Extracting $B(\omega\rightarrow\pi^+\pi^-)$ from the timelike pion form factor

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We extract the G-parity-violating branching ratio $B(\omega\rightarrow\pi^+\pi^-)$ from the effective ρ^0-ω mixing matrix element $\tilde{\Pi}_{\rho\omega}(s)$, determined from $e^+e^-\rightarrow\pi^+\pi^-$ data. The $\omega\rightarrow\pi^+\pi^-$ partial width can be determined either from the timelike pion form factor or through the constraint that the mixed physical propagator $D^{\mu\nu}_{\rho\omega}(s)$ possesses no poles. The two procedures are inequivalent in practice, and we show why the first is preferred, to find finally $B(\omega\rightarrow\pi^+\pi^-)=1.9\pm0.3\%$. [S0556-2821(99)05005-5]

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I. INTRODUCTION

The presence of the ω resonance in $e^+e^-\rightarrow\pi^+\pi^-$, in the region dominated by the ρ^0, signals the presence of the G-parity-violating decay $\omega\rightarrow\pi^+\pi^-$. Our purpose is to extract the value of $B(\omega\rightarrow\pi^+\pi^-)$ from fits to $e^+e^-\rightarrow\pi^+\pi^-$ in the ρ^0-ω interference region. To do so, we must consider the relationship between the partial width $\Gamma(\omega\rightarrow\pi^+\pi^-)$ and the effective ρ^0-ω mixing matrix element $\tilde{\Pi}_{\rho\omega}(m^2_\omega)$, determined in our earlier fits [1] to $e^+e^-\rightarrow\pi^+\pi^-$ data [2,3]. The $e^+e^-\rightarrow\pi^+\pi^-$ cross section $\sigma(s)$ can be written as $\sigma(s)=\sigma_{\rm em}(s)|F_\pi(s)|^2$, where $\sigma_{\rm em}(s)$ is the cross section for the production of a structureless $\pi^+\pi^-$ pair and s is the usual Mandlestam variable. The timelike pion form factor $F_\pi(s)$ can in turn be written, to leading order in isospin violation, as [1]

$$F_\pi(s)=F_\rho(s)\left[1+\frac{1}{3}\left(\frac{\tilde{\Pi}_{\rho\omega}(s)}{s-m^2_\omega+im_\omega\Gamma_\omega}\right)\right], \qquad (1)$$

where $F_\rho(s)$ parametrizes the ρ^0 resonance and $\tilde{\Pi}_{\rho\omega}(s)$ is the effective ρ^0-ω mixing matrix element noted earlier. $\Gamma(\omega\rightarrow\pi^+\pi^-)$ is determined by the effective $\omega\rightarrow\pi^+\pi^-$ coupling constant $g^{\rm eff}_{\omega\pi\pi}$, which can be extracted either from the timelike pion form factor or from the relationship between the physical and isospin-perfect vector meson fields, determined through the constraint that the mixed physical propagator $D^{\mu\nu}_{\rho\omega}(s)$ possesses no poles. We evaluate not only the relationship between these two different methods, but also the impact of the uncertainty in the ρ^0 mass and width on $B(\omega\rightarrow\pi^+\pi^-)$ before reporting our final results. Despite the close connection between $B(\omega\rightarrow\pi^+\pi^-)$ and $\tilde{\Pi}_{\rho\omega}(s)$, we believe this work represents the first attempt to determine both simultaneously from $e^+e^-\rightarrow\pi^+\pi^-$ data.

II. $\Gamma(\omega\rightarrow\pi^+\pi^-)$ AND ρ^0-ω MIXING

If isospin symmetry were perfect, the ρ and ω resonances would be exact eigenstates of G parity, so that the ρ, of even G-parity, would decay to two, but not three, pions and the ω, of odd G-parity, would decay to three, but not two, pions. Yet this is not strictly so, for ρ^0-ω interference in $e^+e^-\rightarrow\pi^+\pi^-$ is observed in nature [4]. Nevertheless, it is useful to introduce an isospin-perfect basis ρ^0_I and ω_I in which to describe the physical ρ^0 and ω. In this basis, G-parity can be violated either through "mixing," $\langle\omega_I|H^{\rm mix}|\rho_I\rangle$, where $H^{\rm mix}$ represents isospin-violating terms in the effective Hamiltonian in the vector meson sector, or through the direct decay $\langle\omega_I|H^{\rm mix}|\pi^+\pi^-\rangle$. The vector mesons in $e^+e^-\rightarrow\pi^+\pi^-$ couple to a conserved current, so that we can write their propagators as $D^{\mu\nu}_{VV}(s)\equiv g^{\mu\nu}D_{VV}(s)$, thereby defining the scalar part of the propagator, $D_{VV}(s)$. The propagator possesses a pole in the complex plane at $s=z_V$, so that in the vicinity of this pole we have $D_{VV}(s)=1/(s-z_V)\equiv1/s_V$. The difference between the diagonal scalar propagator in the physical and isospin-perfect bases, i.e., between $D_{VV}(s)$ and $D^I_{VV}(s)$, is of non-leading-order in isospin violation, so that $D^I_{VV}(s)=1/s_V$ as well. Consequently, the pion form factor in the resonance region in the isospin-perfect basis can be written, to leading order in isospin violation, as

$$F_\pi(s)=\frac{g_{\rho_I\pi\pi}f_{\rho_I\gamma}}{s_\rho}+\frac{g_{\omega_I\pi\pi}f_{\omega_I\gamma}}{s_\omega}+\frac{g_{\rho_I\pi\pi}\Pi^I_{\rho\omega}(s)f_{\omega_I\gamma}}{s_\rho s_\omega}, \qquad (2)$$

where $g_{V_I\pi\pi}$ and $f_{V_I\gamma}$ are the vector-meson–pion-pion and vector-meson–photon coupling constants, respectively. The first term reflects the dominant process $\gamma\rightarrow\rho^0\rightarrow\pi^+\pi^-$, whereas the G-parity-violating terms reflect the direct decay $\omega\rightarrow\pi^+\pi^-$ and ρ^0-ω mixing, $\omega\rightarrow\rho^0\rightarrow\pi^+\pi^-$, respectively, noting the mixing matrix element $\Pi^I_{\rho\omega}(s)$. Defining $G\equiv g_{\omega_I\pi\pi}/g_{\rho_I\pi\pi}$ we can rewrite Eq. (2) as

$$F_\pi(s)=\frac{g_{\rho_I\pi\pi}f_{\rho_I\gamma}}{s_\rho}+\frac{g_{\rho_I\pi\pi}f_{\omega_I\gamma}}{s_\rho s_\omega}(G(s-z_\rho)+\Pi^I_{\rho\omega}(s))$$
$$\equiv\frac{f_{\rho_I\gamma}g_{\rho_I\pi\pi}}{s_\rho}\left[1+\frac{f_{\omega_I\gamma}}{f_{\rho_I\gamma}}\left(\frac{\tilde{\Pi}_{\rho\omega}(s)}{s-z_\omega}\right)\right]. \qquad (3)$$

Note that we have defined the effective mixing matrix element $\tilde{\Pi}_{\rho\omega}(s)$, as G and $\Pi^I_{\rho\omega}(s)$ cannot be meaningfully separated in a fit to data [5,6], for both terms are s-dependent

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[7,8]. As $\Gamma_\omega \ll m_\omega$ a Breit-Wigner lineshape may be used to model the ω resonance, but the large width of the ρ relative to its mass obliges a more sophisticated treatment. Rather than adopting $s_\rho = s - z_\rho$, appropriate for $s \approx z_\rho$, for the entire resonance region, we replace $f_{\rho_I\gamma} g_{\rho_I\pi\pi}/s_\rho$ by $F_\rho(s)$, a function constructed to incorporate the constraints imposed on the form factor by time-reversal invariance, unitarity, analyticity, and charge conservation. For further details, see Ref. [1] and references therein. Using

$$F_\pi(s) = F_\rho(s)\left[1 + \frac{f_{\omega_I\gamma}}{f_{\rho_I\gamma}}\left(\frac{\tilde{\Pi}_{\rho\omega}(s)}{s - m_\omega^2 + i m_\omega \Gamma_\omega}\right)\right] \tag{4}$$

with the SU(6) value of $f_{\omega_I\gamma}/f_{\rho_I\gamma} = 1/3$, we find $\tilde{\Pi}_{\rho\omega}(m_\omega^2) = -3500 \pm 300$ MeV^2, where the systematic error due to the ρ^0 parametrization adopted is negligible [1]. Note that both the imaginary part of $\tilde{\Pi}_{\rho\omega}(s)$, $\mathrm{Im}\,\tilde{\Pi}_{\rho\omega}(m_\omega^2) = -300 \pm 300$ MeV^2, and its s-dependence about $s = m_\omega^2$, $\tilde{\Pi}_{\rho\omega}(s) = \tilde{\Pi}_{\rho\omega}(m_\omega^2) + (s - m_\omega^2)\tilde{\Pi}'_{\rho\omega}(m_\omega^2)$ with $\tilde{\Pi}'_{\rho\omega}(m_\omega^2) = 0.03 \pm 0.04$, are also negligible [1].

Equation (3) can also be used to define an effective, isospin-violating coupling constant, $g_{\omega\pi\pi}^{\mathrm{eff}}(s)$, such that

$$F_\pi(s) = \frac{g_{\rho_I\pi\pi} f_{\rho_I\gamma}}{s_\rho} + \frac{g_{\omega\pi\pi}^{\mathrm{eff}}(s) f_{\omega_I\gamma}}{s_\omega}, \tag{5}$$

so that $g_{\omega\pi\pi}^{\mathrm{eff}}(s) \equiv g_{\rho_I\pi\pi}\tilde{\Pi}_{\rho\omega}(s)/s_\rho$. To determine the partial width $\Gamma(\omega \to \pi^+\pi^-)$, and hence $B(\omega \to \pi^+\pi^-)$, we must relate it to the effective coupling constant $g_{\omega\pi\pi}^{\mathrm{eff}}(s)$.

In a Lagrangian model in which the pion is an elementary field and $g_{V\pi\pi}$ denotes the vector meson coupling constant to two pions, the vector meson self-energy $\Pi_{VV}(s)$, noting $D_{VV}^{-1}(s) = s - m^2 - \Pi_{VV}(s)$, can be approximated as a sum of iterated bubble diagrams, where each bubble contains a two-pion intermediate state [9]. Here $g_{V\pi\pi}$ is a simple constant, and direct calculation yields [9]

$$\mathrm{Im}\,\Pi_{VV}(s) = -g_{V\pi\pi}^2 \frac{(s - 4m_\pi^2)^{3/2}}{48\pi\sqrt{s}}\,\Theta(s - 4m_\pi^2). \tag{6}$$

Finally, noting $\lim_{s\to m_V^2} \mathrm{Im}\,\Pi_{VV}(s) = -m_V \Gamma(V \to \pi^+\pi^-)$, then [9,10]

$$\Gamma(V \to \pi^+\pi^-) = \frac{g_{V\pi\pi}^2}{48\pi}\,\frac{(m_V^2 - 4m_\pi^2)^{3/2}}{m_V^2}. \tag{7}$$

Replacing $g_{\omega\pi\pi}$ by $|g_{\omega\pi\pi}^{\mathrm{eff}}(s)|$, one finds that $B(\omega \to \pi^+\pi^-)$, to leading order in isospin violation, is given by

$$B(\omega \to \pi^+\pi^-) = \frac{1}{48\pi}\,\frac{(m_\omega^2 - 4m_\pi^2)^{3/2}}{m_\omega^2 \Gamma_\omega}\left|\frac{g_{\rho_I\pi\pi}\tilde{\Pi}_{\rho\omega}(s)}{s_\rho}\right|^2_{s=m_\omega^2}. \tag{8}$$

Another relation for $g_{\omega\pi\pi}^{\mathrm{eff}}(s)$ emerges through consideration of the pion form factor in the physical basis [11]. To leading order in isospin violation, we have [5,6]

$$F_\pi(s) = g_{\rho\pi\pi} D_{\rho\rho} f_{\rho\gamma} + g_{\rho\pi\pi} D_{\rho\omega} f_{\omega\gamma} + g_{\omega\pi\pi} D_{\omega\omega} f_{\omega\gamma}, \tag{9}$$

where we introduce a ρ^0-ω mixing matrix element, $\Pi_{\rho\omega}(s)$, such that [8]

$$D_{\rho\omega}^I(s) = D_{\omega\omega}^I(s)\Pi_{\rho\omega}(s) D_{\rho\rho}^I(s). \tag{10}$$

To relate the physical states ρ and ω to the isospin perfect ones ρ_I and ω_I, we introduce two mixing parameters, ϵ_1 and ϵ_2, such that [5,6]

$$\rho = \rho_I - \epsilon_1 \omega_I; \quad \omega = \epsilon_2 \rho_I + \omega_I. \tag{11}$$

Requiring the mixed physical propagator $D_{\rho\omega}(s)$ to possess no poles, ϵ_1 and ϵ_2 are determined to be [5,6]

$$\epsilon_1 = \frac{\Pi_{\rho\omega}(z_\omega)}{z_\omega - z_\rho}, \quad \epsilon_2 = \frac{\Pi_{\rho\omega}(z_\rho)}{z_\omega - z_\rho}. \tag{12}$$

Using Eqs. (9) and (11), and $D_{VV} = D_{VV}^I = 1/s_V$ for s in the vicinity of z_ρ, z_ω yields

$$F_\pi(s) = \frac{g_{\rho_I\pi\pi} f_{\rho_I\gamma}}{s_\rho} + \frac{g_{\omega_I\pi\pi}}{s_\omega s_\rho} \times \left[G(s - z_\rho) + \frac{(\Pi_{\rho\omega}(z_\rho) - \Pi_{\rho\omega}(z_\omega))}{z_\omega - z_\rho} \times (s - z_\omega + s - z_\rho) + \Pi_{\rho\omega}(s) \right] f_{\omega_I\gamma}. \tag{13}$$

Comparison with Eq. (2) shows that $\Pi_{\rho\omega}^I(s)$ and $\Pi_{\rho\omega}(s)$ are only equivalent if $\Pi_{\rho\omega}(z_\rho) = \Pi_{\rho\omega}(z_\omega)$. We can then write $g_{\omega\pi\pi}^{\mathrm{eff}}$ found above as

$$g_{\omega\pi\pi}^{\mathrm{eff}}(s) = \frac{g_{\rho_I\pi\pi}}{s_\rho}\tilde{\Pi}_{\rho\omega}(s) = \frac{g_{\rho_I\pi\pi}}{s_\rho}\left[G(s - z_\rho) + \frac{(\Pi_{\rho\omega}(z_\rho) - \Pi_{\rho\omega}(z_\omega))}{z_\omega - z_\rho} \times (s - z_\omega + s - z_\rho) + \Pi_{\rho\omega}(s) \right]. \tag{14}$$

We could also have defined $g_{\omega\pi\pi}^{\mathrm{eff}}$ directly from the relation between the physical and isospin perfect bases, Eq. (11):

$$g_{\omega\pi\pi}^{\mathrm{eff}} = g_{\omega_I\pi\pi} + \epsilon_2 g_{\rho_I\pi\pi} = \frac{g_{\rho_I\pi\pi}}{z_\omega - z_\rho}[G(z_\omega - z_\rho) + \Pi_{\rho\omega}(z_\rho)]. \tag{15}$$

These two possible definitions of $g_{\omega\pi\pi}^{\mathrm{eff}}$ are *identical* at the ω pole, $s = z_\omega$. However, fits to the time-like pion form factor data yield $\tilde{\Pi}_{\rho\omega}(s)$ merely at real values of s, so that Eq. (14)

TABLE I. The results of our fits [1] to the pion form-factor and the corresponding values of $\Gamma(\rho^0\to e^+e^-)$, noting Eq. (26), and $B(\rho^0\to e^+e^-)$. Also shown are the ρ parameters, $\overline{m}_\rho$ and $\overline{\Gamma}_\rho$, defined from the pole position z_ρ, as in Eq. (20).

Fit	m_ρ MeV [1]	Γ_ρ MeV [1]	$\Gamma(\rho\to e^+e^-)$ (keV)	$B(\rho\to e^+e^-)\times(10^5)$	$\overline{m}_\rho$ (MeV)	$\overline{\Gamma}_\rho$ (MeV)
A	763.1 ± 3.9	153.8 ± 1.2	7.27 ± 0.08	4.73 ± 0.05	756.3 ± 1.2	141.9 ± 3.1
B	771.3 ± 1.3	156.2 ± 0.4	7.24 ± 0.08	4.63 ± 0.06	757.0 ± 1.0	141.7 ± 3.0
C	773.9 ± 1.2	157.0 ± 0.4	7.19 ± 0.08	4.58 ± 0.05	757.0 ± 1.0	141.7 ± 3.0
D	773.9 ± 1.2	146.9 ± 3.4	6.73 ± 0.10	4.58 ± 0.05	757.0 ± 1.0	141.7 ± 3.0

is the only practicable definition of $g^{\rm eff}_{\omega\pi\pi}$. The two expressions differ in general as isospin-violating pieces are present in $f_{\rho\gamma}$ as well; they vanish, however, at $s=z_\omega$.

Interestingly, if we were to demand as in Ref. [6] that $\Pi^I_{\rho\omega}(s)\equiv\Pi_{\rho\omega}(s)$, implying that Eq. (11) cannot be used to relate $f_{V\gamma}$ to $f_{V_I\gamma}$ and $g_{V\pi\pi}$ to $g_{V_I\pi\pi}$ unless $\epsilon_1=\epsilon_2$ [6], then Eq. (14) would become $g^{\rm eff}_{\omega\pi\pi}(s)=g_{\rho_I\pi\pi}[G(s-z_\rho)+\Pi_{\rho\omega}(s)]/s_\rho$. This latter definition of $g^{\rm eff}_{\omega\pi\pi}(s)$ would be inconsistent with Eq. (15) at $s=z_\omega$. We prefer the analysis yielding Eq. (14).

To determine $B(\omega\to\pi^+\pi^-)$ using Eq. (8) we must evaluate $g_{\rho_I\pi\pi}/s_\rho$ at $s=m_\omega^2$. As $s_\rho=s-z_\rho$ only for $s\approx z_\rho$, it is appropriate to replace $g_{\rho_I\pi\pi}/s_\rho$ by $F_\rho(s)/f_{\rho_I\gamma}$, noting Eqs. (3) and (4), to yield finally

$$B(\omega\to\pi^+\pi^-)=\frac{(m_\omega^2-4m_\pi^2)^{3/2}}{48\pi m_\omega^2\Gamma_\omega f^2_{\omega_I\gamma}}\left|F_\rho(m_\omega^2)\frac{1}{3}\tilde{\Pi}_{\rho\omega}(m_\omega^2)\right|^2. \tag{16}$$

In the fit to data using Eq. (4), $(f_{\omega_I\gamma}/f_{\rho_I\gamma})\tilde{\Pi}_{\rho\omega}(s)$ appears as a single fitting parameter. Choosing $f_{\omega_I\gamma}/f_{\rho_I\gamma}=1/3$, then, allows us to use our earlier value of $\tilde{\Pi}_{\rho\omega}=-3500\ \text{MeV}^2$ [1]. Equation (16) defines the branching ratio in terms of the phenomenologically well-constrained fitting functions $F_\rho(s)$ and $\tilde{\Pi}_{\rho\omega}/3$ and thus avoids the explicit introduction of ρ resonance parameters. The model dependence of Eq. (16) is therefore minimal, and for this reason it is our preferred definition.

To assess its utility, we shall compare it with other definitions in the literature. We may also use Eq. (7) to replace $g_{\rho_I\pi\pi}$ and write $s_\rho=s-m_\rho^2+im_\rho\Gamma_\rho$ to find

$$B^{(2)}(\omega\to\pi^+\pi^-)=\frac{m_\rho^2(m_\omega^2-4m_\pi^2)^{3/2}}{m_\omega^2(m_\rho^2-4m_\pi^2)^{3/2}}\frac{\Gamma_\rho}{\Gamma_\omega}\left|\frac{\tilde{\Pi}_{\rho\omega}(m_\omega^2)}{m_\omega^2-m_\rho^2+im_\rho\Gamma_\rho}\right|^2, \tag{17}$$

where we have used $\Gamma_\rho=\Gamma(\rho\to\pi^+\pi^-)$. If we set $m_\omega=m_\rho$, Eq. (17) becomes that used in Ref. [12] to extract $\tilde{\Pi}_{\rho\omega}(m_\omega^2)=-4520\ \text{MeV}^2$ [13], a value commonly used in the literature [14]. We prefer determining both $\tilde{\Pi}_{\rho\omega}(m_\omega^2)$ and $B(\omega\to\pi^+\pi^-)$ directly from our fits to the $e^+e^-\to\pi^+\pi^-$ data. Yet another expression for $B(\omega\to\pi^+\pi^-)$ results if we consider Eq. (15) in place of Eq. (14) for $g^{\rm eff}_{\omega\pi\pi}$; that is,

$$B^{(3)}(\omega\to\pi^+\pi^-)=\frac{m_\rho^2(m_\omega^2-4m_\pi^2)^{3/2}}{m_\omega^2(m_\rho^2-4m_\pi^2)^{3/2}}\frac{\Gamma_\rho}{\Gamma_\omega}\left|\frac{\overline{\Pi}_{\rho\omega}(m_\omega^2)}{z_\omega-z_\rho}\right|^2, \tag{18}$$

where $\overline{\Pi}_{\rho\omega}(m_\omega^2)\equiv G(z_\omega-z_\rho)+\Pi_{\rho\omega}(z_\rho)$. $\overline{\Pi}_{\rho\omega}(m_\omega^2)$ is not determined directly in fits to $e^+e^-\to\pi^+\pi^-$ data and thus we favor Eqs. (16) or (17). Nevertheless, as we found no significant s-dependence to $\tilde{\Pi}_{\rho\omega}$ in our fits to $e^+e^-\to\pi^+\pi^-$ data [1], we will replace $\overline{\Pi}_{\rho\omega}(m_\omega^2)$ by $\tilde{\Pi}_{\rho\omega}(m_\omega^2)$ in our subsequent numerical estimates. Neglecting terms of $\mathcal{O}[(m_\omega-m_\rho)/m^{\rm av}]$, with $m^{\rm av}=(m_\rho+m_\omega)/2$, and setting $z_\rho=m_\rho^2+im_\rho\Gamma_\rho$, Eq. (18) yields

$$\Gamma(\omega\to\pi^+\pi^-)=\frac{|\overline{\Pi}_{\rho\omega}(m_\omega^2)|^2}{4m_\rho^2\left((m_\omega-m_\rho)^2+\frac{1}{4}(\Gamma_\omega-\Gamma_\rho)^2\right)}\Gamma_\rho, \tag{19}$$

and is thus equivalent to Eq. (B12) in Ref. [15]. So far we have freely changed from one realization of s_ρ to another; i.e., we have written both $s_\rho=s-z_\rho$ and $s_\rho=s-m_\rho^2+im_\rho\Gamma_\rho$. Yet it is important to recognize that for a broad resonance, such as the ρ (but unlike the ω), these realizations are not necessarily equivalent. A parametrization of $F_\rho(s)$ which explicitly suits the constraint of unitarity and time-reversal invariance, obliging its phase to be that of $l=1$, $I=1$ π-π scattering for s where the scattering is elastic [17,18,1], results in an s-dependent width [19]. Effectively, then, $(F_\rho(s))^{-1}\propto s-m_\rho^2+im_\rho\Gamma_\rho(s)$, where the m_ρ and Γ_ρ we have used thus far satisfy $\Gamma_\rho\equiv\Gamma_\rho(m_\rho^2)$. However, the ρ pole, z_ρ, in the complex s plane is determined by requiring $(F_\rho(z_\rho))^{-1}=0$. Thus, in the presence of a s-dependent width, $\Gamma_\rho(s)$, $z_\rho\neq m_\rho^2-im_\rho\Gamma_\rho$. If we parametrize z_ρ as

$$z_\rho\equiv\overline{m}_\rho^2-i\overline{m}_\rho\overline{\Gamma}_\rho, \tag{20}$$

then $\overline{m}_\rho$ and $\overline{\Gamma}_\rho$ differ substantially from m_ρ and Γ_ρ [20], as illustrated in Table I. Moreover, $\overline{m}_\rho$ and $\overline{\Gamma}_\rho$ are independent of the parametrization of $F_\rho(s)$ [21–23,20], whereas m_ρ and Γ_ρ are *not* [24–26,1]. In marked contrast to m_ρ and Γ_ρ given in Table I, the average values of $\overline{m}_\rho$ and $\overline{\Gamma}_\rho$,

$$\overline{m}_\rho = 757.0 \pm 1.1\,\text{MeV}, \quad \overline{\Gamma}_\rho = 141.3 \pm 3.1\,\text{MeV}, \tag{21}$$

are within one standard deviation of the $\overline{m}_\rho$ and $\overline{\Gamma}_\rho$ found in each and every model. This is in excellent agreement with Ref. [27], where the ρ parameters are found to be $\overline{m}_\rho = 757.5 \pm 1.5$ MeV and $\overline{\Gamma}_\rho = 142.5 \pm 3.5$ MeV. The stability shown here is that of the S-matrix pole position, z_ρ, which is model independent [21–23,20]. The separation of z_ρ into a ''mass'' and ''width,'' as in Eq. (20), though useful [21], is somewhat artificial, as Re($\sqrt{z_\rho}$) and $\sqrt{\text{Re}\, z_\rho}$ could equally well serve as the mass [28]. We shall consider the consequence of $z_\rho \neq m_\rho^2 - i m_\rho \Gamma_\rho$ on the numerical values of $B^{(3)}(\omega \to \pi^+\pi^-)$.

It should also be noted that the value of $\tilde{\Pi}_{\rho\omega}(m_\omega^2)$ to be used in Eqs. (17) and (18) can be determined from our previous, averaged result [1], noting Eq. (4), through

$$\tilde{\Pi}_{\rho\omega}(m_\omega^2) = \frac{1}{3}\frac{f_{\rho_I\gamma}}{f_{\omega_I\gamma}}(-3500 \ \ \text{MeV}^2). \tag{22}$$

We must therefore now determine the leptonic couplings $f_{\rho_I\gamma}$ and $f_{\omega_I\gamma}$.

III. VECTOR MESON ELECTROMAGNETIC COUPLINGS

We have related the branching ratio $B(\omega \to \pi^+\pi^-)$ to the effective mixing term $\tilde{\Pi}_{\rho\omega}(s)$ and various vector-meson parameters, yet in order to fix $\tilde{\Pi}_{\rho\omega}$ in a fit to $e^+e^- \to \pi^+\pi^-$ data, we need to determine the ratio $r_\gamma \equiv f_{\rho_I\gamma}/f_{\omega_I\gamma}$. In the SU(6) limit $r_\gamma = 3$, but this relation is broken at the $\sim 10\%$ level [10] by the large ρ width [19,29]. In this section we discuss the extraction of $f_{\rho_I\gamma}$ and $f_{\omega_I\gamma}$.

The vector-meson–photon coupling constant $f_{V\gamma}$ is related to the leptonic decay width $\Gamma(V \to l^+ l^-)$ through

$$\Gamma(V \to \ell^+\ell^-) = \frac{4\pi\alpha^2}{3m_V^3} f_{V\gamma}^2, \tag{23}$$

noting that lepton masses enter at $\mathcal{O}((m_\ell/m_V)^4)$ [10]. The cross-section for $e^+e^- \to \pi^+\pi^-$, proceeding solely through $e^+e^- \to \rho^0 \to \pi^+\pi^-$, that is, assuming no background, for $s = m_\rho^2$ is

$$\begin{aligned}\sigma(e^+e^- \to \rho_I \to \pi^+\pi^-) &= \frac{\pi\alpha^2}{3}\frac{(s-4m_\pi^2)^{3/2}}{s^{5/2}}\frac{(f_{\rho_I\gamma}g_{\rho_I\pi\pi})^2}{(s-m_\rho^2)^2 + m_\rho^2\Gamma_\rho^2}\Bigg|_{s=m_\rho^2} \\ &= 12\pi\frac{\Gamma(\rho_I \to e^+e^-)\Gamma(\rho_I \to \pi^+\pi^-)}{m_\rho^2\Gamma_\rho^2},\end{aligned} \tag{24}$$

where we have used Eqs. (7) and (23). This is a particular case of the Cabibbo-Gatto relation for a resonant, spin-one interaction [30], valid for any hadronic final state. Thus, an analogous ''Cabibbo-Gatto'' formula exists for $e^+e^- \to \omega \to \pi^+\pi^0\pi^-$. In this manner, $\Gamma(\omega \to e^+e^-)$ and $f_{\omega_I\gamma}$, via Eq. (23), can both be inferred from the $e^+e^- \to \pi^+\pi^0\pi^-$ data [31]. We use $\Gamma(\omega \to e^+e^-) = 0.60 \pm .02$ keV [26] in what follows. We can now calculate $\Gamma(\rho^0 \to e^+e^-)$ and hence $f_{\rho_I\gamma}$. Recalling Eq. (4), we find

$$\sigma(e^+e^- \to \rho_I \to \pi^+\pi^-) = \frac{\pi\alpha^2}{3}\frac{(s-4m_\pi^2)^{3/2}}{s^{5/2}}|F_\rho(s)|^2\Bigg|_{s=m_\rho^2}, \tag{25}$$

which when combined with Eq. (24) yields

$$\Gamma(\rho_I^0 \to e^+e^-) = \frac{\alpha^2}{36}\frac{(m_\rho^2 - 4m_\pi^2)^{3/2}}{m_\rho^3}|F_\rho(m_\rho^2)|^2\Gamma_\rho, \tag{26}$$

where $\Gamma_\rho = \Gamma(\rho \to \pi^+\pi^-)$, allowing us to determine $f_{\rho_I\gamma}$ from Eq. (23).

We note in passing that it is quite common in the literature to see the ω contribution to the pion form-factor expressed in terms of ω partial widths [2,32]. Such an expression follows from our Eq. (5), in concert with Eqs. (7) and (23), to yield

$$F_\pi(s) = F_\rho(s) + \sqrt{\frac{36\Gamma(\omega \to e^+e^-)\Gamma(\omega \to \pi^+\pi^-)}{m_\omega^2\alpha^2\beta_\omega^3}} \times \frac{m_\omega^2}{s - m_\omega + i m_\omega\Gamma_\omega}, \tag{27}$$

where $\beta_\omega = (1 - 4m_\pi^2/m_\omega^2)^{1/2}$ and we replace $f_{\rho_I\gamma}g_{\rho_I\pi\pi}/(s - m_\rho + i m_\rho\Gamma_\rho)$ with $F_\rho(s)$ as earlier. Thus, our Eq. (16) is explicitly equivalent to the determinations of $B(\omega \to \pi^+\pi^-)$ found in Refs. [2,32].

IV. RESULTS AND DISCUSSION

We now use our fits of Ref. [1] to compute $B(\omega \to \pi^+\pi^-)$, $\Gamma(\rho \to e^+e^-)$, and other associated parameters, in addition to their errors. Our fits to the pion form-factor data [3], noting Eq. (4), adopt parametrizations of $F_\rho(s)$ consistent with the following theoretical constraints. That is, analyticity requires that $F_\rho(s)$ be real below threshold, $s = 4m_\pi^2$, charge conservation requires $F_\rho(0) = 1$, and unitarity and time-reversal invariance requires its phase be that of $l=1$, $I=1$ π-π scattering for s where the latter is elastic [18]. For the present work we shall concentrate on four of these choices for $F_\rho(s)$, labeled, as per Ref. [1], A, B, C, and D, in which $\tilde{\Pi}_{\rho\omega}$ is an explicit fitting parameter. These four fits assume $\tilde{\Pi}_{\rho\omega}$ to be a real constant in the resonance region, for the current $e^+e^- \to \pi^+\pi^-$data supports neither a phase nor s-dependent pieces [1].

Table I shows our results for $\Gamma(\rho^0 \to e^+e^-)$ and $B(\rho^0$

TABLE II. Results for the effective ρ^0-ω mixing element, $\tilde{\Pi}_{\rho\omega}$, and the branching ratio $B(\omega\to\pi^+\pi^-)$, from Eq. (16), using the fits of Ref. [1]. $f_{\omega_I\gamma}$ follows from Eq. (23) and the parameters of Ref. [26]. We also show the value of $\tilde{\Pi}_{\rho\omega}$ which results from using Eq. (22) with $f_{\rho_I\gamma}/f_{\omega_I\gamma}$, as per Eqs. (23) and (24), again using the fits of Ref. [1].

Fit	$\tilde{\Pi}_{\rho\omega}(m_\omega^2)$(MeV^2) [1]	$f_{\rho_I\gamma}$ (GeV^2)	$f_{\rho_I\gamma}/f_{\omega_I\gamma}$	$\tilde{\Pi}_{\rho\omega}(m_\omega^2)$(MeV^2)	$B(\omega\to\pi^+\pi^-)$
A	-3460 ± 290	0.120 ± 0.001	3.36 ± 0.07	-3870 ± 320	$1.87\pm0.30\%$
B	-3460 ± 290	0.122 ± 0.001	3.40 ± 0.06	-3920 ± 330	$1.87\pm0.30\%$
C	-3460 ± 290	0.122 ± 0.001	3.41 ± 0.06	-3930 ± 330	$1.87\pm0.30\%$
D	-3460 ± 290	0.118 ± 0.001	3.30 ± 0.06	-3800 ± 330	$1.87\pm0.30\%$

$\to e^+e^-)\equiv\Gamma(\rho^0\to e^+e^-)/\Gamma_\rho$ as determined from Eq. (26). We find the following average values:

$$\Gamma(\rho^0\to e^+e^-)=7.11\pm0.08\pm0.25\,\text{keV},$$

$$B(\rho^0\to e^+e^-)=(4.63\pm0.05\pm0.07)\times10^{-5}, \qquad (28)$$

where the second error on $\Gamma(\rho^0\to e^+e^-)$ is the theoretical systematic error associated with the model choice [33], and all other errors are statistical. Γ_ρ from Fit D is significantly lower than those from the other fits and leads to a significantly lower value for $\Gamma(\rho^0\to e^+e^-)$, indeed one commensurate with the value of $6.77\pm0.10\pm0.30$ keV reported in Ref. [2]. This is likely consequent to the choice of the Gounaris-Sakurai form factor [19] in both fits; our other fits use a Heyn-Lang form factor [18]. Such model dependence also plagues the extraction of the ρ parameters m_ρ and Γ_ρ, as discussed following Eq.(20).

Using $\Gamma(\rho_I\to e^+e^-)$ of Table I and Eq. (23) yields $f_{\rho_I\gamma}$ and r_γ, using $f_{\omega_I\gamma}$ computed from $\Gamma(\omega\to e^+e^-)$ of Ref. [26]. In the SU(6) limit r_γ is 3; the ''finite width'' correction [19,29], as seen in Table II, is $\sim10\%$, as also found in Ref. [10], and hence significant. Including this correction as per Eq. (22) gives us perhaps a more realistic value of $\tilde{\Pi}_{\rho\omega}(m_\omega^2)$ [34], and its model dependence appears to be modest, allowing us to determine an average value of

$$\tilde{\Pi}_{\rho\omega}(m_\omega^2)=-3900\pm300\ \ \text{MeV}^2, \qquad (29)$$

again some 10% larger than our value of $\tilde{\Pi}_{\rho\omega}(m_\omega^2)=-3500\pm300\,\text{MeV}^2$ in Ref. [1] using $r_\gamma=3$.

Our preferred determination of $B(\omega\to\pi^+\pi^-)$, Eq. (16), does not require r_γ, and we find

$$B(\omega\to\pi^+\pi^-)=1.9\pm0.3\%. \qquad (30)$$

Barkov *et al.*, noting Eq. (27) and the discussion thereafter, obtain $B(\omega\to\pi\pi)=2.3\pm0.4\%$ [2] with the same data set [3] used here. We agree closely, however, with the result of Bernicha *et al.*, $1.85\pm0.30\%$ [27], obtained from the same data [3]. Their relation for the branching ratio, Eq. (42) [27], is our Eq. (17), though they use the parameters m_ρ and Γ_ρ, noting Eq. (20), in place of m_ρ and Γ_ρ and use $\Gamma_{\rho\to e^+e^-}=6.77$ keV to compute the leptonic coupling $f_{\rho_I\gamma}$ [27]. The latter effects compensate, so that we would expect to find a branching ratio comparable to theirs. The data set we have adopted [3] contains 30 data points for center of mass energies between 750 and 810 MeV, the region likely most relevant for the determination of $\Gamma(\omega\to\pi^+\pi^-)$. The older work of Benaksas *et al.* [32], which uses Eq. (27), and Quenzer *et al.* [35] find $B(\omega\to\pi\pi)=3.6\pm0.4\%$ and $B(\omega\to\pi\pi)=1.6\pm0.9\%$, respectively, though both experiments possess less than 10 data points in the energy region of interest.

We can also compute $B(\omega\to\pi^+\pi^-)$ using Eqs. (17) or (18) and (22), as shown in Table III. Apparently it makes little difference whether we use Eq. (16) or Eq. (17), though the former, our preferred analysis, possesses essentially no parametrization dependence. $B^{(3)}(\omega\to\pi\pi)$, from Eq. (18), is substantially larger, though this may be an artifact of using the true S-matrix pole position z_ρ in Eq. (18). If we were to

TABLE III. The branching ratio $B(\omega\to\pi^+\pi^-)$ from our preferred method, Eq. (16), compared with the alternatives $B^{(2)}(\omega\to\pi^+\pi^-)$, Eq. (17), and $B^{(3)}(\omega\to\pi^+\pi^-)$, Eq. (18). In parentheses we give the values for the branching ratio as determined by Eq. (18), but replace z_ρ with $m_\rho^2-im_\rho\Gamma_\rho$, noting the discussion preceding Eq. (20) and the results of Table I.

Fit	$B(\omega\to\pi^+\pi^-)$	$B^{(2)}(\omega\to\pi^+\pi^-)$	$B^{(3)}(\omega\to\pi^+\pi^-)$
A	$1.87\pm0.30\%$	$1.93\pm0.32\%$	$2.41\pm0.39\%$ ($2.15\pm0.35\%$)
B	$1.87\pm0.30\%$	$1.97\pm0.32\%$	$2.50\pm0.40\%$ ($2.19\pm0.35\%$)
C	$1.87\pm0.30\%$	$1.96\pm0.32\%$	$2.51\pm0.40\%$ ($2.19\pm0.35\%$)
D	$1.87\pm0.30\%$	$1.95\pm0.32\%$	$2.20\pm0.37\%$ ($2.20\pm0.35\%$)

replace z_ρ with $m_\rho^2 - i m_\rho \Gamma_\rho$, noting the discussion surrounding Eq. (20), the values, as shown in parentheses, would differ less, even though we were obliged to assume that $\overline{\Pi}_{\rho\omega}(m_\omega^2)$ and $\tilde{\Pi}_{\rho\omega}(m_\omega^2)$ are the same.

In summary, we have elucidated the connection between $\tilde{\Pi}_{\rho\omega}(m_\omega^2)$ and $B(\omega\to\pi^+\pi^-)$ and shown how different methods of determining $B(\omega\to\pi^+\pi^-)$ would be equivalent were it possible to evaluate $\tilde{\Pi}_{\rho\omega}(z_\omega)$. In practice, the methods are different, yet, nevertheless, it seems that a plurality of methods of computing $B(\omega\to\pi^+\pi^-)$ yield roughly comparable results.

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