Flavor-symmetry restoration and Symanzik improvement for staggered quarks

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We resolve contradictions in the literature concerning the origins and size of unphysical flavor-changing strong interactions generated by the staggered-quark discretization of QCD. We show that the leading contributions are tree level in $\mathcal{O}(a^2)$ and that they can be removed by adding three correction terms to the link operator in the standard action. These corrections are part of the systematic Symanzik improvement of the staggered-quark action. We present a new improved action for staggered quarks that is accurate up to errors of $\mathcal{O}(a^4,a^2\alpha_s)$—more accurate than most, if not all, other discretizations of light-quark dynamics. [S0556-2821(99)00107-1]

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I. INTRODUCTION

The staggered-quark discretization of QCD has several advantages for numerical simulations. In particular, numerical evaluation of quark propagators is far more efficient with staggered quarks than with any other standard discretization. This is especially true for small quark masses where a chiral symmetry of the staggered-quark action prohibits additive mass renormalizations, in close analogy with continuum QCD. This chiral symmetry also implies that errors caused by the nonzero grid spacing, a, are automatically quadratic in a, rather than linear as in Wilson's formulation. These advantages are offset by the fact that staggered quarks always come in groups of four identical flavors, which is too many for QCD. The multiplication of flavors is not in itself a major problem; the effective number of flavors is easily adjusted in simulations. The real problem is that the standard staggered-quark action allows relatively large flavor-changing strong interactions, which are completely absent in real QCD. These flavor-changing interactions greatly complicate the interpretation of staggered-quark simulations; a deeper understanding of their origins is essential for precision work with staggered quarks, particularly at larger lattice spacings.

Recently, in [1,2], it was observed that the dominant flavor-changing interaction is due to one-gluon exchange between quarks. It was also observed that this interaction is a short-distance lattice artifact of relative order a^2 which can be removed by tree-level modifications of the lattice action. A more recent study, however, contradicts this conclusion by arguing that the one-gluon mechanism is redundant in $\mathcal{O}(a^2)$, and therefore that flavor-changing interactions are higher order in a or α_s [3]. In this paper we resolve this contradiction by showing that one-gluon exchange is indeed responsible for the dominant flavor-changing interaction, as originally argued; the flavor-changing effects are not redundant in $\mathcal{O}(a^2)$. Furthermore, these effects can be canceled by a simple modification of the quark-gluon interaction, similar to but more complicated than the ''fat-link'' improvement shown to reduce flavor-changing effects in [4]. Our fat link is similar to that in [2], but less complicated.

Fat-link improvement was first introduced as a heuristic scheme for reducing flavor-changing interactions in staggered quarks. In fact it is part of the systematic Symanzik improvement of the lattice quark action. As mentioned above, the standard action has no errors in $\mathcal{O}(a)$ (for light quarks). There are only two sources of $\mathcal{O}(a^2)$ error at the tree level. One is the flavor-changing interaction due to one-gluon exchange; the other is a kinetic term common to all standard quark discretizations [5,3]. In this paper, we show how to correct for both of these. The result is a lattice quark action that is accurate up to errors of $\mathcal{O}(a^4,a^2\alpha_s)$—the most accurate to date [6].

II. FLAVOR-CHANGING INTERACTIONS

The staggered-quark formalism is derived from the simplest discretization of the quark action in QCD, the ''naive'' quark action:

$$\mathcal{S}=\sum_x \bar{\psi}(x)(\gamma\cdot\Delta+m)\psi(x). \tag{1}$$

Here Δ_μ is a discrete version of the covariant derivative,

$$\Delta_\mu\psi(x)\equiv\frac{1}{2au_0}(U_\mu(x)\psi(x+a\hat{\mu}) - U_\mu^\dagger(x-a\hat{\mu})\psi(x-a\hat{\mu})), \tag{2}$$

$U_\mu(x)$ is the gluon link-field, and u_0 is the mean-link tadpole improvement [7,1]. This action has an exact ''doubling'' symmetry under the transformation

$$\psi(x)\to\tilde{\psi}(x)\equiv i\gamma_5\gamma_\rho(-1)^{x_\rho/a}\psi(x) = i\gamma_5\gamma_\rho\exp(ix_\rho\pi/a)\psi(x). \tag{3}$$

Thus any low energy-momentum mode, $\psi(x)$, of the theory is equivalent to another mode, $\tilde{\psi}(x)$, that has momentum $p_\rho\approx\pi/a$, the maximum allowed on the lattice. This new mode is one of the ''doublers'' of the naive quark action. The doubling transformation can be applied successively in two or more directions; the general transformation is

$$\psi(x)\to\tilde{\psi}(x)\equiv\prod_{\rho}\ (i\gamma_5\gamma_\rho)^{\zeta_\rho}\exp(ix\cdot\zeta\pi/a)\psi(x) \qquad (4)$$

where ζ is a vector with one or more components equal to 1 and all the others 0. Consequently there are 15 doublers in all (in four dimensions), which we label with the fifteen different ζ's.

As a consequence of the doubling symmetry, the standard low-energy mode and the fifteen doubler modes must be interpreted as sixteen equivalent flavors of quark. (The sixteen flavors are reduced to four by staggering the quark field, as we discuss at the end.) This unusual implementation of quark flavors has surprising consequences. Most striking is that a low-energy quark that absorbs momentum close to $\zeta\pi/a$, for one of the fifteen ζ's, is not driven far off energy-shell. Rather it is turned into a low-energy quark of another flavor. Thus the simplest process by which a quark changes flavor is the emission of a single gluon with momentum $q\approx\zeta\pi/a$. This gluon is highly virtual, with $q^2=\mathcal{O}((\pi/a)^2)$, and therefore it must immediately be reabsorbed by another quark, whose flavor will also change. Flavor changes necessarily involve highly virtual gluons, and so are perturbative for typical lattice spacings. Consequently one-gluon exchange, with gluon momentum $q\approx\zeta\pi/a$, is the dominant flavor-changing interaction since it is lowest order in $\alpha_s(\pi/a)$.

Flavor-changing gluon exchanges between quarks are in effect identical to flavor-changing quark-quark contact interactions since the exchanged gluons are highly virtual. Thus the effects of these exchanges can be canceled by adding four-quark interactions to the quark action [1]. The contact terms required at the tree level were explicitly constructed in [3]; these could be added to the staggered-quark action and the leading flavor-changing interactions removed. There is, however, a simpler modification of the action that accomplishes the same goal. This is to change the quark-gluon coupling in the naive action so as to suppress gluon momenta near $\zeta\pi/a$ for each of the ζ's [8]. For example, defining a covariant second derivative that acts on link operators,

$$\begin{aligned}\Delta^{(2)}_\rho U_\mu(x)\equiv\frac{1}{u_0^2a^2}(&U_\rho(x)U_\mu(x+a\hat{\rho})U^\dagger_\rho(x+a\hat{\mu})\\&-2u_0^2U_\mu(x)+U^\dagger_\rho(x-a\hat{\rho})U_\mu(x-a\hat{\rho})\\&\times U_\rho(x-a\hat{\rho}+a\hat{\mu})),\end{aligned} \qquad (5)$$

the operator

$$\left(1+\sum_{\rho\neq\mu}\frac{a^2\Delta^{(2)}_\rho}{4}\right)U_\mu(x) \qquad (6)$$

is identical to the link operator for low gluon momenta, up to errors of $\mathcal{O}(a^2)$, but vanishes when a single gluon with momentum $q_\rho=\pi/a$ is extracted. Replacing the links in the naive action by this operator would remove flavor-changing interactions with $\zeta^2=1$ (that is, one component equal to 1 and all others 0). This is essentially what was done in the fat-link improvement scheme presented in [4]. To remove all leading-order flavor-changing interactions, we replace

$$U_\mu(x)\to V_\mu(x)\equiv\left(1+\sum_{\zeta}\ (1-\zeta_\mu)c(\zeta^2)\mathcal{P}(\zeta)\right)U_\mu(x) \qquad (7)$$

in the naive action, where

$$\mathcal{P}(\zeta)\equiv\prod_\rho\left(\frac{a^2\Delta^{(2)}_\rho}{4}\right)^{\zeta_\rho}\Bigg|_{\text{symmetrized}} \qquad (8)$$

is symmetrized over all possible orderings of the operators. We drop corrections to V_μ with $\zeta_\mu=1$ since the other parts of the corresponding quark-gluon vertex vanish when the gluon has $q_\mu=\pi/a$ (just as in the original action). The coefficients $c(\zeta^2)$ have perturbative expansions:

$$c(\zeta^2)=1+\mathcal{O}(\alpha_s(\pi/a)). \qquad (9)$$

Tree-level values, with tadpole-improved operators, should be sufficiently accurate for most applications at typical lattice spacings. Alternatively, we can tune the coefficients nonperturbatively, for example to remove flavor splittings between pions. (In four dimensions there are three independent c's just as there are three independent flavor-nonsinglet pion splittings for small quark masses [9].) Such tuning maximizes the cancellation of residual tadpole effects. Taking just the tree-level values for the couplings, V_μ simplifies to

$$V_\mu(x)=\prod_{\rho\neq\mu}\left(1+\frac{a^2\Delta^{(2)}_\rho}{4}\right)\Bigg|_{\text{symm.}}U_\mu(x)\quad(\text{tree level}). \qquad (10)$$

Here the prefactors vanish when acting on a gluon field that has any momentum component other than q_μ equal to π/a; as mentioned above, momenta with $q_\mu=\pi/a$ are suppressed by the remainder of the quark-gluon vertex. The action in [2] is similar to ours, but has three additional operators, for suppressing $q_\mu=\pi/a$, that are unnecessary.

Our prescription for removing the leading flavor-changing interactions is to replace the link field U_μ in the naive action by the link field V_μ defined above. As in the standard case, our improved naive action is equivalent to four identical, uncoupled staggered-quark theories. Three of these can be removed by staggering the quark spinors over nearby lattice sites, thereby reducing the number of flavors to four (in four dimensions). Our final staggered theory is identical to the standard theory but with the substitution: $U_\mu\to V_\mu$. The additional computing required to implement this improvement is negligible.

III. $\mathcal{O}(A^2)$ SYMANZIK IMPROVEMENT

As mentioned in the previous section, the leading flavor-changing interactions can be canceled by adding four-quark contact interactions to the lattice action. This means that the errors introduced by these interactions are $\mathcal{O}(a^2)$ at the tree level in the original theory [1,2]. In [3] the relevant contact interactions were converted into quark bilinears by trans-

forming the gluon field in the path integral. The bilinears that resulted were of the form

$$a^2\overline{\psi}(x)\gamma_\mu(\mathcal{P}(\zeta)\Delta^{(2)}U_\mu(x))\psi(x+a\hat{\mu}). \tag{11}$$

Such operators appear to be of relative order a^4 and higher, and thus they were dropped in the $\mathcal{O}(a^2)$ analysis of [3]. It was argued that the original contact terms could be transformed away, that they were redundant in $\mathcal{O}(a^2)$. This is incorrect. The quark bilinear, despite appearances, affects the theory in $\mathcal{O}(a^2)$. While the factors of $a^2\Delta_\rho^{(2)}$ acting on the link field $U_\mu(x)$ in the bilinear operator strongly suppress quark-gluon interactions for small gluon momenta, these same factors are of order unity for the large, flavor-changing gluon momenta that leave the quark on-shell. Thus the flavor-changing parts of these terms are of the same order as the flavor-changing part of the original quark-gluon vertex; that is, they are $\mathcal{O}(a^2)$.

Conventional power-counting, as assumed in [3] and elsewhere, is incorrect for a theory in which a highly virtual gluon can connect two on-shell quarks. Just about any quark bilinear will contribute flavor-changing interactions of this sort. In such situations additional factors of $a^2\Delta_\rho^{(2)}$ acting on the gluon field bring no additional suppression of the bilinear's contribution. The correct power counting for the flavor-changing part of a quark bilinear can always be determined by examining the corresponding quark-quark contact terms.

The $\mathcal{O}(a^2)$ corrections introduced when U_μ is replaced by V_μ cancel all tree-level flavor-changing interactions in that order. It is straightforward to remove the remaining errors in that order to obtain an $\mathcal{O}(a^2)$ Symanzik improved quark action. First we must cancel a flavor-*conserving* order a^2 error due to the $\zeta^2=1$ parts of V_μ. This is removed by further modifying V_μ:

$$V_\mu(x)\to V'_\mu(x)\equiv V_\mu(x)-\sum_{\rho\neq\mu}\frac{a^2(\Delta_\rho)^2}{4}U_\mu(x). \tag{12}$$

Here the $(\Delta_\rho)^2$'s are squares of discretized covariant first-order derivatives for links, defined analogously to Eq. (2); they cancel the low-energy effects of the single $\Delta_\rho^{(2)}$'s in V_μ without affecting their high-momentum behavior. Second we must correct the discretization of the derivative through $\mathcal{O}(a^2)$ by replacing

$$\Delta\cdot\gamma\to\left(\Delta_\mu-\frac{a^2}{6}(\Delta_\mu)^3\right)\gamma_\mu \tag{13}$$

in the action [5,3]. While the first of these terms requires the $V'_\mu(x)$ link field in place of $U_\mu(x)$, this is unnecessary in the Δ_μ^3 term. The additional first-order derivatives in that term connect quark fields separated by an even number of lattice spacings, and therefore they suppress both flavor-conserving and flavor-changing interactions [to $\mathcal{O}(a^4,a^2\alpha_s)$]. Thus our $\mathcal{O}(a^2)$-accurate action for naive quarks is

$$\mathcal{S}=\sum_x\ \overline{\psi}(x)\left(\gamma\cdot\Delta'-\frac{a^2}{6}\gamma\cdot\Delta^3+m\right)\psi(x) \tag{14}$$

where

$$\Delta'_\mu\psi(x)\equiv\frac{1}{2au_0}(V'_\mu(x)\psi(x+a\hat{\mu}) - V'^{\dagger}_\mu(x-a\hat{\mu})\psi(x-a\hat{\mu})). \tag{15}$$

The quark fields for this action can again be staggered in the standard fashion to obtain an a^2-accurate staggered-quark action for only four flavors of quark (in four dimensions):

$$\mathcal{S}_{\text{stagg.}}=\sum_x\ \eta_\mu(x)\overline{\chi}(x)\left(\Delta'_\mu-\frac{a^2}{6}\Delta_\mu^3\right)\chi(x)+m\overline{\chi}(x)\chi(x) \tag{16}$$

where $\chi(x)$ is a one-component quark field and

$$\eta_\mu(x)\equiv(-1)^{x_1/a+\cdots+x_{\mu-1}/a}. \tag{17}$$

We have already noted that the coupling constants in V_μ (and V'_μ) can be tuned nonperturbatively, if desired, to remove flavor-splittings between different pions in the staggered theory. The kinetic correction can also be tuned nonperturbatively using a symmetry, in this case Lorentz invariance (relativity). For example, the coefficient of this term might be tuned to adjust

$$c^2(\mathbf{p})\equiv\frac{E^2(\mathbf{p})-m^2}{\mathbf{p}^2} \tag{18}$$

to $c^2=1$ for the pion with the lowest nonzero momentum.

IV. CONCLUSIONS

In this paper we have established that the leading flavor-changing interactions in the naive and staggered-quark actions are short-distance, tree-level, $\mathcal{O}(a^2)$ effects. They are most easily removed by modifying the link field in the original action. The minimal number of correction terms required is three. Other schemes for fattening the link operators have shown some success at reducing flavor-splittings [4,2]. Our scheme should be at least as successful, and probably substantially more so if the coefficients of the correction terms are tuned to remove all residual tadpole effects (for example, by completely removing the flavor-nonsinglet pion splittings).

The flavor-changing interactions in the standard action are not redundant at tree-level in $\mathcal{O}(a^2)$. They must be corrected to obtain an a^2-accurate, Symanzik-improved action. The fat link we introduce for this purpose is not heuristic, but rather a consequence of the systematic Symanzik improvement of the action. In this paper we presented a new Symanzik-improved action for staggered quarks that is accurate through the tree level in $\mathcal{O}(a^2)$. As such it is more accurate than most other discretizations of quark dynamics, and is therefore a prime candidate for use on lattices with large lattice spacings, in the range 0.1–0.3 fm.

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