Gravitationally induced neutrino oscillation phases in static spacetimes

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We critically examine the recent claim of a ''new effect'' of gravitationally induced quantum mechanical phases in neutrino oscillations. Because this claim has generated some discussion in the literature we present here a straightforward calculation of the phase and clarify some of the conceptual issues involved, particularly in relation to the equivalence principle. When expressed in terms of the asymptotic energy of the neutrinos E and Schwarzschild radial coordinates r, the lowest order at which such a gravitational effect appears is $(GM\Delta m^4/\hbar E^3)\ln(r_B/r_A)$. [S0556-2821(99)01204-7]

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Two years ago Ahluwalia and Burgard claimed to have discovered a ''new effect from an hitherto unexplored interplay of gravitation and the principle of the linear superposition of quantum mechanics,'' in a static, spherically symmetric gravitational potential [1]. In fact, the calculation of the quantum mechanical phase of a particle propagating in the vacuum Schwarzschild geometry appears in several textbooks on general relativity [2]. More to the point, the claimed results in Eqs. (6) to (8) of Ref. [1] appear to be at variance with the standard treatment found in these texts. Subsequently several authors have discussed the propagation of neutrinos including also the effects of matter [3] and nonradial propagation [4]. Finally, the authors of Ref. [1] have speculated on the relevance of the gravitational effect on neutrino oscillation phases for type-II supernovae [5]. Our purpose in this Brief Report is to present a treatment of phase interference in the Schwarzschild geometry in a manifestly coordinate invariant way, clarifying the conceptual issues raised by these works.

Let us begin by making some general remarks, first about quantum interference and second about the equivalence principle. Strictly speaking, in any quantum interference phenomenon such as neutrino oscillations, one should always deal with coherent wave packets whose shape may change during the propagation from the point of creation A to the point of detection B. The overlap of different components of the wave packet controls the visibility of the interference pattern at the spacetime point B. We will be concerned here only with the phase *not* the amplitude of the interference, and therefore we can avoid an explicit discussion of wave packet propagation. Of course no interference at all will be observable at B, unless the wave packet of a single neutrino created in a weak flavor eigenstate at A remains coherent for a time long enough for the components of the wavepacket transmitted from A at different velocities to interfere coherently at B. As a first approximation we will assume that at the point of emission A the flavor of the neutrino is independent of time, i.e. that the coherence time of the source is effectively infinite. This assumption allows us to concentrate on the superposition of neutrino mass eigenstates at a definite monochromatic energy E in the geometrical optics limit. A quantitative lower bound for the necessary coherence time Δt and therefore, an upper bound for the energy spread of the wave packet emitted by the source will emerge in Eq. (8) below.

Next we remark that the equivalence principle (EP) requires that all reference to the gravitational potential can always be removed locally. Since the EP is built into the metric formulation of general relativity this means that there can be no *local* observational consequences of a gravitational potential in Einstein's theory. In this Brief Report we are interested in neutrinos propagating between two different spacetime points A and B, which is not strictly a local process, and the natural variable in which to express our result for the neutrinos' phase Φ is their conserved asymptotic energy E, which is not strictly a local quantity. Hence a nonzero gravitational effect on the phase in terms of E does not contradict the EP. However, the equivalence principle does guarantee that there is no effect of the local gravitational potential on any observable expressed entirely in terms of local quantitites. This fact alone should alert us to the importance of defining the variables in which we express our result for the phase precisely. Otherwise one can change to local variables in which terms proportional to the gravitational potential necessarily disappear. Clearly, these changes of variables can have no physical consequences on a measurement, *unless* one can define and measure the gravitational and nongravitational contributions to the phase *independently*. In other words when assessing whether or not we have a measurable gravitational effect in the phase, the crucial question is: *with respect to what*?

In the geometrical optics limit the quantum mechanical phase accumulated by a particle propagating from point A to point B in the gravitational field described by the metric $g_{\mu\nu}$ is given by the action of the particle along its classical trajectory, namely,

$$\Phi_{AB}=\frac{1}{\hbar}\int_A^B m\,ds=\frac{1}{\hbar}\int_A^B p_\mu dx^\mu$$

$$=\frac{1}{\hbar}\int_A^B(-E\,dt+p_i dx^i) \tag{1}$$

where $p_\mu = m g_{\mu\nu}(dx^\nu/ds)$ is the four momentum conjugate to x^μ and ds is an element of proper length of the particle's worldline. The integrand of Eq. (1) is obviously invariant under coordinate transformations. However, this line integral does depend on coordinate changes at the end-points A and

B and therefore is not a physically meaningful quantity as it stands. Equation (1) is the same as Eq. (4) of Ref. [1] with which those authors begin.

The authors of Ref. [1] address the radial propagation of relativistic neutrinos in the potential of a spherically symmetric non-rotating star which is described by the Schwarzschild line element,

$$ds^2=-\left(1-\frac{2GM}{r}\right)dt^2+\left(1-\frac{2GM}{r}\right)^{-1}dr^2+r^2d\Omega^2.$$

We note that the semiclassical phase for radial motion in a spherically symmetric background does not depend on the spin of the particle, as can be verified by explicit calculation using the spin connection in the Dirac equation in this background [6]. Hence Eq. (1) applies equally well to neutrinos as to scalar particles in the case of radial motion, and we specialize to this case as it was the only one considered in Refs. [1] and [5].

Because the Schwarzschild spacetime has a timelike Killing vector, $\partial/\partial t$, the momentum conjugate to t is time independent, i.e.

$$E\equiv -p_t=m\left(1-\frac{2GM}{r}\right)\frac{dt}{ds}=\text{const.} \tag{2}$$

The value of this constant E is the asymptotic energy of the neutrino at $r=\infty$. For radial motion, the mass shell constraint $p_\mu g^{\mu\nu}p_\nu+m^2=0$ is

$$-\left(1-\frac{2GM}{r}\right)^{-1}E^2+\left(1-\frac{2GM}{r}\right)p_r^2+m^2=0 \tag{3}$$

from which we obtain

$$p_r\left(1-\frac{2GM}{r}\right)=\sqrt{E^2-m^2+\frac{2GMm^2}{r}}. \tag{4}$$

Making use of the definitions,

$$p_r=m\left(1-\frac{2GM}{r}\right)^{-1}\frac{dr}{ds}, \tag{5}$$

and Eq. (2), we have also

$$\frac{dt}{dr}=\left(1-\frac{2GM}{r}\right)^{-2}\frac{E}{p_r}. \tag{6}$$

We regard the Schwarzschild radial coordinates r_A and r_B as fixed and express the phase in these coordinates:

$$\begin{aligned}\Phi_{AB}&=\frac{1}{\hbar}\int_{r_A}^{r_B}\left(-E\frac{dt}{dr}+p_r\right)dr\\&=-\frac{1}{\hbar}\int_{r_A}^{r_B}\frac{m^2dr}{\left(1-\frac{2GM}{r}\right)p_r}\\&=-\frac{m^2}{\hbar}\int_{r_A}^{r_B}\frac{dr}{\sqrt{E^2-m^2+\frac{2GMm^2}{r}}}.\end{aligned} \tag{7}$$

This is just the standard expression for the phase [2]. However, since Φ_{AB} is not invariant under coordinate changes of the endpoints we cannot apply this result directly without first carefully specifying the physical situation, and in particular, which variables are to be held fixed in a given interference experiment.

Let us consider the case of neutrinos produced at fixed asymptotic energy E in a weak flavor eigenstate that is a linear superposition of mass eigenstates, m_1 and m_2. Since the energy is fixed but the masses are different, if interference is to be observed at the same final spacetime point (r_B,t_B), the relevant components of the wave function could not both have started from the same initial spacetime point (r_A,t_A), in the geometrical optics approximation. Instead the lighter mass (hence faster moving) component must either have started at the same time from a spatial location $r<r_A$, or started from the same location r_A at a later time $t_A+\Delta t$. Since the source is located at r_A and has been assumed monochromatic at fixed E for all times it is the latter situation which applies. Hence, there is an additional phase difference between the two mass components due to the time lag Δt, quite apart from the phase Φ_{AB} in Eq. (7). This additional initial phase difference may be taken into account by treating the spatial coordinates r_A and r_B as fixed and the time of transit,

$$\Delta t=t_B-t_A=\int_A^B dt=\int_{r_A}^{r_B}\frac{dt}{dr}dr \tag{8}$$

as the dependent variable through Eqs. (4) and (6). The difference of this time of transit between the two mass eigenstates, multiplied by E is precisely the additional phase, $E\Delta t$ which we must add to $\Delta\Phi_{AB}$ to obtain the correct relative phase between the two mass components of the same single neutrino wave function which interfere at (r_B,t_B) with fixed energy E. We note that the Collela-Overhauser-Werner (COW) experiment [7] may be treated by similar reasoning and that many other neutrino oscillation scenarios which may be envisaged lead to the same result.

This additional phase $E\Delta t$ enters for clear kinematic reasons due to the different times of transit of the classical trajectories from A to B of the different mass components and allows us to treat the cases of relativistic and nonrelativistic neutrinos at once and on an equal footing in contrast to the ‘‘light ray’’ approach of Ref. [4], which applies only in the

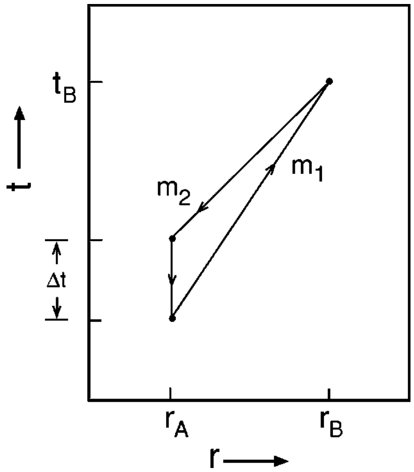

FIG. 1. The spacetime closed contour over which the line integral for the phase $\Delta\Phi_E$ is to be evaluated in Eq. (10).

relativistic limit $E \gg m$. It is also this Δt that determines the lower bound on the coherence time of the wave train emitted at A, for if the coherence time is less than Δt the components of the wave function corresponding to the two different masses could never interfere coherently at B. Thus a necessary condition for interference to be observed at B is that the neutrino emission at A must have an energy width $\Gamma < \hbar/\Delta t$, which is the maximum extent that our assumption of monochromatic neutrino energy can be relaxed.

Hence we are led to compute instead of $\Delta\Phi_{AB}$,

$$\Delta\Phi_E \equiv \Delta\Phi_{AB} + \frac{E}{\hbar}\Delta t = \frac{1}{\hbar}\Delta\int_{r_A}^{r_B} p_r dr \tag{9}$$

where the Δ refers to the difference of the phase for the two mass eigenstates m_1 and m_2. Evidently, this $\Delta\Phi_E$ may also be rewritten as the line integral around the *closed* loop $\mathcal{C}$ pictured in Fig. 1, i.e.

$$\Delta\Phi_E = \oint_{\mathcal{C}} p_\mu dx^\mu, \tag{10}$$

where $p_\mu(x;m_i)$ is to be regarded as a vector field that depends on the mass component m_i for the legs of $\mathcal{C}$ where r is varying but $p_t = -E$ is constant along the return leg where $r = r_A$. In this closed loop representation it is clear that $\Delta\Phi_E$ is a completely coordinate invariant, physically measurable phase difference which must vanish linearly in both $\Delta r \equiv r_A - r_B$ and $\Delta m \equiv m_1 - m_2$.

From Eqs. (4) and (9), we obtain

$$\Delta\Phi_E = \frac{1}{\hbar}\Delta\int_{r_A}^{r_B} \frac{\sqrt{E^2 - m^2 + \frac{2GMm^2}{r}}}{\left(1 - \frac{2GM}{r}\right)} dr. \tag{11}$$

In the weak field expansion this becomes

$$\Delta\Phi_E \simeq \frac{1}{\hbar}\left[(r_B - r_A) + 2GM\ln\left(\frac{r_B}{r_A}\right)\right]\Delta\sqrt{E^2 - m^2} + \frac{GM}{\hbar}\ln\left(\frac{r_B}{r_A}\right)\Delta\frac{m^2}{\sqrt{E^2 - m^2}} + \cdots. \tag{12}$$

Specializing now to the case $E \gg m_i$ the phase difference between two mass eigenstates of relativistic neutrinos with fixed E created at r_A and interfering at r_B is

$$\Delta\Phi_E \simeq \frac{(\Delta m^2)c^3}{2\hbar E}(r_B - r_A) + \frac{(\Delta m^4)c^7}{4\hbar E^3}(r_B - r_A) - \frac{(\Delta m^4)c^5}{2\hbar E^3} GM \ln\left(\frac{r_B}{r_A}\right) + \cdots, \tag{13}$$

where c has been restored to facilitate numerical calculations. We note that in the relativistic limit this result is just minus half of the equivalent quantity computed from the phase Φ_{AB}. The first term ($\Delta\Phi_E^0$) in Eq. (13) is the standard flat space result, well known in both neutrino and strangeness oscillations. The leading order ($G\Delta m^2$) correction to this familiar result has cancelled in the relativistic limit and we are left only with the latter higher order terms in Eq. (13). The second term is a special relativistic correction to the phase which is usually neglected for light neutrinos, and the last term is the effect of the gravitational field of the star in static Schwarzschild coordinates which enters only at the same higher order in $\Delta m^4/E^3$. Numerically its magnitude is

$$3.74\times 10^{-9}\left(\frac{M}{M_\odot}\right)\left(\frac{\Delta m^4}{\mathrm{eV}^4}\right)\left(\frac{\mathrm{MeV}}{E}\right)^3 \ln\left(\frac{r_B}{r_A}\right) \tag{14}$$

which is negligibly small for light neutrinos in typical astrophysical applications.

The authors of Ref. [1] claim to find an effect on the phase, first order in G, of the form

$$\frac{GMc}{\hbar}\left[\int_A^B \frac{dr}{r}\right]\frac{\Delta m^2}{E}. \tag{15}$$

No derivation is given to support this claim, the coordinate dependence of Φ_{AB} is not discussed and the quantity ''E'' is never defined in Refs. [1] or [5]. If E is the constant of motion defined by Eq. (2), this claim disagrees with the standard result (13) rederived here, and is therefore incorrect. On the other hand, if ''E'' is to be identified with $E_{\rm local} = E/\sqrt{-g_{tt}}$, then it cannot be removed from the integral, and Eq. (15) is incorrect for that reason.

There is a sense in which the *first* ($\Delta\Phi_E^0$) term of Eq. (13) has a contribution similar in form to Eq. (15). Let

$$\overline{E}_{\rm local} \equiv \frac{1}{r_B - r_A}\int_A^B \frac{E dr}{\sqrt{1 - \frac{2GM}{r}}} \simeq E + \frac{EGM}{r_B - r_A}\ln\left(\frac{r_B}{r_A}\right) \tag{16}$$

in the weak field limit. Hence we can write

$$\Delta\Phi_E^0 \simeq \frac{\Delta m^2 c^3}{2\hbar \bar{E}_{\rm local}}(r_B - r_A) + \frac{\Delta m^2 GMc}{2\bar{E}_{\rm local}} \ln\left(\frac{r_B}{r_A}\right) \quad (17)$$

which is of the form reported in Ref. [1]. However (and this is the point), this result depends critically on the precise definition of the variable $\bar{E}_{\rm local}$. Expressing the same physical phase difference (12) or (13) in terms of any other variables (such as the local energy of the emitter at r_A or the detector at r_B) will give different expressions again. Moreover by changing to both a local energy and local distance measure, $\Delta r_{\rm local} \equiv \int_A^B \sqrt{g_{rr}} dr$ we can reabsorb the ''new gravitational effect'' completely into these redefinitions and rewrite Eq. (17) in the form,

$$\Delta\Phi_E^0 \simeq \frac{(\Delta m^2) c^3}{2\hbar \bar{E}_{\rm local}} \Delta r_{\rm local}. \quad (18)$$

In fact, for the case of radial motion in general static coordinates for which $g_{rt} = 0$, the expression for each of the phases in Eq. (9) can be rewritten as

$$\Phi_E = -\frac{m^2 c^4}{\hbar} \int_A^B \frac{dt_{\rm local}}{E_{\rm local}} + \frac{1}{\hbar} \int_A^B dt_{\rm local} E_{\rm local} \quad (19)$$

where $dt_{\rm local} = \sqrt{-g_{tt}} dt$ and $E_{\rm local}$ was defined above. Equation (19) is a statement of the EP, and completely in accord with our general discussion at the start of this note, since all reference to the gravitational potential has been absorbed into integrals of local quantities.

Hence, we cannot agree with Ref. [8] that we have ''rederived'' the effect claimed in the original paper [1], since expressed entirely in terms of Schwarzschild or local coordinates there is *no* effect to order $G\Delta m^2$. Likewise the order $G\Delta m^2$ term obtained in Eq. (40) of [4] arises from these authors' use of $E_{\rm local}$ at the point of detection r_B instead of $\bar{E}_{\rm local}$ and it too can be removed by a simple change of variables.

The only indication of the basis for the claim of the result, (15) in Refs. [1] or [5], is a reference to a paper by Stodolsky [9]. However, as Stodolsky himself notes, the split between ''flat'' and ''curved'' space effects in Eq. (2.3) of his paper is coordinate dependent. Hence there is no invariant meaning to the splitting of the phase into these two pieces, and such a splitting is completely misleading for the present application, just as is the splitting of $\Delta\Phi_E^0$ into the two pieces in Eq. (17) above. In addition, the time component of the quantity Stodolsky calls the ''usual four-momentum of special relativity'' is not a constant of motion and cannot be removed from integrals over r. Thus Stodolsky's paper must be read closely to avoid potential pitfalls and sources of confusion. In any case since Stodolsky starts with precisely the same phase Φ_{AB} of Eq. (1) the *sum* of his two pieces is precisely equal to the same result (7) rederived here, as may be checked directly from the definitions in Ref. [9].

The essential point is that there is no physical meaning to decomposing the result into gravitational and nongravitational contributions such as Eq. (17), *unless* such a decomposition is invariant under coordinate transformations, in which case the two contributions should be measurable *separately*. The standard redshift of clocks in a gravitational field satisfies this criterion because one can measure the clock rate at two different locations and compare them. Accordingly, the difference in the clock times at the two locations can be expressed as a line integral, $\int ds$, over a closed contour which is a rectangle with sides at fixed r_A, r_B, t_1 and t_2. Hence the difference in the elapsed times on the two clocks is a coordinate invariant physically measurable effect. Grossman and Lipkin have considered this time delay effect, first order in $G\Delta m^2$, on neutrino oscillations due to a gravitating mass (such as the moon) intervening between the source and the detector [10], in which case the interference can be measured in principle both with and without the external body present. This is a completely different situation from that considered in Refs. [1,5] in which there is *no* nongravitational reference experiment (even in principle) relative to which the gravitational time delay effect on the phase can be measured.

The physical effects of neutrino oscillations on energy transport in supernova explosions are quite indifferent to local redefinitions of length, time and energy scales. If all calculations are done in a relativistically covariant framework, there are no observable consequences for supernova evolution to be deduced from the decomposition in Eq. (17), and even genuine nonlocal gravitational effects such as Eq. (14) are accounted for automatically. Of course, if one does *not* use a relativistically covariant framework for all the calculations, the error made will be of the order of the second term in Eq. (17), which is the order of the standard redshift of clocks in a gravitational field, and not a new effect.

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