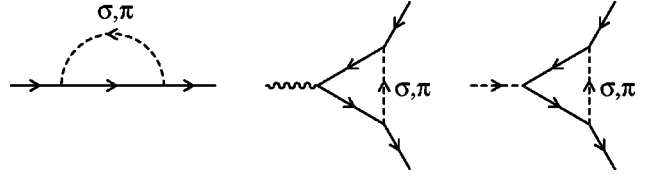


FIG. 4. Dominant topologies contributing to the HTLs of Fig. 3.

the constituent quarks since $m \ll gT$ when we approach the critical temperature. We obtain for the retarded self-energy at HTL order

$$-i\Sigma_{RA}(P)_{|\sigma} = -ig^2 \int \frac{d^3l}{(2\pi)^3} \frac{[n_B(l)+n_F(l)]}{2lP\cdot\hat{L}}\hat{L},$$

$$-i\Sigma_{RA}(P)_{|\pi} = +ig^2 \int \frac{d^3l}{(2\pi)^3} \frac{[n_B(l)+n_F(l)]}{2lP\cdot\hat{L}}\gamma^5\hat{L}\gamma^5,$$

(33)

where we denote $\hat{L} \equiv (1,\hat{\boldsymbol{l}})$. As one can see, the sum of the above two contributions is precisely equal to the standard result of QED with $e^2$ replaced by $g^2$. As a consequence, we know already all the properties of the effective propagator obtained by the resummation of the above self-energies: this resummation introduces a cutoff of order $gT$ in loop involving these effective propagators.

For the sake of completeness, we can give expressions for the HTL part of the above vertices. Starting with the $\gamma q\bar{q}$ vertex, we find

$$\Gamma^{\gamma q\bar{q}}_{ARR}(Q,P,-P-Q)|^\mu_\sigma$$

$$= -ieg^2 \int \frac{d^3l}{(2\pi)^3} \frac{[n_B(l)+n_F(l)]\hat{L}\gamma^\mu\hat{L}}{4lP\cdot\hat{L}R\cdot\hat{L}}$$

(34)

for the contribution of the $\sigma$ field, and

$$\Gamma^{\gamma q\bar{q}}_{ARR}(Q,P,-P-Q)|^\mu_\pi$$

$$= ieg^2 \int \frac{d^3l}{(2\pi)^3} \frac{[n_B(l)+n_F(l)]\gamma^5\hat{L}\gamma^\mu\hat{L}\gamma^5}{4lP\cdot\hat{L}R\cdot\hat{L}}$$

(35)

for the contribution of the $\boldsymbol{\pi}$ field, where we denote $R \equiv P + Q$. If we add the two contributions, we can see that it is equal to the QED HTL vertex with two factors of $e$ replaced by $g$. Exactly in the same way, we can obtain the HTL contribution to the vertex $\pi^0 q\bar{q}$. In fact, the result is obtained by substituting in Eqs. (34) and (35) the last power of $e$ by $g$ and the matrix $\gamma^\mu$ by $i\gamma^5$. Since $\gamma^5$ anticommutes with the Dirac's matrices, we see that the product of matrices entering in this vertex is always proportional to $\hat{L}\hat{L} = L^2 = 0$. There-

fore, the $\pi^0 q\bar{q}$ vertex does not have a HTL contribution at the scale $gT$ (it may have one at the much smaller scale $eT$ though).

### C. Effect of HTLs on $\pi^0 \rightarrow 2\gamma$

This resummation has the effect of giving a thermal mass $m_T$ to the quark, and this thermal mass remains constant[13] as we approach the point of chiral symmetry restoration (i.e., $\lim_{T \rightarrow T_c} m_T \sim gT_c$), while the constituent quark mass $m$ goes to zero ($\lim_{T \rightarrow T_c} m = 0$). In other words, the thermal mass will become the relevant infrared regulator as soon as $m \ll m_T$ (i.e., the relevant infrared regulator is always the biggest one available). Moreover, the thermal mass has the property of not modifying the chiral properties of the propagator which means that if $m \rightarrow 0$, the thermal mass will not change anything to the fact that the Dirac's trace vanishes. On the basis of these arguments, one can expect a modification of the $m/mT$ behavior in the case of real photons. Indeed, if we track the origin of the various $m$ factors in this result, we see that the mass $m$ in the numerator comes from the Dirac's trace, and is closely related to the fact that the chiral symmetry is broken by the mass $m$. On the contrary, the thermal mass $m_T$ does not break chiral symmetry. Therefore, this factor $m$ at the numerator remains unmodified by the resummation of the thermal mass. The mass factor in the denominator has a completely different origin: it comes from the infrared sensitivity of the integration over the loop momentum. This infrared scale is affected by the resummation of the thermal mass. As a consequence, we may expect that the mass $m$ is replaced by the thermal mass in the denominator but not in the numerator, when $m \ll m_T$.

Therefore the resummation of the thermal mass $m_T$ would lead to the behavior $m/m_T T$ for real photons,[14] near the chiral symmetry restoration. As a consequence, the result according to which the pion decay rate for the channel $\pi^0 \rightarrow 2\gamma$ vanishes in the chiral phase at finite temperature survives. More precisely, the situation that emerges from the constituent quark model used in this paper is summarized on Fig. 5. Assuming that the chiral symmetry restoration is a second order phase transition, the constituent quark mass is a function $m = m(T)$ of temperature that vanishes at a certain critical temperature $T_c$. We can also define two additional

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[13]Assuming a second order phase transition, we expect the coupling constant $g$ to depend only logarithmically upon temperature, while the vacuum expectation value of the sigma field, responsible for the mass $m$ of the constituent quarks, vanishes as a power of $T - T_c$ at the critical temperature $T_c$.

[14]This expression is obtained by just replacing $m$ by the thermal mass in the result found in the bare theory. The exact calculation of the effect of HTLs would require the calculation of the diagram of Fig. 3, which is an extremely involved task. Anyway, the point I want to make here is just that the denominator is unlikely to vanish in the chiral limit if the HTL corrections are taken into account. Therefore, knowing in detail the effect of HTL corrections is not really required here.
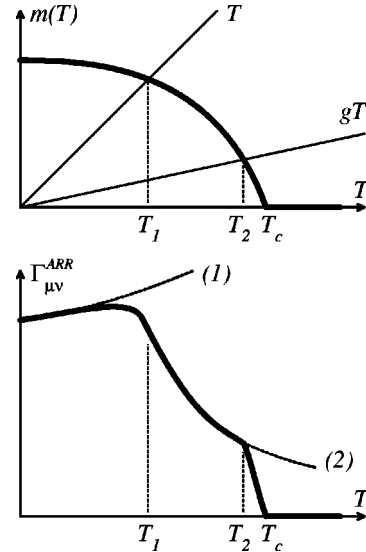


FIG. 5. Shape of the temperature dependence of the $\pi^0 \rightarrow 2\gamma$ decay amplitude in a constituent quark model. The dotted curve (1) is an extrapolation of the $T=0$ result with a $T$ dependent quark mass. The dotted curve (2) is the result obtained with real external photons when one takes into account thermal corrections, but not the resummation of hard thermal loops.

temperature scales $T_1$ and $T_2$ for which $m(T_1) = T_1$ and $m(T_2) = gT_2$,[15] respectively.

Then, for $T \in [0, T_1]$, the temperature has negligible effects, and the decay amplitude is very close to the zero temperature one: it behaves as $1/m$. When the temperature reaches values of order of $T_1$, thermal corrections become important, which has the effect of replacing the factor $1/m^2$ by $1/mT$. As a consequence, the decay amplitude behaves as $1/T$ for $T \in [T_1, T_2]$. Finally, in the domain $[T_2, T_c]$, the resummation of thermal masses plays the dominant role in the regulation of infrared singularities, and the decay amplitude eventually vanishes for $T = T_c$ since it goes as $m/gT^2$.

### V. CONCLUSIONS

First, we have seen that the difference between the results of Pisarski and GN can be interpreted as an effect of the nonuniqueness of the zero-momentum limit of the triangle diagram at finite temperature. It appeared also that Pisarski's result, originally derived in the imaginary-time formalism, corresponds in fact to a situation where the emitted photons are spacelike. The most physical situation corresponding to the case where the emitted photons are both real, a superficial analysis tends to invalidate the result according to which the $\pi^0 \rightarrow 2\gamma$ decay rate vanishes in a hot chirally symmetric phase.

Nevertheless, this calculation seems incomplete in the

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[15]The Fig. 5 assumes that $g$ is smaller than 1, which may not be the case since $g$ comes from strong interactions. This remark does change anything to the necessity of performing the HTL corrections. In fact, the larger the thermal mass, the more important are its effects.

chiral limit since the loop integral is now sensitive to soft momenta, which means that the resummation of hard thermal loops may have important effects. Taking them into account will change the infrared regulator, and modify GN's result in such a way that it now vanishes in the hot chiral phase.

I will add also a word of caution concerning the imaginary time formalism. Since the zero-momentum limit of thermal Green's functions is not unique and depend on the location of the external legs with respect to the light cone, this limit cannot be handled correctly in the imaginary time formalism (there is no light cone in an Euclidean formalism). Indeed, in such situations, this formalism gives a number which corresponds to one particular way of taking the limit, but which is not necessarily the most appropriate for the problem under study. Moreover, the information about the nonlocality of the corresponding effective coupling is lost.

Among the related topics that seem worth studying, I would mention the case of the box diagram appearing in $\pi^0 \sigma \rightarrow \gamma\gamma$, since one can expect here also to have a nonlocal effective coupling in a hot chirally symmetric phase. Another interesting aspect is to find a work-around for the limitations of the imaginary time formalism in the derivation of the anomalous processes near the phase transition from the Wess-Zumino-Witten Lagrangian.

## ACKNOWLEDGMENTS

## APPENDIX A: SIDE EFFECTS OF HAZARDOUS TRANSLATIONS

A superficial inspection of the expression of the vertex function given in Eq. (7) indicates that it may be advantageous to change the integration variable in the first and third term. Namely, changing $P+K_2$ into $P$ in the first term and $P-K_1$ into $P$ in the third term would allow the factorization of a common statistical weight $[n_F(\omega_p) - \theta(-\epsilon)]/2\omega_p$. In addition to this factorization, the major advantage of such a transformation would be to eliminate the $K_{1,2}$ dependence (i.e., the $\lambda$ dependence) in this statistical factor: as a consequence, the subsequent expansions in powers of $\lambda$ would become much simpler. This technique can be applied in GN's situation where it leads directly to Eq. (25), without the need of performing a cumbersome integration by parts (the reason for that lies in the fact that no derivatives of $n_F$ are generated in this second approach, because the statistical prefactor does not contain the expansion parameter $\lambda$). This method works also for the third case studied in Sec. III. Nevertheless, we avoided its use to derive the general limit presented in Eq. (13) because this transformation is not always legitimate.

The reason why the above transformations are sometimes illegitimate comes in fact from the infrared sector. To be more definite, let me focus on the situation of Pisarski, since for this one the above changes of variables are not allowed. The particularity of this configuration is that $k_1^0 = k_2^0 = 0$. Therefore, the denominators in Eq. (7) are combinations of $2p \cdot k_1 - k_1^2$ and $2p \cdot k_2 + k_2^2$. As a consequence, the expansion of these factors in powers of $\lambda$ generates powers of $k_i^2/(p \cdot k_i)$. A simple counting shows that the order $\lambda^0$ behaves as $\int dp/p^2$ in the infrared region, even if we keep $m$ strictly positive. Therefore, each individual term in Eq. (7) is strongly infrared divergent at $p=0$. Of course, a conspiracy of the three terms occur in order to cancel this divergence so that the final result is finite. If one performs different translations on the three terms, then the way this cancellation works is modified and finite terms are modified. The correct answer can only be obtained when the same transformation (or no transformation at all) is applied to the three terms.

An alternative way to see that this transformation is not legitimate in Pisarski's situation is as follows. If we are allowed to perform this transformation, it is obvious that it generates an integrand proportional to $1 - 2n_F(\omega_p)$, without any derivatives of the Fermi-Dirac distribution. Therefore, let us integrate by parts the result of Eq. (17) in order to try to get rid of the derivatives $n_F'$ and $n_F''$, to see if we can recover this simple structure:

$$
\int_0^{+\infty} p^2 \, dp \left\{ \frac{3}{4} \frac{1-2n_F(\omega_p)}{\omega_p^5} + \frac{3}{2} \frac{n_F'(\omega_p)}{\omega_p^4} - \frac{1}{2} \frac{n_F''(\omega_p)}{\omega_p^3} \right\}
$$

$$
= \int_0^{+\infty} p^2 \, dp \left\{ \frac{3}{4\omega_p^5} + \frac{1}{2} \frac{n_F(\omega_p)}{p^4 \omega_p} \right.
$$

$$
\left. + \frac{1}{2} \frac{n_F(m)}{\omega_p^5} \left[ 6 - \frac{3\omega_p^2}{p^2} - \frac{\omega_p^4}{p^4} \right] \right\}
$$

$$
= \frac{1-2n_F(m)}{4m^2} + \int_0^{+\infty} dp \, \frac{n_F(\omega_p) - n_F(m)}{2p^2 \omega_p}. \qquad (A1)
$$

As one can see, some terms in $n_F(m)$ appear in this integration by parts, which are absolutely necessary to ensure the infrared finiteness of the integral. There is no way of transforming this integral in order to have the temperature dependence only in a factor $1 - 2n_F(\omega_p)$, as it would be if the changes of variables $P+K_2 \rightarrow P, P-K_1 \rightarrow P$ were possible.[16]

---

[16]The condition to be able to eliminate the $n_F(m)$ terms is

$$
\lim_{p \rightarrow 0} \frac{\omega_p}{p} \frac{\partial}{\partial p} \left[ \frac{p}{\omega_p^2} \int d\Omega_p B(\hat{K}_1, \hat{K}_2) \prod_{i=1}^{3} \frac{1}{(\mathcal{P}_+ \cdot \hat{K}_i)(\mathcal{P}_- \cdot \hat{K}_i)} \right] = 0,
\qquad (A2)
$$

which is satisfied in GN's case but not in Pisarski's one. A more extensive study seems to indicate that this condition is satisfied by all configurations, excepted the configuration that corresponds to Pisarski's calculation (see also Section III E).

### APPENDIX B: FUNCTIONS $A(K_1,K_2)$ AND $B(K_1,K_2)$

$$A(K_1,K_2)=k_1^{03}k_1^{05}(3k_1^0+k_2^0)(k_1^0\boldsymbol{k}_2^2+k_2^0\boldsymbol{k}_1^2)\omega_p^{12}+k_1^0k_2^{04}[k_1^0k_2^{04}(\boldsymbol{p}\cdot\boldsymbol{k}_1)^2+k_1^{02}k_2^{03}(\boldsymbol{p}\cdot\boldsymbol{k}_1)^2+2k_1^{02}k_2^{03}(\boldsymbol{p}\cdot\boldsymbol{k}_1)(\boldsymbol{p}\cdot\boldsymbol{k}_2)$$

$$+k_2^{03}\boldsymbol{k}_1^2(\boldsymbol{p}\cdot\boldsymbol{k}_1)^2+10k_1^{03}k_2^{02}(\boldsymbol{p}\cdot\boldsymbol{k}_1)(\boldsymbol{p}\cdot\boldsymbol{k}_2)-2k_1^{03}k_2^{02}(\boldsymbol{p}\cdot\boldsymbol{k}_1)^2-k_1^{03}k_2^{02}(\boldsymbol{p}\cdot\boldsymbol{k}_2)^2+2k_1^0k_2^{02}\boldsymbol{k}_1^2(\boldsymbol{p}\cdot\boldsymbol{k}_1)(\boldsymbol{p}\cdot\boldsymbol{k}_2)$$

$$-3k_1^0k_2^{02}(\boldsymbol{p}\cdot\boldsymbol{k}_1)^2\boldsymbol{k}_2^2+6k_1^0k_2^{02}\boldsymbol{k}_1^2(\boldsymbol{p}\cdot\boldsymbol{k}_1)^2+4k_1^{02}k_2^0\boldsymbol{k}_1^2(\boldsymbol{p}\cdot\boldsymbol{k}_1)(\boldsymbol{p}\cdot\boldsymbol{k}_2)-10k_1^{02}k_2^0\boldsymbol{k}_2^2(\boldsymbol{p}\cdot\boldsymbol{k}_1)^2+8k_1^{04}k_2^0(\boldsymbol{p}\cdot\boldsymbol{k}_1)(\boldsymbol{p}\cdot\boldsymbol{k}_2)$$

$$-2k_1^{02}k_2^0\boldsymbol{k}_2^2(\boldsymbol{p}\cdot\boldsymbol{k}_1)(\boldsymbol{p}\cdot\boldsymbol{k}_2)+10k_1^{02}k_2^0\boldsymbol{k}_1^2(\boldsymbol{p}\cdot\boldsymbol{k}_1)^2-3k_1^{04}k_2^0(\boldsymbol{p}\cdot\boldsymbol{k}_2)^2-3k_1^{02}k_2^0\boldsymbol{k}_1^2(\boldsymbol{p}\cdot\boldsymbol{k}_2)^2-10k_1^{03}\boldsymbol{k}_1^2(\boldsymbol{p}\cdot\boldsymbol{k}_2)^2$$

$$+5k_1^{03}\boldsymbol{k}_2^2(\boldsymbol{p}\cdot\boldsymbol{k}_2)^2]\omega_p^{10}+k_2^{03}[4k_1^{03}(\boldsymbol{p}\cdot\boldsymbol{k}_1)\boldsymbol{k}_2^2(\boldsymbol{p}\cdot\boldsymbol{k}_2)^3-8k_1^{03}\boldsymbol{k}_1^2(\boldsymbol{p}\cdot\boldsymbol{k}_1)(\boldsymbol{p}\cdot\boldsymbol{k}_2)^3-(\boldsymbol{p}\cdot\boldsymbol{k}_1)^4k_1^{05}$$

$$-2k_2^{04}k_1^0(\boldsymbol{p}\cdot\boldsymbol{k}_1)^3(\boldsymbol{p}\cdot\boldsymbol{k}_2)-20k_2^{03}k_1^{02}(\boldsymbol{p}\cdot\boldsymbol{k}_1)^3(\boldsymbol{p}\cdot\boldsymbol{k}_2)-3k_2^{04}k_1^0(\boldsymbol{p}\cdot\boldsymbol{k}_1)^4-26k_2^0k_1^{04}(\boldsymbol{p}\cdot\boldsymbol{k}_2)^3(\boldsymbol{p}\cdot\boldsymbol{k}_1)-9k_2^0k_1^{04}(\boldsymbol{p}\cdot\boldsymbol{k}_2)^4$$

$$+7k_2^{02}k_1^{03}(\boldsymbol{p}\cdot\boldsymbol{k}_1)^2(\boldsymbol{p}\cdot\boldsymbol{k}_2)^2-36k_2^{02}k_1^{03}(\boldsymbol{p}\cdot\boldsymbol{k}_1)^3(\boldsymbol{p}\cdot\boldsymbol{k}_2)+3k_2^0(\boldsymbol{p}\cdot\boldsymbol{k}_2)^2k_1^{04}(\boldsymbol{p}\cdot\boldsymbol{k}_1)^2-12k_2^{03}k_1^{02}(\boldsymbol{p}\cdot\boldsymbol{k}_1)^4-18k_2^{02}(\boldsymbol{p}\cdot\boldsymbol{k}_1)^4k_1^{03}$$

$$+k_2^{03}k_1^{02}(\boldsymbol{p}\cdot\boldsymbol{k}_1)^2(\boldsymbol{p}\cdot\boldsymbol{k}_2)^2-4k_2^{02}k_1^{03}(\boldsymbol{p}\cdot\boldsymbol{k}_1)(\boldsymbol{p}\cdot\boldsymbol{k}_2)^3-12k_1^{03}(\boldsymbol{p}\cdot\boldsymbol{k}_1)^2\boldsymbol{k}_2^2(\boldsymbol{p}\cdot\boldsymbol{k}_2)^2$$

$$+3k_1^{03}\boldsymbol{k}_1^2(\boldsymbol{p}\cdot\boldsymbol{k}_2)^4-4k_2^{02}k_1^0\boldsymbol{k}_1^2(\boldsymbol{p}\cdot\boldsymbol{k}_1)^3(\boldsymbol{p}\cdot\boldsymbol{k}_2)+2k_2^{02}k_1^0(\boldsymbol{p}\cdot\boldsymbol{k}_1)^3\boldsymbol{k}_2^2(\boldsymbol{p}\cdot\boldsymbol{k}_2)+7k_2^{02}k_1^0(\boldsymbol{p}\cdot\boldsymbol{k}_1)^4\boldsymbol{k}_2^2-5k_2^0k_1^{02}\boldsymbol{k}_1^2(\boldsymbol{p}\cdot\boldsymbol{k}_1)^4$$

$$-3k_2^{02}k_1^0\boldsymbol{k}_1^2(\boldsymbol{p}\cdot\boldsymbol{k}_1)^2(\boldsymbol{p}\cdot\boldsymbol{k}_2)^2-5k_2^{02}\boldsymbol{k}_1^2k_1^0(\boldsymbol{p}\cdot\boldsymbol{k}_1)^4-k_2^{03}\boldsymbol{k}_1^2(\boldsymbol{p}\cdot\boldsymbol{k}_1)^4+2k_2^{03}(\boldsymbol{p}\cdot\boldsymbol{k}_1)^4\boldsymbol{k}_2^2+7k_2^0(\boldsymbol{p}\cdot\boldsymbol{k}_1)^4k_1^{02}\boldsymbol{k}_2^2$$

$$-6k_2^0k_1^{02}\boldsymbol{k}_1^2(\boldsymbol{p}\cdot\boldsymbol{k}_1)(\boldsymbol{p}\cdot\boldsymbol{k}_2)^3-9k_2^0k_1^{02}(\boldsymbol{p}\cdot\boldsymbol{k}_1)^2\boldsymbol{k}_2^2(\boldsymbol{p}\cdot\boldsymbol{k}_2)^2-4k_2^0\boldsymbol{k}_1^2k_1^{02}(\boldsymbol{p}\cdot\boldsymbol{k}_1)^3(\boldsymbol{p}\cdot\boldsymbol{k}_2)$$

$$-4k_2^0k_1^{02}(\boldsymbol{p}\cdot\boldsymbol{k}_1)^3\boldsymbol{k}_2^2(\boldsymbol{p}\cdot\boldsymbol{k}_2)-12k_2^0k_1^{02}\boldsymbol{k}_1^2(\boldsymbol{p}\cdot\boldsymbol{k}_1)^2(\boldsymbol{p}\cdot\boldsymbol{k}_2)^2]\omega_p^8+(\boldsymbol{p}\cdot\boldsymbol{k}_1)k_2^{02}[6k_2^0(\boldsymbol{p}\cdot\boldsymbol{k}_1)^3\boldsymbol{k}_1^2k_1^0(\boldsymbol{p}\cdot\boldsymbol{k}_2)^2$$

$$+3k_2^0(\boldsymbol{p}\cdot\boldsymbol{k}_1)k_1^0\boldsymbol{k}_1^2(\boldsymbol{p}\cdot\boldsymbol{k}_2)^4-4k_2^0(\boldsymbol{p}\cdot\boldsymbol{k}_1)^2k_1^0\boldsymbol{k}_2^2(\boldsymbol{p}\cdot\boldsymbol{k}_2)^3+6(\boldsymbol{p}\cdot\boldsymbol{k}_1)^5k_2^{04}+8(\boldsymbol{p}\cdot\boldsymbol{k}_1)^4(\boldsymbol{p}\cdot\boldsymbol{k}_2)k_2^{04}+2(\boldsymbol{p}\cdot\boldsymbol{k}_1)^3k_1^0(\boldsymbol{p}\cdot\boldsymbol{k}_2)^2k_2^{03}$$

$$+17(\boldsymbol{p}\cdot\boldsymbol{k}_1)k_2^{02}k_1^{02}(\boldsymbol{p}\cdot\boldsymbol{k}_2)^4+20(\boldsymbol{p}\cdot\boldsymbol{k}_1)^4k_2^{03}(\boldsymbol{p}\cdot\boldsymbol{k}_2)k_1^0+4(\boldsymbol{p}\cdot\boldsymbol{k}_1)^2k_2^{03}k_1^0(\boldsymbol{p}\cdot\boldsymbol{k}_2)^3+56(\boldsymbol{p}\cdot\boldsymbol{k}_1)^2k_2^{02}k_1^{02}(\boldsymbol{p}\cdot\boldsymbol{k}_2)^3$$

$$+13(\boldsymbol{p}\cdot\boldsymbol{k}_1)^5k_1^0k_2^{03}+13k_2^{02}(\boldsymbol{p}\cdot\boldsymbol{k}_1)^5k_1^{02}+34k_2^{02}(\boldsymbol{p}\cdot\boldsymbol{k}_1)^4(\boldsymbol{p}\cdot\boldsymbol{k}_2)k_1^{02}+2k_2^0k_1^{03}(\boldsymbol{p}\cdot\boldsymbol{k}_2)^5$$

$$+31k_2^0(\boldsymbol{p}\cdot\boldsymbol{k}_1)k_1^{03}(\boldsymbol{p}\cdot\boldsymbol{k}_2)^4+40k_2^0(\boldsymbol{p}\cdot\boldsymbol{k}_1)^2k_1^{03}(\boldsymbol{p}\cdot\boldsymbol{k}_2)^3+16(\boldsymbol{p}\cdot\boldsymbol{k}_1)^2k_1^{02}\boldsymbol{k}_2^2(\boldsymbol{p}\cdot\boldsymbol{k}_2)^3+6k_1^{02}\boldsymbol{k}_1^2(\boldsymbol{p}\cdot\boldsymbol{k}_2)^5+42k_2^{02}(\boldsymbol{p}\cdot\boldsymbol{k}_1)^3$$

$$\times(\boldsymbol{p}\cdot\boldsymbol{k}_2)^2k_1^{02}+14(\boldsymbol{p}\cdot\boldsymbol{k}_1)k_1^{02}\boldsymbol{k}_1^2(\boldsymbol{p}\cdot\boldsymbol{k}_2)^4+11(\boldsymbol{p}\cdot\boldsymbol{k}_1)k_1^{02}(\boldsymbol{p}\cdot\boldsymbol{k}_2)^4\boldsymbol{k}_2^2-k_2^{02}(\boldsymbol{p}\cdot\boldsymbol{k}_1)^5\boldsymbol{k}_1^2-2k_2^{02}(\boldsymbol{p}\cdot\boldsymbol{k}_1)^3\boldsymbol{k}_1^2(\boldsymbol{p}\cdot\boldsymbol{k}_2)^2$$

$$+4k_2^{02}(\boldsymbol{p}\cdot\boldsymbol{k}_1)^3\boldsymbol{k}_2^2(\boldsymbol{p}\cdot\boldsymbol{k}_2)^2+4k_2^{02}(\boldsymbol{p}\cdot\boldsymbol{k}_1)^4\boldsymbol{k}_2^2(\boldsymbol{p}\cdot\boldsymbol{k}_2)-k_2^0(\boldsymbol{p}\cdot\boldsymbol{k}_1)^5\boldsymbol{k}_2^2k_1^0+4k_2^0(\boldsymbol{p}\cdot\boldsymbol{k}_1)^4\boldsymbol{k}_2^2(\boldsymbol{p}\cdot\boldsymbol{k}_2)k_1^0$$

$$-2k_2^0(\boldsymbol{p}\cdot\boldsymbol{k}_1)^4\boldsymbol{k}_1^2(\boldsymbol{p}\cdot\boldsymbol{k}_2)k_1^0-4k_2^{02}(\boldsymbol{p}\cdot\boldsymbol{k}_1)^4\boldsymbol{k}_1^2(\boldsymbol{p}\cdot\boldsymbol{k}_2)+2k_2^0(\boldsymbol{p}\cdot\boldsymbol{k}_1)^3k_1^0\boldsymbol{k}_2^2(\boldsymbol{p}\cdot\boldsymbol{k}_2)^2+8k_2^0(\boldsymbol{p}\cdot\boldsymbol{k}_1)^2k_1^0\boldsymbol{k}_1^2(\boldsymbol{p}\cdot\boldsymbol{k}_2)^3]\omega_p^6$$

$$-(\boldsymbol{p}\cdot\boldsymbol{k}_1)^2k_2^0[10k_2^{03}(\boldsymbol{p}\cdot\boldsymbol{k}_1)^5(\boldsymbol{p}\cdot\boldsymbol{k}_2)+22k_2^{03}(\boldsymbol{p}\cdot\boldsymbol{k}_1)^4(\boldsymbol{p}\cdot\boldsymbol{k}_2)^2+24k_2^{03}(\boldsymbol{p}\cdot\boldsymbol{k}_1)^3(\boldsymbol{p}\cdot\boldsymbol{k}_2)^3+3k_2^{03}(\boldsymbol{p}\cdot\boldsymbol{k}_1)^6$$

$$+8k_2^{03}(\boldsymbol{p}\cdot\boldsymbol{k}_1)^2(\boldsymbol{p}\cdot\boldsymbol{k}_2)^4+23k_2^{02}(\boldsymbol{p}\cdot\boldsymbol{k}_1)^4(\boldsymbol{p}\cdot\boldsymbol{k}_2)^2k_1^0+44k_2^{02}(\boldsymbol{p}\cdot\boldsymbol{k}_1)^3k_1^0(\boldsymbol{p}\cdot\boldsymbol{k}_2)^3+2k_2^{02}(\boldsymbol{p}\cdot\boldsymbol{k}_1)k_1^0(\boldsymbol{p}\cdot\boldsymbol{k}_2)^5$$

$$+19k_2^{02}(\boldsymbol{p}\cdot\boldsymbol{k}_1)^2k_1^0(\boldsymbol{p}\cdot\boldsymbol{k}_2)^4+12k_2^0(\boldsymbol{p}\cdot\boldsymbol{k}_1)^3\boldsymbol{k}_2^2(\boldsymbol{p}\cdot\boldsymbol{k}_2)^3-4k_2^0(\boldsymbol{p}\cdot\boldsymbol{k}_1)^3\boldsymbol{k}_1^2(\boldsymbol{p}\cdot\boldsymbol{k}_2)^3+25k_2^0k_1^{02}(\boldsymbol{p}\cdot\boldsymbol{k}_2)^6$$

$$+76k_2^0(\boldsymbol{p}\cdot\boldsymbol{k}_1)k_1^{02}(\boldsymbol{p}\cdot\boldsymbol{k}_2)^5+2k_2^0(\boldsymbol{p}\cdot\boldsymbol{k}_1)^5\boldsymbol{k}_2^2(\boldsymbol{p}\cdot\boldsymbol{k}_2)-3k_2^0(\boldsymbol{p}\cdot\boldsymbol{k}_1)^2\boldsymbol{k}_1^2(\boldsymbol{p}\cdot\boldsymbol{k}_2)^4+8k_2^0(\boldsymbol{p}\cdot\boldsymbol{k}_1)^4(\boldsymbol{p}\cdot\boldsymbol{k}_2)^2\boldsymbol{k}_2^2$$

$$-k_2^0(\boldsymbol{p}\cdot\boldsymbol{k}_1)^4(\boldsymbol{p}\cdot\boldsymbol{k}_2)^2\boldsymbol{k}_1^2+48k_2^0(\boldsymbol{p}\cdot\boldsymbol{k}_1)^2(\boldsymbol{p}\cdot\boldsymbol{k}_2)^4k_1^{02}+6k_2^0(\boldsymbol{p}\cdot\boldsymbol{k}_1)^2\boldsymbol{k}_2^2(\boldsymbol{p}\cdot\boldsymbol{k}_2)^4+4(\boldsymbol{p}\cdot\boldsymbol{k}_1)k_1^0\boldsymbol{k}_1^2(\boldsymbol{p}\cdot\boldsymbol{k}_2)^5+k_1^0(\boldsymbol{p}\cdot\boldsymbol{k}_2)^6\boldsymbol{k}_1^2$$

$$+(\boldsymbol{p}\cdot\boldsymbol{k}_1)^2k_1^0\boldsymbol{k}_2^2(\boldsymbol{p}\cdot\boldsymbol{k}_2)^4-2(\boldsymbol{p}\cdot\boldsymbol{k}_1)k_1^0(\boldsymbol{p}\cdot\boldsymbol{k}_2)^5\boldsymbol{k}_2^2]\omega_p^4+6[(\boldsymbol{p}\cdot\boldsymbol{k}_2)+(\boldsymbol{p}\cdot\boldsymbol{k}_1)]k_2^0(\boldsymbol{p}\cdot\boldsymbol{k}_1)^4(\boldsymbol{p}\cdot\boldsymbol{k}_2)^2$$

$$\times[k_2^0(\boldsymbol{p}\cdot\boldsymbol{k}_1)^3+3(\boldsymbol{p}\cdot\boldsymbol{k}_1)^2k_2^0(\boldsymbol{p}\cdot\boldsymbol{k}_2)+4k_2^0(\boldsymbol{p}\cdot\boldsymbol{k}_1)(\boldsymbol{p}\cdot\boldsymbol{k}_2)^2+2k_2^0(\boldsymbol{p}\cdot\boldsymbol{k}_2)^3+2k_1^0(\boldsymbol{p}\cdot\boldsymbol{k}_2)^3]\omega_p^2$$

$$-3(\boldsymbol{p}\cdot\boldsymbol{k}_1)^4(\boldsymbol{p}\cdot\boldsymbol{k}_2)^6[(\boldsymbol{p}\cdot\boldsymbol{k}_2)+(\boldsymbol{p}\cdot\boldsymbol{k}_1)][3(\boldsymbol{p}\cdot\boldsymbol{k}_1)+(\boldsymbol{p}\cdot\boldsymbol{k}_2)]+(k_1^0,\boldsymbol{k}_1)\leftrightarrow(k_2^0,\boldsymbol{k}_2),$$

$$B(K_1,K_2)=k_1^0k_2^{02}[k_1^0(\boldsymbol{p}\cdot\boldsymbol{k}_2)^2+k_2^0(\boldsymbol{p}\cdot\boldsymbol{k}_1)^2]\omega_p^4-k_2^0(\boldsymbol{p}\cdot\boldsymbol{k}_1)^2\{(k_1^0+k_2^0)(\boldsymbol{p}\cdot\boldsymbol{k}_2)^2-k_2^0(\boldsymbol{p}\cdot\boldsymbol{k}_1)[\boldsymbol{p}\cdot(\boldsymbol{k}_1+\boldsymbol{k}_2)]\}\omega_p^2$$

$$-(\boldsymbol{p}\cdot\boldsymbol{k}_1)^2(\boldsymbol{p}\cdot\boldsymbol{k}_2)^3[\boldsymbol{p}\cdot(\boldsymbol{k}_1+\boldsymbol{k}_2)]+(k_1^0,\boldsymbol{k}_1)\leftrightarrow(k_2^0,\boldsymbol{k}_2).$$

## APPENDIX C: CALCULATION OF $\int_0^{+\infty}dx\,x^n\ln(x)[\exp(x)+1]^{-p}$

In Sec. III B, we need to evaluate integrals of the form

$$I_{n,p}\equiv\int_0^{+\infty}dx\,\frac{x^n\ln(x)}{(e^x+1)^p},\qquad(C1)$$

where $n,p$ are positive integers. The starting point is to expand $(e^x+1)^{-p}$ in powers of $e^{-x}$, which gives

$$I_{n,p}=\frac{(-1)^p}{(p-1)!}\sum_{m=1}^{+\infty}(-1)^m\sum_{i=0}^{p-1}\alpha_{p-1,i}m^i$$

$$\times\int_0^{+\infty}dx\,x^n\ln(x)e^{-mx},\qquad(C2)$$

where the numbers $\alpha_{p-1,i}$ are the coefficients of the polynomial[17]

$$Q_{p-1}(x)\equiv(x-1)(x-2)\cdots(x-p+1)\equiv\sum_{i=0}^{p-1}\alpha_{p-1,i}x^i.$$
$$(C3)$$

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[17]To be complete, one must define also $Q_0(x)\equiv1$.

We need then

$$\int_0^{+\infty}dx\,x^n\ln(x)e^{-mx}=\frac{A_n-n!\,[\gamma+\ln(m)]}{m^{n+1}},\qquad(C4)$$

where $\gamma$ is the Euler's constant, and $A_n$ are integers defined recursively[18] by

$$A_0=0,\quad A_{n+1}=n!+(n+1)A_n.\qquad(C5)$$

It is now straightforward to collect the various pieces in order to obtain the following expression:

$$I_{n,p}=\frac{(-1)^p n!}{(p-1)!}\sum_{i=0}^{p-1}\alpha_{p-1,i}\left[(2^{i-n}-1)\zeta'(n+1-i)\right.$$

$$-2^{i-n}\ln(2)\zeta(n+1-i)+(2^{i-n}-1)$$

$$\left.\times\left(\frac{A_n}{n!}-\gamma\right)\zeta(n+1-i)\right].\qquad(C6)$$

---

[18]The solution of the recursion is $A_n=\gamma n!+\Gamma'(n+1)$, where $\Gamma(z)=\int_0^{+\infty}dt\,e^{-t}t^{z-1}$, but this expression does not make obvious the fact that $A_n$ is an integer. Moreover, one usually needs $A_n$ for small fixed values of $n$, which makes the use of Eq. (C5) very convenient in practice.

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