More on membranes in matrix theory

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We study noncompact and static membrane solutions in matrix theory. Demanding axial symmetry on a membrane embedded in three spatial dimensions, we obtain a wormhole solution whose shape is the same with the catenoidal solution of Born-Infeld theory. We also discuss another interesting class of solutions, membranes embedded holomorphically in four spatial dimensions, which are 1/4 BPS. [S0556-2821(99)01306-5]

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Matrix theory [1], formulated from the U(N) gauge supersymmetric quantum mechanics describing motions of N $D0$-branes, is now well established as a nonperturbative formulation of M-theory. The basic properties of M-theory which could be readily checked were that it becomes 11-dimensional supergravity at low energy, it contains extended objects such as a membrane and its magnetic dual five-brane, and the dynamics of D-particles of type-IIA superstring theory converges to the supermembrane action in the light cone frame, formulated in [2]. In addition to the rather trivial flat branes, compact branes, in particular, with spherical topologies, were obtained and studied also [3,4,5]. These branes with finite size are not static, but oscillating in time. Recently, an interesting class of static membrane solutions was presented in [6], viz., holomorphically embedded membranes. In this Brief Report we investigate the possibility of another static solution of matrix theory. We find one embedded in three dimensions which describes wormhole membranes, made by a string between branes and antibranes. A similar system was already discussed using Born-Infeld (BI) theory [7,8], which is low-energy effective theory for Dp-branes. It is amusing to see that our solution has the same shape as the electrostatic solution of three-dimensional Born-Infeld theory, describing two $D2$-branes connected by a throat.

In this Brief Report it is sufficient to consider only the bosonic sector of the matrix theory Lagrangian:

$$L=\mathrm{Tr}\left(\frac{1}{2g}(D_tX^I)^2+\frac{1}{4g}([X^I,X^J]^2)\right), \tag{1}$$

where X^I, $I=1,\ldots,9$, are $N\times N$ Hermitian matrices and g is the string coupling constant. $D_tX\equiv\partial_tX-i[A,X]$, where $A=X^0$ is the gauge field of our gauged quantum mechanics. It is well known that matrix theory, in the large-N limit, converges to the 11-dimensional supermembrane theory in the light cone frame. In other words, Eq. (1) becomes

$$L=\frac{p_{11}}{2}\int dp\,dq(D_tX^I)^2+\frac{1}{4p_{11}}\int dp\,dq(\{X^I,X^J\})^2; \tag{2}$$

viz., the trace is approximated by integration over two variables and the matrix commutator by i times Poisson brackets,

$$\{X(p,q),Y(p,q)\}\equiv i\left(\frac{\partial X}{\partial q}\frac{\partial Y}{\partial p}-\frac{\partial X}{\partial p}\frac{\partial Y}{\partial q}\right). \tag{3}$$

q,p are the world surface coordinates of the membrane. The equation of motion for the above Lagrangian is

$$D_t^2X^I+[X^J,[X^J,X^I]]=0 \tag{4}$$

or, in the large-N limit,

$$D_t^2X^i+\{X^j,\{X^j,X^i\}\}=0. \tag{5}$$

For simplicity, we set the coupling constants to 1 from now on.

The simplest solution is the flat membrane

$$X^1=\sqrt{\frac{q}{2}}\cos p,$$

$$X^2=\sqrt{\frac{q}{2}}\sin p,$$

$$X^J=0,\quad J\neq 1,2, \tag{6}$$

where p is an angular variable and q can take any real number. It is obvious that the above parametrization expands the entire range of the 12-plane. For matrix theory we take the ladder operators a, $a^\dagger$ of the harmonic oscillator and set $X^1=(a+a^\dagger)/\sqrt{2}$, $X^2=i(a-a^\dagger)/\sqrt{2}$. This basic membrane solution is 1/2 Bogomol'nyi-Prasad-Sommerfield (BPS) of matrix theory.

Another interesting example is that of spherical shape:

$$X^1=r(t)\sqrt{1-q^2}\cos p,$$

$$X^2=r(t)\sqrt{1-q^2}\sin p,$$

$$X^3=r(t)q,$$

$$X^J=0,\quad J\neq 1,2,3. \tag{7}$$

We have S^2 with radius $r(t)$, which is not a constant, but oscillates according to the equation $\ddot{r}+2r^3=0$. Note that in this case we can obtain a finite-dimensional representation of the solution, i.e., $X^i=r(t)J_i$ $(i=1,2,3)$, where J_i are the familiar angular momentum operators. But for the infinite flat membranes, we cannot satisfy the equation of motion in terms of matrices with finite size.

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Lately, Cornalba and Taylor studied the problem of finding static solutions in matrix theory [6]. They studied solutions which can be represented by holomorphic curves, i.e., membranes embedded holomorphically in $\mathbb{C}^4$. To define holomorphically embedded membranes, we introduce four complex coordinates

$$\begin{aligned} Z_1 &= X^1 + iX^2, \\ Z_2 &= X^3 + iX^4, \\ Z_3 &= X^5 + iX^6, \\ Z_4 &= X^7 + iX^8, \end{aligned} \tag{8}$$

and truncate the last spatial coordinate $X^9=0$. By a holomorphic curve we mean they are all holomorphic functions of one complex variable, $Z_A = f_A(z)$. After quantization, z is traded into an operator or matrix. In terms of the above complex variables, the potential term of the membrane theory can be written as

$$V = -\frac{1}{16}\sum_{A,B=1,\ldots,4}\int dq\,dp(\{Z_A,Z_B\}\{\overline{Z}_A,\overline{Z}_B\} + \{Z_A,\overline{Z}_B\}\{\overline{Z}_A,Z_B\}). \tag{9}$$

Assuming $\{z,\overline{z}\} = F(z,\overline{z})$ and minimizing the potential, we have

$$F(z,\overline{z})\left(\sum_A f'_A(z) f'_A(\overline{z})\right) = C, \tag{10}$$

where C is a constant. Note that this condition means

$$\sum_A \{Z_A,\overline{Z}_A\} = \sum_A \{f_A(z), f_A(\overline{z})\} = C. \tag{11}$$

This was introduced as a *gauge condition* in [6], but as we have seen from above this can be obtained as a result of the equation of motion for the membrane theory, and we can show that even in matrix theory, where Z_A become noncommutative, this condition makes the equation of motion trivially satisfied. So we admit the following is a BPS condition for a static membrane in matrix theory:

$$\sum_A [Z_A,\overline{Z}_A] = C. \tag{12}$$

In fact, this condition guarantees preservation of some proportion of supersymmetry. Consider the supersymmetry transformation of fermion fields in matrix theory,

$$\delta\theta = \frac{1}{2}\Gamma_I D_t X^J \epsilon + \frac{i}{4}\Gamma_{IJ}[X^I,X^J]\epsilon + \epsilon', \tag{13}$$

where $\Gamma_{IJ} = \frac{1}{2}[\Gamma_I,\Gamma_J]$ and Γ_I, $I=1,\ldots,9$ are nine-dimensional gamma matrices which satisfy $\Gamma_I\Gamma_J + \Gamma_J\Gamma_I = \delta_{IJ}\mathbf{1}_{16\times 16}$. There are two 16-component supersymmetry parameters ϵ, ϵ'; the former is an ordinary dynamic supersymmetry parameter, while the latter is called a kinematic one, which is common in light cone formulations. Since here we are interested in static BPS solutions without a nontrivial gauge field background, the first term vanishes. Assume we have a membrane solution holomorphically embedded in $\mathbb{C}^2$, viz., $Z = X^1 + iX^2$, $W = X^3 + iX^4$,

$$[Z,W] = [\overline{Z},\overline{W}] = 0, \tag{14}$$

and expand the second term of Eq. (13):

$$\begin{aligned} \frac{1}{4}\Gamma_{IJ}[X^I,X^J]\epsilon &= \frac{i}{4}(\Gamma_{12}[Z,\overline{Z}] + \Gamma_{34}[W,\overline{W}])\epsilon \\ &+ \frac{1}{8}(\Gamma_{13}+\Gamma_{24})([Z,\overline{W}] + [\overline{Z},W])\epsilon \\ &+ \frac{i}{8}(\Gamma_{14}-\Gamma_{23})([Z,\overline{W}] - [\overline{Z},W])\epsilon. \end{aligned} \tag{15}$$

Now, if we have

$$\Gamma_1\Gamma_2\Gamma_3\Gamma_4\epsilon = -\epsilon, \tag{16}$$

the second and third terms vanish. Now applying the BPS condition, Eq. (12),

$$[Z,\overline{Z}] + [W,\overline{W}] = C, \tag{17}$$

we obtain an additional condition for the supersymmetric solution:

$$-\frac{1}{4}C\Gamma_{12}\epsilon + \epsilon' = 0. \tag{18}$$

With Eqs. (16) and (18), we see that 1/4 of the supersymmetry is conserved by a holomorphic membrane solution of matrix theory when it is embedded in $\mathbb{C}^2$.

As the simplest example, we can consider membranes intersecting at a right angle, represented by $Z=z$, $W=1/z$. We take the ansatz $z = f(q)e^{ip}$ and $\{q,p\} = i$. Then it is straightforward that Eq. (11) means

$$\left(f^2 - \frac{1}{f^2}\right)' = C. \tag{19}$$

Solving it, we have the following solution for a static membrane:

$$X^1 = \sqrt{\frac{Cq + \sqrt{C^2q^2+4}}{2}}\cos p, \tag{20}$$

$$X^2 = \sqrt{\frac{Cq + \sqrt{C^2q^2+4}}{2}}\sin p,$$

$$X^3 = \sqrt{\frac{-Cq + \sqrt{C^2q^2+4}}{2}}\cos p,$$

$$X^4=\sqrt{\frac{-Cq+\sqrt{C^2q^2+4}}{2}}\sin p,$$

$$X^I=0,\quad I\neq 1,2,3,4.$$

As $q\rightarrow\infty$, the membrane lies on the 12-plane, while in the limit $q\rightarrow-\infty$ it lies on the 34-plane. Thus this solution represents membranes intersecting at a point, and the fact that this is 1/4 BPS is consistent with the general result of [9], which studied supersymmetric intersecting brane configurations of matrix theory.

Because of the ordering problem, the above solution in itself does not turn out to be useful in obtaining a solution of matrix theory directly. Instead, we get a hint from the flat membrane solution, Eq. (6), and its interpretation in terms of harmonic oscillator operators. We set

$$X^1=\frac{1}{2}(\delta_{i,j+1}x_j+\delta_{i,j-1}x_i),$$

$$X^2=\frac{i}{2}(\delta_{i,j+1}x_j-\delta_{i,j-1}x_i),\tag{21}$$

and X^3, X^4 according to the curve equation $W=1/Z$. Now we solve the equation of motion from the matrix theory Lagrangian to obtain the following difference equation:

$$x_{i+1}^2-\frac{1}{x_{i+1}^2}=x_i^2-\frac{1}{x_i^2}+C,\tag{22}$$

which is obviously the *quantized* version of the membrane equation (19). With a constant C and initial value of x_i, we can calculate every term of the array recursively. Calculating the eigenvalues of the coordinates, we note that

$$(X^1)^2+(X^2)^2=\frac{1}{2}\mathrm{Diag}(\ldots,x_{i+1}^2+x_i^2,\ldots),$$

$$(X^3)^2+(X^4)^2=\frac{1}{2}\mathrm{Diag}(\ldots,x_{i+1}^{-2}+x_i^{-2},\ldots).\tag{23}$$

When C is positive, $x_i^2\rightarrow\infty$ as $i\rightarrow\infty$ and $x_i^2\rightarrow 0$ as $i\rightarrow-\infty$; so the *D0*-branes with label i very large are confined near the origin of the 34-plane, but very far from the origin of the 12-plane, and oppositely when i goes to negative infinity. This again leads us to interpret the solution as intersecting membranes.

Now we turn to finding a membrane embedded nonholomorphically in three spatial dimensions. As an ansatz we demand axial symmetry, and this time we try a solution with nontrivial excitation of the gauge field. Since matrix theory describes M theory in the light cone frame, the time component of the gauge field corresponds to the excitation of the membrane in the X^- direction. This should be exactly the way to get a fundamental string from M theory in DLCQ (discrete light cone quantization) formalism. We will find that our solution corresponds to the BI solution of fundamental string attached to a D-brane:

$$\begin{aligned}X^0&=h(q),\\ X^1&=f(q)\cos p,\\ X^2&=f(q)\sin p,\\ X^3&=g(q),\\ X^I&=0,\quad I\neq 0,1,2,3.\end{aligned}\tag{24}$$

Using the equation of motion, the functions f,g,h should satisfy

$$\frac{1}{2}(f^2)''=(g')^2-(h')^2,$$

$$(f^2g')'=(f^2h')'=0,\tag{25}$$

where a prime denotes differentiation with respect to the variable q. Being autonomous, this set of coupled nonlinear differential equations is easily integrated. We have

$$g=\pm\int\frac{C\,df}{\sqrt{k^2f^2-(C^2-D^2)}}=\pm\frac{C}{k}\log[f+\sqrt{f^2-(C^2-D^2)/k^2}],\tag{26}$$

$$h=\pm\int\frac{D\,df}{\sqrt{k^2f^2-(C^2-D^2)}}=\pm\frac{D}{k}\log[f+\sqrt{f^2-(C^2-D^2)/k^2}],\tag{27}$$

$$q=\int\frac{2f^2df}{\sqrt{k^2f^2-(C^2-D^2)}}=\frac{1}{k}\left(f\sqrt{f^2-(C^2-D^2)/k^2}-\frac{C^2-D^2}{k^2}\times\log[f+\sqrt{f^2-(C^2-D^2)/k^2}]\right),\tag{28}$$

where C,D,k are integration constants.

Let us have a look at the supersymmetry transformation rule, Eq. (13), under the ansatz, Eq. (24):

$$\delta\theta=-\frac{1}{2}fh'(\sin p\Gamma_1-\cos p\Gamma_2)\epsilon-\frac{1}{2}ff'\Gamma_{12}\epsilon-\frac{1}{2}fg'\sin p\Gamma_{13}\epsilon+\frac{1}{2}fg'\cos p\Gamma_{23}\epsilon+\epsilon'.\tag{29}$$

When $C=D$, the solution simplifies into

$$f=\sqrt{kq},$$

$$g=h=\pm\frac{C}{2k}\log q,\tag{30}$$

and it is straightforward to see that when $C=D$ the solution is 1/4 BPS with the following condition of unbroken supersymmetry:

$$(1+\Gamma_3)\epsilon=0,$$

$$-\frac{k}{4}\Gamma_{12}\epsilon+\epsilon'=0. \tag{31}$$

The *M*-theory membrane is dual to the $D2$-brane of type-IIA string theory. Since Born-Infeld theory describes the low-energy effective dynamics of *D*-branes, we expect that both theories may allow the same solutions.[1] In [7,8] solutions with transverse excitations with electromagnetic charge were found:

$$X(r)=\int_r^\infty \frac{B}{\sqrt{r^{2p-2}-r_0^{2p-2}}}dr, \tag{32}$$

$$E=F_{0r}=\frac{A}{\sqrt{r^{2p-2}-r_0^{2p-2}}}, \tag{33}$$

where X represents one of the transverse directions of the p-brane and E is the radial component of the electric field. r is the radial coordinate on the world volume, $r_0^{2p-2}=B^2-A^2$, and the BPS condition corresponds to $A/B\to 1$ or $r_0\to 0$. When $r_0=0$ this solution represents a string attached on *D*-branes, while when $r_0\neq 0$ we have a catenoidal solution of the brane antibrane bound state. We find that the form of Eqs. (32),(33) is the same as Eqs. (26),(27), and furthermore in the same limit of the point charge solution, the two solutions become BPS, as expected.

For the matrix representation we follow the same reasoning which was used for the holomorphic curve $W=1/Z$, and set

$$X^1=\delta_{i,j+1}x_j+\delta_{i,j-1}x_i, \tag{34}$$

$$X^2=i(\delta_{i,j+1}x_j-\delta_{i,j-1}x_i),$$

$$X^3=\delta_{i,j}y_i,$$

$$X^0=\delta_{i,j}z_i,$$

$$X^I=0, \quad I\neq 0,1,2,3,$$

and try to solve the equation of motion for the static configuration. We have the following coupled differential equations for x_i, y_i, z_i:

$$x_{i+1}^2-2x_i^2+x_{i-1}^2=\frac{1}{2}(y_{i+1}-y_i)^2-\frac{1}{2}(z_{i+1}-z_i)^2,$$

$$x_i^2(y_{i+1}-y_i)=x_{i-1}^2(y_i-y_{i-1}),$$

$$x_i^2(z_{i+1}-z_i)=x_{i-1}^2(z_i-z_{i-1}). \tag{35}$$

It is evident that this is just the discretized version of the coupled differential equations (25); so it is reasonable to admit the solution of the above difference equations as a quantized version of the continuum solution. The above equations can be rewritten as

$$x_{i+1}^2=2x_i^2-x_{i-1}^2+2\frac{C^2-D^2}{x_i^4},$$

$$y_{i+1}=y_i+\frac{C}{x_i^2},$$

$$z_{i+1}=z_i+\frac{D}{x_i^2}. \tag{36}$$

This matrix solution also becomes BPS when $C=D$.

It would be excellent if we repeat the study of dynamic issues treated in [7] with the matrix theory solution and find coincidence with supergravity again. But unfortunately the BI theory result is claimed to be incompatible with the supergravity calculation except for $D3$ and $D4$-branes [11].

One interesting topic for further study is the extension to higher-dimensional branes. In [5] four-dimensional spherical branes in matrix theory were constructed using SO(5) gamma matrices. It would be exciting if we could obtain four-dimensional static solutions which have the same shape as $(4+1)$-dimensional Born-Infeld theory solutions. Since in that case the result of BI fluctuation studies is identical to the supergravity result, we might do the same calculation in terms of matrix theory and check if it is consistent with BI theory or supergravity.

[1]This fact was noted also in [10], that the three-dimensional Born-Infeld action can be derived from the action of relativistic membrane moving in $\mathbb{R}^3$ through gauge fixing.

[1] T. Banks, W. Fischler, S. H. Shenker, and L. Susskind, Phys. Rev. D **55**, 5112 (1997).

[2] B. de Wit, J. Hoppe, and H. Nicolai, Nucl. Phys. **B305** [FS23], 545 (1988).

[3] D. Kabat and W. Taylor, Adv. Theor. Math. Phys. **2**, 181 (1998).

[4] S.-J. Rey, hep-th/9711081.

[5] J. Castelino, S. Lee, and W. Taylor, Nucl. Phys. **B526**, 334 (1998).

[6] L. Cornalba and W. Taylor, Nucl. Phys. **B536**, 513 (1998).

[7] C. G. Callan and J. M. Maldacena, Nucl. Phys. **B513**, 198 (1998).

[8] G. W. Gibbons, Nucl. Phys. **B514**, 603 (1998).

[9] M. de Roo, S. Panda, and J. P. van der Schaar, Phys. Lett. B **426**, 50 (1998).

[10] M. Bordemann and J. Hoppe, Phys. Lett. B **325**, 359 (1994).

[11] S. Lee, A. Peet, and L. Thorlacius, Nucl. Phys. **B514**, 161 (1998).