UV-IR relations in AdS dynamics

Amanda W. Peet* and Joseph Polchinski[†]

Institute for Theoretical Physics, University of California, Santa Barbara, California 93106-4030

We point out that two distinct distance-energy relations have been discussed in the AdS-CFT correspondence. In conformal backgrounds they differ only in normalization, but in nonconformal backgrounds they differ in functional form. We discuss the relation to probe processes, the holographic principle, and black hole entropies. [S0556-2821(98)07924-7]

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An important feature of the recently discovered anti–de Sitter space (AdS) conformal field theory (CFT) duality is a correspondence between long distances in the AdS space and high energies in the CFT [1,2]. In fact, at least two quite distinct distance-energy relations have been discussed. While this point may have been noted implicitly elsewhere, we believe that it is instructive to discuss it in some detail. In Sec. I we consider various conformally invariant spaces (D3, M5, and M2). In Sec. II we consider conformally noninvariant Dp-brane spaces, where new issues arise.

I. CONFORMAL THEORIES

A. The D3-brane

For illustration let us consider the near-horizon geometry of N D3-branes, the AdS$_5 \times$S$_5$ space with string metric

$$ds^2 = \alpha' \left[\frac{U^2}{g_{\mathrm{YM}} N^{1/2}} dx_\parallel^2 + \frac{g_{\mathrm{YM}} N^{1/2}}{U^2} (dU^2 + U^2 d\Omega_5^2) \right] \quad (1)$$

and a constant dilaton $e^\Phi = g_s = g_{\mathrm{YM}}^2$. We use the conventions of Refs. [1,3] but omit all numerical constants. Susskind and Witten [2] argue that imposing an upper cutoff U on the AdS radius translates into an upper cutoff E on the CFT, where

$$E = \frac{U}{g_{\mathrm{YM}} N^{1/2}}. \quad (2)$$

One way to obtain this relation is to consider a local change in the boundary conditions at $U = \infty$. At radius U the fields are then perturbed in a region of size[1]

$$\delta x_\parallel = \frac{g_{\mathrm{YM}} N^{1/2}}{U}, \quad (3)$$

inverse to the energy (2). The relation (2) leads to a holographic result for the number of states of the string theory [2]: wrapping the system on a torus of side L, the area of surface in Planck units is

$$\alpha'^{-4} (e^{-2\Phi} \alpha'^4 g_{\mathrm{YM}} L^3 U^3 N^{1/2}) = L^3 E^3 N^2, \quad (4)$$

which is the indeed the entropy of the cut off gauge system.

Consider on the other hand a string stretched from the origin $U = 0$ to a probe D3-brane at a radius U as in the original discussion of Maldacena [1]. The world-sheet measure $(G_{tt} G_{UU})^{1/2} = \alpha'$ just offsets the string tension so that the energy is the coordinate length U,

$$E = U. \quad (5)$$

Relations (2) and (5) are both linear, a consequence of conformal symmetry. Although the physics of the CFT is independent of the scale, these relations are physically distinct: if one uses the relation (5) to evaluate the density of states one obtains the wrong holographic relation, a fact which led to the current investigation.

There is no contradiction or ambiguity here—there is no particular reason that the characteristic energy of probe processes should be related to the cutoff scale. It does, however, seem to contradict a naive renormalization group interpretation of the AdS-CFT correspondence. A remark in Ref. [2] helps to resolve this. The energy of the stretched string is interpreted in the gauge theory as the self-energy of a point charge. This self-energy will be proportional to $\delta x_\parallel = E^{-1}$ but also to the effective strength [4] $g_{\mathrm{YM}} N^{1/2}$ of the Coulomb interaction. Thus the distinct distance-energy relations (2) and (5) are consistent with the single distance-distance relation (3), at least when (as here) the effective description is given by low energy supergravity.

The energy (5) is the characteristic gauge theory energy governing the effective action of a D3-brane probe at a position U (see however the discussion at the end of this subsection). Similarly the holographic relation (2) corresponds to a probe by one of the massless supergravity fields. For an s-wave scalar ψ with longitudinal momentum k the wave equation is

$$\left[-k^2 \frac{g_{YM}^2 N}{U^2} + U^{-3} \partial_U (U^5 \partial_U) \right] \psi = 0. \quad (6)$$

By a scaling argument the solution depends only on $k^2 g_{YM}^2 N / U^2$, and so the characteristic radial dependence of the solution has the holographic relation (2) with the energy. This result is robust—it is the scaling such that all terms in the metric are of the same order—and holds for all other

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*Email address: peet@itp.ucsb.edu

[†]Email address: joep@itp.ucsb.edu

[1]This follows from consideration of the wave operator, given in Eq. (6) below.

fields and partial waves as well. This holographic relation arises in any context where the supergravity fields control the physics, including the temperature-radius relation for the black hole [2] and the relation between gauge instanton size and D-instanton position [5].

We should note that the D3-brane probe action is not a simple Wilsonian action at the scale $U=E$. Although it appears to be obtained by integrating out stretched strings of this energy, the discussion above shows that the size (3) of these states in the gauge theory is larger than their Compton wavelength for large $g_{YM}N^{1/2}$; thus they should not be treated as elementary in loops. Certain loop amplitudes, such the celebrated v^4 term [6], are protected by supersymmetry and are correctly given, but higher momentum dependences from the loop graph are incorrect [7].

We should also note several other relevant papers, including Refs. [8,9] which emphasize the differences between probes, and Refs. [10–16] which discuss the UV-IR relation from various points of view including the renormalization group interpretation.

B. The M-branes

The M5-brane metric

$$ds^2 = l_{11}^2 \left[\frac{U^2}{N^{1/3}} dx_\parallel^2 + \frac{N^{2/3}}{U^2} (dU^2 + U^2 d\Omega_4^2) \right] \quad (7)$$

and the M2-brane metric

$$ds^2 = l_{11}^2 \left[\frac{U^2}{N^{2/3}} dx_\parallel^2 + \frac{N^{1/3}}{U^2} (dU^2 + U^2 d\Omega_7^2) \right] \quad (8)$$

each lead to the holographic relation

$$E = \frac{U}{N^{1/2}}. \quad (9)$$

This is obtained either from the geodesic equation or the wave equation for a supergravity probe. These lead respectively to the densities of states[2]

$$S = l_{11}^{-9} (l_{11}^9 L^5 U^5 N^{1/2}) = L^5 E^5 N^3 \quad (10)$$

and

$$S = l_{11}^{-9} (l_{11}^9 L^2 U^2 N^{1/2}) = L^2 E^2 N^{3/2} \quad (11)$$

as expected from the nonextremal black hole entropies [17].

For the M5-brane system, the tension of an M2-brane stretched to an M5-brane probe at a position U is simply $\tau = U^2$, giving the scale

$$E = \tau^{1/2} = U. \quad (12)$$

This is the characteristic mass gap for an M5-brane probe, and as in the D3 case it is much larger than the holographic

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[2]This extension of Ref. [2] has been noted in Refs. [12,16] and in various unpublished remarks.

scale. For the M2-brane system, we know of no simple picture of the stretched state that is responsible for the mass gap in the M2-probe system. Conformal invariance determines that $E \propto U$ but not the coefficient. With the assumption that the scale is N-independent when expressed in terms of the usual D-brane coordinate $r = U^{1/2} l_{11}^{3/2}$, one obtains again the large scale $E = U$.

II. NONCONFORMAL Dp-BRANES

Although the existence of distinct distance-energy relations does not lead to an immediate contradiction, it can do so indirectly. Reference [3] analyzes various nonconformal near-horizon Dp-brane backgrounds, determining the effective theory governing the dynamics at various radii. It is a satisfying result of that paper that at every radius one can identify a useful effective theory, but that at no radius is there more than one weakly coupled effective theory (which would have been a contradiction). The effective coupling of the gauge theory depends on energy and so on the assumed distance-energy relation. The stretched-string relation (5) was used, so it follows that the analysis is relevant for a Dp-brane probe but not for a supergravity probe. We therefore extend the analysis to the latter case.

The Dp-brane near-horizon string metric and dilaton are

$$ds_p^2 = \alpha' \left[\frac{U^{(7-p)/2}}{g_{YM}N^{1/2}} dx_\parallel^2 + \frac{g_{YM}N^{1/2}}{U^{(7-p)/2}} (dU^2 + U^2 d\Omega_{8-p}^2) \right],$$

$$e^\Phi = g_{YM}^2 \left(\frac{U^{7-p}}{g_{YM}^2 N} \right)^{(p-3)/4}. \quad (13)$$

Here $g_{YM}^2 = g_s \alpha'^{(p-3)/2}$. The wave equation for a massless scalar ψ with angular momentum l, minimally coupled to the Einstein metric $ds_p'^2 = g_s^{1/2} e^{-\Phi/2} ds_p^2$, is

$$\left[-\frac{\partial^2}{\partial U^2} + \frac{(2l+8-p)(2l+6-p)}{4U^2} + \frac{k^2 g_{YM}^2 N}{U^{7-p}} \right] U^{(8-p)/2} \psi$$

$$= 0. \quad (14)$$

The energy-distance relations are then

$$\text{holographic-supergravity:} \quad E = \frac{U^{(5-p)/2}}{g_{YM}N^{1/2}},$$

$$\text{D}p\text{-brane:} \quad E = U. \quad (15)$$

For the supergravity probe, this is obtained from the wave equation, or again by requiring that the two terms in the metric have a common scaling. For the Dp-brane probe it is again determined by the energy of a stretched string. The effective gauge coupling is $g_{eff}^2 = g_{YM}^2 N E^{p-3}$ so that

$$\text{holographic-supergravity:} \quad g_{eff}^2 = [g_{YM}^2 N U^{p-3}]^{(5-p)/2},$$

$$\text{D}p\text{-brane:} \quad g_{eff}^2 = g_{YM}^2 N U^{p-3}. \quad (16)$$

In the absence of conformal invariance, the relations (15) in general no longer have the same functional form, and so we distinguish several cases.

A. $p \leqslant 4$

For $p \leqslant 4$, energy of the supergravity probe increases with distance, as in the conformal case. Moreover, the effective couplings (16), though distinct, are related in a simple way. The conditions $g_{\text{eff}}^2 \leqslant 1$ are equivalent for the two kinds of probe, and so the results in Ref. [3] for the effective description of a Dp-brane probe apply to the supergravity probe as well. The effective descriptions given in that paper thus cover the full range of U for both kinds of probe, with one subtlety of the $p = 1$ case to be discussed below.

It is interesting to extend the analysis of the holographic principle to these nonconformal cases. The area of the surface at radius U, in Planck units, is

$$A/G_{\text{N}} = e^{-2\Phi} L^p \left[\frac{U^{(7-p)/2}}{g_{\text{YM}} N^{1/2}} \right]^{p-4} U^{8-p}$$

$$= L^p E^{(9-p)/(5-p)} N^2 [g_{\text{YM}}^2 N]^{(p-3)/(5-p)}. \quad (17)$$

This is the same as the nonextremal Dp-brane entropy [17] at the corresponding temperature $T = E$, generalizing a result of Ref. [2]. The interpretation of this entropy has been discussed in Ref. [18]. In particular it has been noted that for $p = 4$ the $E^5 N^3$ behavior agrees with the expectation for a wrapped M5-brane. Curiously for $p = 1$ the $E^2 N^{3/2}$ behavior matches that of an M2-brane, even though there is no scale at which the D1 system is of this form.

Let us add the observation that for $p \leqslant 3$ the short-distance description is in terms of a dual gauge theory [3], and that the supergravity result (17) is consistent with this in two nontrivial respects: the N-dependence agrees with 't Hooft scaling, and the parameters α' and g_{s} appear only in the combination g_{YM}. These are not new results, in that they follow from the scaling of the action as already discussed in Ref. [1], but it is interesting to contrast this with the understanding obtained from the correspondence principle [19]. The latter gives a microscopic understanding of the entropy at one temperature, the boundary between the supergravity and gauge regimes. The duality with gauge theory, on the other hand, restricts the functional form throughout the supergravity regime. It does not determine the full form—for that one needs to understand the exponent of E (the exponent of g_{YM} then follows by dimensional analysis), which depends on the interactions.

More on $p = 1$

For $p = 1$ there is another issue.[3] In this case the effective supergravity description breaks down in both the low and high energy limits. The high energy description is a gauge theory as discussed above. The low energy description is in terms of long free strings. The latter description is effective only up to some cutoff energy, so the effective range in U depends on the relevant distance-energy relation.

The following discussion is equivalent to that in Ref. [3], though we will try to be more explicit about certain points including the energy-distance relation. The upper limit of the supergravity description is $U = g_{\text{YM}} N^{1/2}$ from $g_{\text{eff}} = 1$. For $U < g_{\text{YM}} N^{1/6}$, e^Φ is greater than one and the effective string theory is the S-dual of the original. The local tension of the dual $\widetilde{\text{F}}$-string$=$D-string in terms of the original string metric is $\tilde{\alpha}'(U)^{-1} = e^{\Phi(U)} \alpha'^{-1}$. The condition that the curvature be small compared to this is

$$1 > \tilde{\alpha}'(U) R = g_{\text{YM}}^2 / U^2, \quad (18)$$

or $U > g_{\text{YM}}$.

At shorter distances the effective description is in terms of long $\widetilde{\text{F}}$-strings, the so-called free orbifold CFT, whose most relevant interaction is [20]

$$\frac{1}{g_{\text{YM}}} \int d^2x \, V_{ij} \quad (19)$$

which reconnects the strings i and j. We can verify that the coefficient here is N-independent by considering the large-g_{s} limit in which $\widetilde{\text{F}}$-string perturbation theory is a good description: the coupling in this limit (where $e^\Phi = g_{\text{s}}$ and the metric is flat) is $\tilde{g}_{\text{s}} \tilde{\alpha}'^{1/2} = g_{\text{s}}^{-1/2} \alpha'^{1/2} = g_{\text{YM}}^{-1}$. Dimensionally the effective coupling is then $E N^{1/2} / g_{\text{YM}}$, in either a thermal situation ($E = T$) or a scattering process at energy E. The factor of $N^{1/2}$ is combinatoric: for example, the zeroth order free energy is of order N, while the second order correction in V_{ij} is of order N^2.

It follows that the $\widetilde{\text{F}}$-string picture is effective for

$$E < E_{\text{DVV}} \equiv g_{\text{YM}} N^{-1/2}. \quad (20)$$

(Note that throughout we are measuring energies in terms of the original coordinates $x_\parallel$.) Using the holographic relation $E = U^2 / g_{\text{YM}} N^{1/2}$, this becomes $U < g_{\text{YM}}$. Thus, for either supergravity probes or thermal properties the three descriptions (free CFT, supergravity, gauge theory) cover the full range of U without overlap. This was also the conclusion in Ref. [3], which in this discussion implicitly used the holographic distance-energy relation.

On the other hand, the brane probe relation $E = U$ appears to leave a gap, $U < g_{\text{YM}}$ while $E > E_{\text{DVV}}$, in which both the supergravity and free CFT descriptions are ineffective. However, we have already emphasized that this distance-energy relation is much less universal than the holographic relation. In the present case, the effective D-string description of the probe is replaced at these short distances with an effective $\widetilde{\text{F}}$-string description, and there appears to be no relevance to the scale $E = U$ (which would correspond to the loops of $\widetilde{\text{D}}$-strings).

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[3] We would like to thank N. Itzhaki for bringing this to our attention. The value $p = 1$ is special because it is the only nonconformal case with odd $p \leqslant 4$. For even p the very low energy description is in terms of $d = 11$ supergravity [3].

B. $p = 5$

This case has recently been discussed in Ref. [9], and our analysis will overlap with that paper. Reference [10] also notes that holography appears to break down for $p \geq 5$.

For $p = 5$ the holographic-supergravity relation between energy and radius degenerates: a supergravity field of given energy does not probe a characteristic radius. Rather, an on-shell field propagates along the throat indefinitely. In the small-curvature region $U > (g_{YM} N^{1/2})^{-1/2}$ the supergravity description is valid (though with an S-duality crossover in terms of the underlying string theory [3]). There are two solutions in this region, $\chi \propto U^{\beta_\pm}$ with

$$(\beta + 3/2)(\beta + 1/2) = (l + 3/2)(l + 1/2) + k^2 g_{YM}^2 N. \quad (21)$$

The reflection coefficient depends on physics in the small-U high-curvature region. For small $k^2 g_{YM}^2 N$ this will be determined by a weakly coupled gauge theory on the D5-branes. Note that also in this regime the energy-dependence from the supergravity solution (21) is weak. For large $k^2 g_{YM}^2 N$, there is no effective theory for the high-curvature region. Here, however, the energy-dependence in the supergravity region is strong and one might expect that this is the dominant effect. Thus there are effective descriptions for all energies.

Let us make a further remark about the holographic idea in this context.[4] Consider an upper cutoff U_0, and imagine a perturbation of the boundary condition which is local in $x_\parallel$. At $U < U_0$ it follows from the wave operator (14) that the fields are perturbed in a region of size

$$\Delta x_\parallel \sim g_{YM} N^{1/2} \ln \frac{U_0}{U}. \quad (22)$$

In order to relate the boundary field perturbation to a renormalized local operator in the boundary quantum theory [21,22] we must take U_0 to infinity while holding the fields fixed at a "renormalization scale" U. Unlike the AdS case (and the $p \leq 4$ branes) this is not possible: according to Eq. (22) the fields at U spread indefinitely. Thus renormalized

local operators do not exist.[5] One can also see this in momentum space: the generic solution grows as

$$U^{\beta_+} \sim U^{(g_{YM}^2 N k^2)^{1/2}} \quad (23)$$

at large U. The boundary data must scale in this way in order to produce a renormalized result; however, this has no Fourier transform. For $p \leq 4$ there is no such momentum-dependent renormalization. This may shed light on the point made in Ref. [9], that the boundary theory cannot be a normal field theory. Fourier modes of quantum fields exist, but not the local fields themselves.

C. $p = 6$

In this case the radius probed by a supergravity field varies inversely with the energy, in contrast with the usual expectation. Correspondingly, the effective gauge couplings (16) for the two probes have opposite weak coupling regimes. When $g_{YM}^2 N E^3$ is large there is no effective theory: the supergravity background is highly curved and also the gauge theory is strongly coupled. When $g_{YM}^2 N E^3$ is small there is the opposite problem: both the gauge and supergravity descriptions are weakly coupled.

This is the classic contradiction, whose avoidance in most circumstance is one of the striking evidences for duality [24]: the existence of distinct weakly coupled descriptions would surely lead to inconsistent results for some observables. In this case, however, we believe that the resolution is rather prosaic. Namely, the low energy Hilbert space has two sectors, a gauge theory which describes D6-brane probes close to the origin and a supergravity theory which describes supergravity probes far away. These sectors are isolated from one another, being respectively at U^3 less than and greater than $g_{YM}^{-2} N^{-1}$. The unusual behavior of this example, not seen in any of the others, is likely related to the nonexistence of an underlying field theory [3].

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[4]Similar observations have been made by Simon Ross.

[5]A similar pathology occurs in another context in which the asymptotic geometry is flat [23].

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