Four-point Green functions in the Schwinger model

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The evaluation of the four-point Green functions in the 1+1 Schwinger model is presented both in momentum and coordinate space representations. The crucial role in our calculations is played by two Ward identities: (i) the standard one and (ii) the chiral one. We demonstrate how the infinite set of Dyson-Schwinger equations is simplified, and is so reduced that a given *n*-point Green function is expressed only through itself and lower ones. For the four-point Green function, with two bosonic and two fermionic external ''legs,'' a compact solution is given both in momentum and coordinate space representations. For the four-fermion Green function a self-consistent equation is written down in the momentum representation and a concrete solution is given in the coordinate space. This exact solution is further analyzed and we show that it contains a pole corresponding to the Schwinger boson. All detailed considerations given for various four-point Green functions are easily generizable to higher functions. [S0556-2821(99)07304-X]

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I. INTRODUCTION

Massless quantum electrodynamics in 1+1 space-time dimensions, known as the Schwinger [1] model (SM), proved to be a very fruitful example of quantum field theory. Thanks to its symmetries it is a completely solvable model, and therefore it is particularly well suited for studying nonperturbative effects.

One of the most important and well-known observations is that the initially massless boson, called hereafter a ''photon'' (if one may think of ''photons'' in two dimensions), aquires a mass — the so-called Schwinger mass. As a consequence of this, the electromagnetic potential becomes a function exponentially decreasing in space and proportional to $e^{-\mu|x|}$, where $\mu = e/\sqrt{\pi}$ represents the Schwinger mass of the dressed photon. The vacuum polarization, which totally shields the charge is responsible for this effect [2]. The effect of charge screening is also known from perturbative calculations in ordinary, four-dimensional QED, giving rise to a weak deviation from the Coulomb law, particularly for small distances [3], while in the SM the change is dramatic. The interpretation of this massive state (composite versus elementary) depends on the particular field variables chosen to describe the model.

The photon mass generation mechanism appears already on the diagrammatical level, because the exact (nonperturbative) vacuum polarization scalar $\Pi(k^2)$ posesses a first order pole at $k^2=0$, with the residuum equal to μ^2. This is commonly known as the *Schwinger mechanism*. On the other hand, from the mathematical point of view, the nonzero photon mass results in the SM from the noninvariance of the path integral fermion measure with respect to the local chiral gauge transformations — the $U_A(1)$ group — which in turn is a reflection of the presence of anomaly in the model [4–7].

This vector meson mass generation through screening effects is of interest in electroweak theory, where the additional, and still unobserved, Higgs field has to be introduced ''by hand,'' to ensure the simultaneous renormalizability of theory and the nonzero masses of the intermediate bosons $W^{\pm}$ and Z^0.

Another important property of the SM is the absence of the asymptotic fermionic states [2]. This in turn is interesting from the point of view of hadron structure investigations [8], where the permanent quark confinement and asymptotic freedom of QCD, giving rise to the nonperturbative mass scale Λ_{QCD}, as the necessary mathematical ingredient of the logarithmic falloff, also precludes the appearence of asymptotic quark states. Yet another similarity between the SM and QCD is the existence of a fermion condensate [9–11], though this requires considering a nontrivial instanton sector. The above features of the SM are also preserved in a generalization of the SM, by allowing fermions to have a nonzero mass [12,13].

Thanks to its full solvability, the SM, on an equal footing with other models, as for instance the Thirring model [14], may also be used to test various assumptions in nonperturbative calculations in quantum field theories: For example, (i) the postulated infrared form of the vertex function in four-dimensional massive electrodynamics [15], applied later in the so-called *gauge technique* [16] and other works in the context of nonperturabative solutions of QED (for which the transverse corrections may be found in the SM explicitly [17]), (ii) renormalization group methods [18], or (iii) even the very formulation of quantum field theory [9,19,20]. One should also mention in this context generalized versions of the SM, formulated on the compact manifolds as a two-sphere [10,21] or torus [22] instead of the flat space as well as the light-cone formulation [23].

Although a number of papers have already been devoted to the investigation of propagators in the SM, a relatively small interest, up to our knowledge, has been paid to higher order Green functions [24]. In this paper we plan to fill the gap with particular interest paid to the four-point functions.

In the following sections we show how they can systematically be found. In Sec. II we consider the Ward identities in momentum space and show how an infinite set of Dyson-

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Schwinger equations can be reduced to only one, fully solvable equation. A particularly simple solution is given in Sec. II B for the function corresponding to ''Compton scattering.'' For the four-fermion Green function we derive in Sec. II C the integral equation which has a closed form (it does not contain any higher Green functions). In Sec. III we consider the same question in coordinate space. Following Schwinger in the quoted work [1], we find explicit solutions for both four-point Green functions: the four-fermion and the two-photon–two-fermion one. Both of them are expressible through the known scalar factors of the fermion propagator. For the most interesting case of four fermions we use the derived formula to show that the function contains a pole at $p^2=\mu^2$, i.e., corresponding to the Schwinger boson. We also give a formula for the form factor of the appropriate residue. In the Appendix we give definitions of all the Green functions considered in the present work.

II. MOMENTUM SPACE FOUR-POINT GREEN FUNCTIONS

In this section we are concentrating on the momentum space equations for the four-point Green functions. First, we deal with two-fermion–two-boson function. We recapitulate Ward identities which allow us to represent it as the appropriate combination of the three-point functions. These, however, are already known and expressible, once again due to Ward identities, through the full fermion propagator [25].

Second, we consider four-fermion Green function. In this case the situation is much more difficult since we do not have at our disposal any identity which would permit us to reduce the problem to lower functions. Therefore, we consider the Dyson-Schwinger equation which couples the four-point function to a five-point one (with one boson and four fermion ''legs''). Next, we apply both Ward identities to the latter, and as a consequence obtain a self-consistent integral equation which contains only the four-fermion function (and lower ones).

A. Notation and definitions

The SM may be defined through the two-dimensional Lagrangian density

$$\mathcal{L}(x)=\bar{\Psi}(x)[i\gamma^{\mu}\partial_{\mu}-eA^{\mu}(x)\gamma_{\mu}]\Psi(x)-\frac{1}{4}F^{\mu\nu}(x)F_{\mu\nu}(x)-\frac{\lambda}{2}[\partial_{\mu}A^{\mu}(x)]^2, \qquad (1)$$

where λ is the gauge fixing parameter. For our calculations it will be convenient to choose later the Landau gauge by setting $\lambda\to\infty$. For the Dirac gamma matrices the following convention will be used,

$$\gamma^0=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},\quad \gamma^1=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix},\quad \gamma^5=\gamma^0\gamma^1=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},$$

and for the metric tensor

$$g^{00}=-g^{11}=1.$$

The totally antisymmetric symbol $\varepsilon^{\mu\nu}$ is defined by

$$\varepsilon^{01}=-\varepsilon^{10}=1,\quad \varepsilon^{00}=\varepsilon^{11}=0.$$

Definitions of all the Green function that appear in the formulas below are collected together in the Appendix.

B. Calculation of the two-boson and two-fermion functions

We start this section with deriving Ward identities satisfied by the relevant Green function. The standard procedure in this derivation is to perform, under the functional integral (A1), the following infintesimal local gauge transformation,

$$A^{\mu}(x)\to A^{\mu}(x)+\partial^{\mu}\omega(x),$$
$$\Psi(x)\to\Psi(x)-ie\,\omega(x)\Psi(x), \qquad (2)$$
$$\bar{\Psi}(x)\to\bar{\Psi}(x)+ie\,\omega(x)\bar{\Psi}(x),$$

and consider the resulting variational equation. Doing in this way we get the relation satisfied by the generating functional $W(\eta,\bar{\eta},J)$:

$$-\lambda\Box_x\partial_x^{\mu}\frac{\delta W}{\delta J^{\mu}(x)}-\partial_x^{\mu}J_{\mu}(x)-ie\,\bar{\eta}_a(x)\frac{\delta W}{\delta\bar{\eta}_a(x)}+ie\,\eta_a(x)\frac{\delta W}{\delta\eta_a(x)}=0. \qquad (3)$$

Now we have to functionally differentiate both sides of this equation over $J^{\nu}(y)$, $\bar{\eta}_b(z)$, and $\eta_c(u)$. After having put all the external currents at zero value we obtain the following equation for the four-point Green function $\Gamma^{\mu\nu}$, defined in the Appendix:

$$\begin{aligned}i\lambda\Box_x\partial_x^{\mu}\int & d^2w_1d^2w_2d^2w_3d^2w_4D_{\mu\alpha}(x-w_1)S(z-w_3)\\ &\times\Gamma^{\alpha\beta}(w_1,w_2;w_3,w_4)S(w_4-u)D_{\beta\nu}(w_2-y)\\ &=-ie^2\int d^2w_1d^2w_2d^2w_3S(z-w_2)\Gamma^{\alpha}(w_1;w_2,w_3)\\ &\quad\times S(w_3-u)D_{\alpha\nu}(w_1-y)[\delta^{(2)}(x-z)-\delta^{(2)}(x-u)],\end{aligned} \qquad (4)$$

where we omitted the obvious spinor indices. If we now make use of the well-known [26] Ward identity satisfied by the photon propagator

$$\lambda\Box_x\partial_x^{\mu}D_{\mu\nu}(x-y)=\partial_{\nu}^{x}\delta^{(2)}(x-y), \qquad (5)$$

which stresses that only the transverse part of $D^{\mu\nu}$ is influenced by the interaction, and rewrite the expression in momentum space using the definitions of Fig. 1, we obtain, after removing the common factors on both sides,

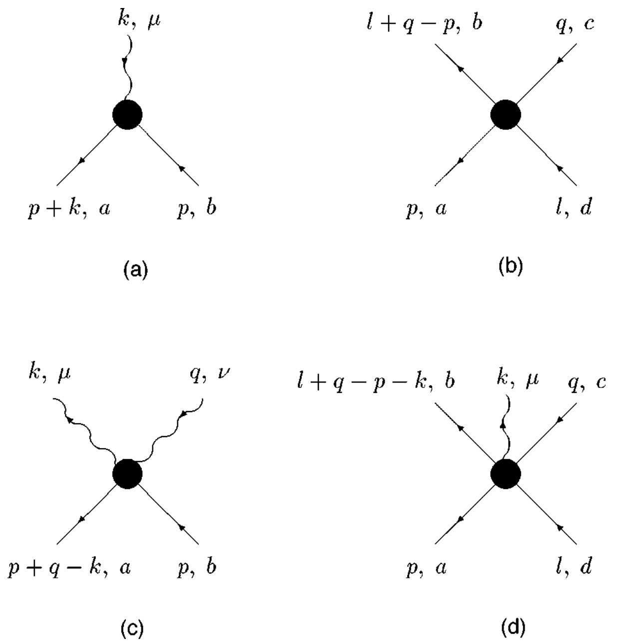

FIG. 1. Definitions of arguments in the vertex function and four- and five-point Green functions: (a) $\Gamma^{\mu}_{ab}(k,p)$, (b) $\Gamma_{ab,cd}(p,q,l)$, (c) $\Gamma^{\mu\nu}_{ab}(k,q,p)$, (d) $\Gamma^{\mu}_{ab;cd}(k,p,q,l)$.

$$ik_\mu S(p+q-k)\Gamma^{\mu\nu}(k,q,p)S(p) = e^2[S(p+q)\Gamma^{\nu}(q,p)S(p)-S(p+q-k)\times\Gamma^{\nu}(q,p-k)S(p-k)]. \tag{6}$$

Obviously, this equation does not define $\Gamma^{\mu\nu}$ entirely, but only its longitudinal part (in index μ). Fortunately, as a result of the vanishing electron mass, the Lagrangian $\mathcal{L}$ is invariant also with respect to the local *chiral* gauge transformations. In the infinitesimal version they read

$$A^{\mu}(x)\to A^{\mu}(x)+\varepsilon^{\mu\nu}\partial_\nu\omega(x),$$
$$\Psi(x)\to\Psi(x)-ie\omega(x)\gamma^5\Psi(x), \tag{7}$$
$$\bar{\Psi}(x)\to\bar{\Psi}(x)-ie\omega(x)\bar{\Psi}(x)\gamma^5.$$

Similarly as it was done to obtain Eq. (3) we can derive the following equation for the generating functional W:

$$\left(\Box_x+\frac{e^2}{\pi}\right)\varepsilon^{\mu\nu}\partial^x_\mu\frac{\delta W}{\delta J^{\nu}(x)}-\varepsilon^{\mu\nu}\partial^x_\nu J_\mu(x)-ie\bar{\eta}_a(x)\times\gamma_5^{ab}\frac{\delta W}{\delta\bar{\eta}_b(x)}-ie\eta_a(x)\gamma_5^{ba}\frac{\delta W}{\delta\eta_b(x)}=0. \tag{8}$$

One important difference in comparison with Eq. (3), which should be noted here, is the presence of the mass equal to e^2/π in the first term. This term results, as mentioned in the Introduction, from the noninvariance of the path integral measure with respect to the group of transformations (7) and constitutes the well-known chiral anomaly [27,5,7]. Following the same procedure as above, and using the chiral version of Eq. (5),

$$\left(\Box_x+\frac{e^2}{\pi}\right)\varepsilon^{\mu\nu}\partial^x_\mu D_{\nu\alpha}(x-y)=-\varepsilon_{\alpha\nu}\partial^\nu_x\delta^{(2)}(x-y), \tag{9}$$

we get, in momentum space,

$$i\varepsilon_{\mu\alpha}k^{\alpha}S(p+q-k)\Gamma^{\mu\nu}(k,q,p)S(p) = e^2[\gamma^5S(p+q)\Gamma^{\nu}(q,p)S(p)+S(p+q-k)\times\Gamma^{\nu}(q,p-k)S(p-k)\gamma^5]. \tag{10}$$

This, together with Eq. (6), determines uniquely $\Gamma^{\mu\nu}$ because in two dimensions there are only two independent space-time two-vectors. Therefore we can write

$$g^{\mu\nu}=\frac{1}{k^2}(k^{\mu}k^{\nu}-\varepsilon^{\mu\alpha}k_\alpha\varepsilon^{\nu\beta}k_\beta), \tag{11}$$

and as a consequence, any tensor $A^{\mu\nu}$ may be written as

$$A^{\mu\nu}=g^{\mu\lambda}A^{\nu}_{\lambda}=\frac{k^{\mu}}{k^2}(k_\lambda A^{\lambda\nu})-\frac{\varepsilon^{\mu\alpha}k_\alpha}{k^2}(\varepsilon_{\lambda\beta}k^{\beta}A^{\lambda\nu}). \tag{12}$$

Applying this to the four-point function $\Gamma^{\mu\nu}$, we find

$$S(p+q-k)\Gamma^{\mu\nu}(k,q,p)S(p) = -\frac{ie^2}{k^2}[\not{k}\gamma^{\mu}S(p+q)\Gamma^{\nu}(q,p)S(p)-S(p+q-k)\times\Gamma^{\nu}(q,p-k)\not{k}\gamma^{\mu}S(p-k)]. \tag{13}$$

In deriving this equation we made use of the fact that $k^{\mu}-\varepsilon^{\mu\alpha}k_\alpha\gamma^5=\not{k}\gamma^{\mu}$, as well as that the propagator S is linear in gamma matrices [1], and consequently $\{S,\gamma^5\}=0$ (at least in the zero-instanton sector to which we restrict ourselves in the present paper). In that way the four-point function is given in terms of the vertex function and the propagator. The vertex function, however, thanks to the analogous Ward identities, can be further reduced [17,25] to the form

$$\Gamma^{\nu}(q,p)=\frac{1}{q^2}[S^{-1}(p+q)-S^{-1}(p)]\not{q}\gamma^{\nu}. \tag{14}$$

Applying this, we obtain our final equation for the four-point Green function:

$$S(p+q-k)\Gamma^{\mu\nu}(k,q,p)S(p) = -\frac{ie^2}{k^2q^2}\not{k}\gamma^{\mu}[S(p)-S(p+q)-S(p-k)+S(p+q-k)]\gamma^{\nu}\not{q}. \tag{15}$$

From Eq. (15) we see that the four-point function is entirely expressible through two-point functions which are already known. In the same way one can reduce to fermion propaga-

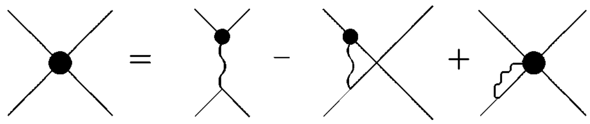

FIG. 2. Graphic representation of the Dyson-Schwinger equation for the four-point Green function: $\Gamma_{ab;cd}$. Thick lines correspond to full propagators and thin lines to free ones.

tors, by successively applying both Ward identities, any Green function with two fermion and n boson ''legs.''

One has to make use of a different approach while dealing with functions with more than two fermions, since only the photon ''legs'' may be removed the above way. We consider this question in the following paragraph.

The simple structure of $\Gamma^{\mu\nu}(k,p,q)$ reflects the fact that in the Schwinger model the external photons (i.e., nonperturbative, massive photons) are always coupled directly to the electron line, without an intermediate fermion loop, since any loop with more than two external photons turns out to be zero, if we consider all possible permutations of vertices.

C. Self-consistent equation for the four-fermion function

In the case of the four-fermion function the exploiting of the ordinary and the chiral Ward identities does not solve the problem completely (although it is still very useful), since we cannot reduce fermion ''legs'' anyway. However, we are able to obtain a self-consistent equation for this function. To do so we start from the Dyson-Schwinger equation for the four-fermion function which can be derived in a standard way. We consider the functional derivative over field $\overline{\Psi}(x)$ [3], and write

$$\int D\Psi D\overline{\Psi}DA\frac{\delta}{\delta\overline{\Psi}(x)}\exp\left(i\int d^2x[\mathcal{L}(x)+\overline{\eta}(x)\Psi(x)+\overline{\Psi}(x)\eta(x)+J^{\mu}(x)A_{\mu}(x)]\right)=0. \tag{16}$$

This gives the following relation for the generating functional:

$$i\gamma^{\mu}_{ab}\partial^{\mu}_{x}\frac{\delta W}{\delta\overline{\eta}_b(x)}-e\frac{\delta W}{\delta J^{\mu}(x)}\gamma^{\mu}_{ab}\frac{\delta W}{\delta\overline{\eta}_b(x)}+ie\gamma^{\mu}_{ab}\frac{\delta^2 W}{\delta J^{\mu}(x)\delta\overline{\eta}_b(x)}+\eta_a(x)=0. \tag{17}$$

Now, one has to differentiate the above equation over the fermionic currents $\eta_d(w)$, $\eta_c(z)$, and $\overline{\eta}_e(y)$, and at the end set all the currents η, $\overline{\eta}$, J equal to zero. The result is

$$\begin{aligned}
&-i\gamma^{\mu}_{ab}\partial^{x}_{\mu}(-i)\int d^2w_1d^2w_2d^2w_3d^2w_4S_{bf}(x-w_1)S_{eg}(y-w_2)\Gamma_{fg;rs}(w_1,w_2;w_3,w_4)S_{rc}(w_3-z)S_{sd}(w_4-w)\\
&=e^2\int d^2w_1d^2w_2d^2w_3D_{\mu\nu}(x-w_1)S_{ef}(y-w_2)\Gamma^{\nu}_{fg}(w_1;w_2,w_3)S_{gc}(w_3-z)\gamma^{\mu}_{ab}S_{bd}(x-w)\\
&-e^2\int d^2w_1d^2w_2d^2w_3D_{\mu\nu}(x-w_1)S_{ef}(y-w_2)\Gamma^{\nu}_{fg}(w_1;w_2,w_3)S_{gd}(w_3-w)\gamma^{\mu}_{ab}S_{bc}(x-z)\\
&+ie\gamma^{\mu}_{ab}\int d^2w_1d^2w_2d^2w_3d^2w_4d^2w_5D_{\mu\nu}(x-w_1)S_{bf}(x-w_2)S_{eg}(y-w_3)\Gamma^{\nu}_{fg;rs}(w_1;w_2,w_3;w_4,w_5)\\
&\times S_{rc}(w_4-z)S_{sd}(w_5-w).
\end{aligned} \tag{18}$$

A schematical representation of this equation is shown in Fig. 2. There the thick lines correspond to full propagators and thin lines to free ones. This equation indicates that the four-point function depends on the five-point one which is, of course, the typical behavior of the set of Dyson-Schwinger equations, since the interacting Lagrangian contains always terms of at least third order in fields, resulting in an infinite interlacement of Green functions. We can, however, use here the method of the previous paragraph and get rid of the five-point function from the right hand side. Equation (18), transformed into momentum space according to the definitions of Fig. 1, reads

$$\begin{aligned}
&i\not{p}_{ab}S_{bf}(p)S_{eg}(q+l-p)\Gamma_{fg;rs}(p,q,l)S_{rc}(q)S_{sd}(l)\\
&=e^2S_{ef}(q+l-p)\Gamma^{\nu}_{fg}(l-p,q)S_{gc}(q)D_{\mu\nu}(p-l)\gamma^{\mu}_{ab}S_{bd}(l)-e^2S_{ef}(q+l-p)\Gamma^{\nu}_{fg}(q-p,l)S_{gd}(l)D_{\mu\nu}(p-q)\\
&\times\gamma^{\mu}_{ab}S_{bc}(q)+ie\gamma^{\mu}_{ab}\int\frac{d^2k}{(2\pi)^2}D_{\mu\nu}(k)S_{bf}(p-k)S_{eg}(q+l-p)\Gamma^{\nu}_{fg;rs}(k,p-k,q,l)S_{rc}(q)S_{sd}(l).
\end{aligned} \tag{19}$$

Since the vertices Γ^{μ}_{ab} are perfectly known, what we need is only a relation which would allow us to express the five-point function $\Gamma^{\mu}_{ab;cd}$ through the four-point one. The method is analogous to that shown in detail in the previous paragraph, and we do not repeat it here. The result is

$$S(p)\otimes S(q+l-p-k)\cdot\Gamma^{\nu}(k,p,q,l)\cdot S(p)\otimes S(l)$$
$$=-\frac{ie}{k^2}\{[\not k\gamma^{\nu}S(p+k)]\otimes S(q+l-p-k)\cdot\Gamma(p+k,q,l)\cdot S(q)\otimes S(l)+S(p)\otimes[\not k\gamma^{\nu}S(q+l-p)]\cdot\Gamma(p,q,l)\cdot S(q)$$
$$\otimes S(l)-S(p)\otimes S(q+l-p-k)\cdot\Gamma(p,q,l-k)\cdot S(q)\otimes[\not k\gamma^{\nu}S(l-k)]$$
$$-S(p)\otimes S(q+l-p-k)\cdot\Gamma(p,q-k,l)\cdot[\not k\gamma^{\nu}S(q-k)]\otimes S(l)\},\tag{20}$$

where, for abbreviation, we have used the obvious notation $S^{(1)}\otimes S^{(2)}\cdot\Gamma\cdot S^{(3)}\otimes S^{(4)}$ for defining an object: $S^{(1)}_{ae}S^{(2)}_{bf}\Gamma_{ef;gh}S^{(3)}_{gc}S^{(4)}_{hd}$. If we substitute Eq. (20), together with Eq. (14), into Eq. (19), we obtain the final equation

$$i\not p_{ab}S_{bf}(p)S_{eg}(q+l-p)\Gamma_{fg;rs}(p,q,l)S_{rc}(q)S_{sd}(l)$$
$$=\frac{e^2}{(l-p)^2}\{(\not l-\not p)\gamma^{\nu}[S(q)-S(q+l-p)]\}_{ec}D_{\mu\nu}(p-l)\gamma^{\mu}_{ab}S_{bd}(l)-\frac{e^2}{(q-p)^2}\{(\not q-\not p)\gamma^{\nu}[S(l)-S(q+l-p)]\}_{ed}$$
$$\times D_{\mu\nu}(p-q)\gamma^{\mu}_{ab}S_{bc}(q)+e^2\gamma^{\mu}_{ab}\int\frac{d^2k}{(2\pi)^2}\frac{D_{\mu\nu}(k)}{k^2}\{[\not k\gamma^{\nu}S(p)]_{bf}S(q+l-p)_{eg}\Gamma_{fg;rs}(p,q,l)S_{rc}(q)S_{sd}(l)$$
$$+S(p-k)_{bf}[\not k\gamma^{\nu}S(q+l-p+k)]_{eg}\Gamma_{fg;rs}(p-k,q,l)S_{rc}(q)S_{sd}(l)-S(p-k)_{bf}S(q+l-p)_{eg}\Gamma_{fg;rs}(p-k,q,l-k)$$
$$\times S_{rc}(q)[\not k\gamma^{\nu}S(l-k)]_{sd}-S(p-k)_{bf}S(q+l-p)_{eg}\Gamma_{fg;rs}(p-k,q-k,l)[\not k\gamma^{\nu}S(q-k)]_{rc}S(l)_{sd}\},\tag{21}$$

in which only the four- and two-point functions are involved.

This derivation shows how the infinite series of coupled Dyson-Schwinger equations may be reduced to only one integral equation, which in principle might be solved. Because of the complicated tensor structure of $\Gamma_{ab;cd}$ (it requires introducing several scalar coefficient function) and perplexing mathematical form of Eq. (21), we do not try to solve it here and will rather concentrate on finding an explicit form of the four-fermion Green function in coordinate space.

It is a common feature of the Schwinger model that coordinate space solutions are much simpler than momentum space ones. In this case it will be even possible to express $\Gamma_{ab;cd}$ through the electron propagator, similarly as was done for $\Gamma^{\mu\nu}$ in Eq. (15). This problem will constitute the subject of the next section.

The method of the present paragraph allows also to find a self-consistent equation for any higher Green function. In particular, a function with $2n_f$ fermionic legs and n_b bosonic legs should first be reduced, thanks to the consecutive n_b applications of both Ward identities, to a purely fermionic $2n_f$-point function, and then the self-consistent equation for the latter can be obtained.

III. FOUR-POINT GREEN FUNCTIONS IN COORDINATE SPACE

In this section we find the explicit formulas for the four-point Green functions in coordinate space. In this case also the four-fermion function may be given a compact form, instead of having it as a solution of an integral equation like Eq. (21). Below we follow the way somehow similar to that of the original Schwinger's work [1], but extend it also to higher functions.

A. Two-boson and two-fermion function

As is known the generating functional $Z(\eta,\overline{\eta},J)$ may be given the following form:

$$Z(\eta,\overline{\eta},J)=\exp\left[-i\int d^2x d^2y\,\overline{\eta}(x)\mathcal{S}(x,y;\delta/i\delta J)\eta(y)\right]$$
$$\times\exp\left[-\frac{i}{2}\int d^2x d^2y J_{\mu}(x)\Delta^{\mu\nu}(x-y;e^2/\pi)\right.$$
$$\left.\times J_{\nu}(y)\right],\tag{22}$$

where $\mathcal{S}(x,y,\mathcal{A})$ is the classical electron propagator in the external electromagnetic field $\mathcal{A}^{\mu}$, and is given by the formula

$$\mathcal{S}(x,y,\mathcal{A})=\mathcal{S}_0(x-y)\exp\{-i[\tilde{\phi}(x,\mathcal{A})-\tilde{\phi}(y,\mathcal{A})]\},\tag{23}$$

with

$$\tilde{\phi}(x,\mathcal{A})=e\int d^2y\Delta(x-y)\gamma^{\nu}\gamma^{\mu}\partial_{\mu}\mathcal{A}_{\nu}(y),\tag{24}$$

$\mathcal{S}_0$ being the free propagator. In the Landau gauge which we now use, $\Delta^{\mu\nu}$ takes the form

$$\Delta^{\mu\nu}(x-y,m^2)=\int d^2z[g^{\mu\nu}\delta^{(2)}(x-z)-\partial^{\mu}_{x}\partial^{\nu}_{x}\Delta(x-z)]$$
$$\times\Delta(z-y;m^2),\tag{25}$$

where $\Delta(x,m^2)$ and $\Delta(x)$ are, respectively, Klein-Gordon and d'Alambert propagators. For the exponential factor in Eq. (23) one often uses the abbreviated and useful form

$$\tilde{\phi}(x,\mathcal{A})-\tilde{\phi}(y,\mathcal{A})=-\int d^2z\mathcal{A}^{\mu}(z)\mathcal{J}_{\mu}(z;x,y), \quad (26)$$

with the (nonconserved) current $\mathcal{J}^{\mu}$ satisfying

$$\partial_z^{\mu}\mathcal{J}_{\mu}(z;x,y)=e[\delta^{(2)}(x-z)-\delta^{(2)}(y-z)]. \quad (27)$$

This current has sources at every point, where charged particles are created or annihilated, and is closely related with the notion of the so-called "compensating current" [28]. Now the four-point Green function, considered in Sec. II B (for functions with no external "legs" amputated we reserve symbol G with appropriate indices), is given by

$$G_{ab}^{\mu\nu}(x_1,x_2;x_3,x_4)=\frac{\delta^4}{\delta J_{\mu}(x_1)\delta J_{\nu}(x_2)\delta\overline{\eta}_a(x_3)\delta\eta_b(x_4)}Z(\overline{\eta},\eta,J)|^{\text{connected}}_{\text{currents}=0}, \quad (28)$$

where we have explicitly written that only connected graphs are considered. This gives

$$G_{ab}^{\mu\nu}(x_1,x_2;x_3,x_4)=i\frac{\delta^2}{\delta J_{\mu}(x_1)\delta J_{\nu}(x_2)}\mathcal{S}_{ab}(x_3,x_4;\delta/i\delta J)\exp\left[-\frac{i}{2}\int d^2xd^2yJ^{\alpha}(x)\Delta_{\alpha\beta}(x-y;e^2/\pi)J^{\beta}(y)\right]\Bigg|^{\text{connected}}_{J=0}. \quad (29)$$

If we make use of the explicit form of $\mathcal{S}$, given in Eqs. (23), (24), leading to the representation in the form of a series of derivatives, and note that $\exp(zd/dx)f(x)=f(x+z)$, we can write

$$G_{ab}^{\mu\nu}(x_1,x_2;x_3,x_4)=i\frac{\delta^2}{\delta J_{\mu}(x_1)\delta J_{\nu}(x_2)}\mathcal{S}_0^{ac}(x_3-x_4)\exp\Bigg[-\frac{i}{2}\int d^2xd^2y[J^{\alpha}(x)+\mathcal{J}^{\alpha}(x;x_3,x_4)]\Delta_{\alpha\beta}(x-y;e^2/\pi)[J^{\beta}(y)$$
$$+\mathcal{J}^{\beta}(y;x_3,x_4)]\Bigg]_{cb}\Bigg|^{\text{connected}}_{J=0}. \quad (30)$$

Since the currents $\mathcal{J}$ defined in Eq. (26) have the matrix structure of the form $A+B\gamma^5$, which means that they commute with each other, the differentiation may be easily performed, and we obtain

$$G_{ab}^{\mu\nu}(x_1,x_2;x_3,x_4)=\mathcal{S}_0^{ac}(x_3-x_4)\Delta^{\mu\nu}(x_1-x_2;e^2/\pi)\exp\left[-\frac{i}{2}\int d^2xd^2y\mathcal{J}^{\alpha}(x;x_3,x_4)\Delta_{\alpha\beta}(x-y;e^2/\pi)\mathcal{J}^{\beta}(y;x_3,x_4)\right]_{cb}$$
$$-i\mathcal{S}_0^{ac}(x_3-x_4)\int d^2zd^2w\Delta_{\mu\lambda}(x_1-w;e^2/\pi)\mathcal{J}^{\lambda}_{cd}(w;x_3,x_4)\Delta_{\nu\rho}(x_2-z;e^2/\pi)\mathcal{J}^{\rho}_{de}(z;x_3,x_4)$$
$$\times\exp\left[-\frac{i}{2}\int d^2xd^2y\mathcal{J}^{\alpha}(x;x_3,x_4)\Delta_{\alpha\beta}(x-y;e^2/\pi)\mathcal{J}^{\beta}_{de}(y;x_3,x_4)\right]_{eb}\Bigg|^{\text{connected}}. \quad (31)$$

Now we recall that the full propagator S has the form [1]

$$S_{ab}(u-w)=\mathcal{S}_0^{ac}(u-w)\exp\left[-\frac{i}{2}\int d^2xd^2y\mathcal{J}^{\alpha}(x;u,v)\Delta_{\alpha\beta}(x-y;e^2/\pi)\mathcal{J}^{\beta}(y;u,v)\right]_{cb}. \quad (32)$$

This allows us to write Eq. (31) in the form

$$G_{ab}^{\mu\nu}(x_1,x_2;x_3,x_4)=S_0^{ab}(x_3,x_4)\Delta^{\mu\nu}(x_1-x_2;e^2/\pi)-iS_{ac}(x_3,x_4)\int d^2zd^2w\Delta_{\mu\lambda}(x_1-w;e^2/\pi)\mathcal{J}^{\lambda}_{cd}(w;x_3,x_4)$$
$$\times\Delta_{\nu\rho}(x_2-z;e^2/\pi)\mathcal{J}^{\rho}_{de}(z;x_3,x_4)|^{\text{connected}}. \quad (33)$$

The first term, constituting the nonconnected contribution, should now be rejected. For the amputated Green function Γ considered in Sec. II B, with the use of the definitions of $\mathcal{J}$ and $\tilde{\phi}$, as well as the fact that $S\gamma^{\mu}\gamma^{\nu}=\gamma^{\nu}\gamma^{\mu}S$, we obtain

$$\int d^2ud^2wS(x_3-u)\Gamma^{\mu\nu}(x_1,x_2;u,w)S(w-x_4)=-ie^2\partial\!\!\!/_{x_1}[\Delta(x_3-x_1)-\Delta(x_4-x_1)]\gamma^{\mu}S(x_3-x_4)\gamma^{\nu}\partial\!\!\!/_{x_2}[\Delta(x_3-x_2)$$
$$-\Delta(x_4-x_2)], \quad (34)$$

where we have omitted the spinor indices. One can easily verify that this is the coordinate space representation of Eq. (15). It is a common feature of all two-fermion Green functions that they can be represented through the fermion propagator in both coordinate and momentum spaces. As was mentioned this is possible since the external photons are coupled directly to the incoming and outgoing electron lines, and no intermediate fermions loops are possible, apart from those producing the photon mass e^2/π. On the other hand, for the four-fermion function we deal with in the following section, a much more complicated structure appears and an explicit and compact expression is possible to be given only in coordinate space.

B. Four-fermion function

Now, instead of Eq. (28), we have

$$G_{ab;cd}(x_1,x_2;x_3,x_4)=\frac{\delta^4}{\delta\overline{\eta}_a(x_1)\,\delta\overline{\eta}_b(x_2)\,\delta\eta_c(x_3)\,\delta\eta_d(x_4)}Z(\overline{\eta},\eta,J)\Big|^{\text{connected}}_{\text{currents}=0}=[\mathcal{S}_{ac}(x_1,x_3;\delta/i\,\delta J)\mathcal{S}_{bd}(x_2,x_4;\delta/i\,\delta J) \\ -\mathcal{S}_{ad}(x_1,x_4;\delta/i\,\delta J)\mathcal{S}_{bc}(x_2,x_3;\delta/i\,\delta J)]Z(J)\Big|^{\text{connected}}_{J=0}. \tag{35}$$

The differentiations over external current J, hidden in propagators $\mathcal{S}$, can be performed similarly as it was done to obtain Eq. (30), although it must be done with greater care than before due to the tensor structure of $\mathcal{J}$. In particular, using the notation of Sec. II C we find

$$\mathcal{S}(x_1,x_3;\delta/i\,\delta J)\otimes\mathcal{S}(x_2,x_4;\delta/i\,\delta J)\cdot Z(J)=\mathcal{S}_0(x_1-x_3)\otimes\mathcal{S}_0(x_2-x_4)\cdot Z[\mathbf{1}\otimes\mathbf{1}\cdot J+\mathcal{J}(x_1,x_3)\otimes\mathbf{1}+\mathbf{1}\otimes\mathcal{J}(x_2,x_4)], \tag{36}$$

where for abbreviation we have not explicitly written the first argument of the currents $\mathcal{J}$ over which the integration in the generating functional Z is taken [see Eqs. (22), (24),(26)]. Now, we concentrate only on the last factor of the above expression (Z), which, after setting $J=0$, takes the form

$$Z[\mathcal{J}(x_1,x_3)\otimes\mathbf{1}+\mathbf{1}\otimes\mathcal{J}(x_2,x_4)]=\exp\left[-\frac{i}{2}\int d^2x d^2y[\mathcal{J}^\mu(x;x_1,x_3)\otimes\mathbf{1}+\mathbf{1}\otimes\mathcal{J}^\mu(x;x_2,x_4)]\Delta_{\mu\nu}(x-y;e^2/\pi)[\mathcal{J}^\nu(y;x_1,x_3) \\ \times\otimes\mathbf{1}+\mathbf{1}\otimes\mathcal{J}^\nu(y;x_2,x_4)]\right]. \tag{37}$$

The expressions for both $\mathcal{J}_\mu$ and $\Delta_{\mu\nu}$ are known, and so it is only a matter of patience to get the formula for the above exponential. Thanks to the fact that in two dimensions $\gamma^\mu\gamma^\nu\gamma_\mu=0$ ''diagonal'' terms of the kind $\mathcal{J}^\mu\otimes\mathbf{1}\cdot\Delta_{\mu\nu}\cdot\mathcal{J}^\nu\otimes\mathbf{1}$ produce only expressions of the tensor structure $\mathbf{1}\otimes\mathbf{1}$, whereas mixed terms such as $\mathcal{J}^\mu\otimes\mathbf{1}\cdot\Delta_{\mu\nu}\cdot\mathbf{1}\otimes\mathcal{J}^\nu$ give both $\mathbf{1}\otimes\mathbf{1}$ and $\gamma^5\otimes\gamma^5$. We skip this calculation here to save the reader's time, and give below only the final result:

$$Z[\mathcal{J}(x_1,x_3)\otimes\mathbf{1}+\mathbf{1}\otimes\mathcal{J}(x_2,x_4)]=\frac{1}{2}(\mathbf{1}\otimes\mathbf{1}+\gamma^5\otimes\gamma^5)\exp\{ie^2[\beta(x_1-x_2)-\beta(x_1-x_3)-\beta(x_1-x_4)-\beta(x_2-x_3)-\beta(x_2-x_4) \\ +\beta(x_3-x_4)]\}+\frac{1}{2}(\mathbf{1}\otimes\mathbf{1}-\gamma^5\otimes\gamma^5)\exp\{ie^2[-\beta(x_1-x_2)-\beta(x_1-x_3)+\beta(x_1-x_4) \\ +\beta(x_2-x_3)-\beta(x_2-x_4)-\beta(x_3-x_4)]\}, \tag{38}$$

where the function β is defined by

$$\beta(x)=\int\frac{d^2p}{(2\pi)^2}(1-e^{ipx})\frac{1}{(p^2-e^2/\pi+i\epsilon)(p^2+i\epsilon)}=\begin{cases}\dfrac{i}{2e^2}\left[-\dfrac{i\pi}{2}+\gamma_E+\ln\sqrt{e^2x^2/4\pi}+\dfrac{i\pi}{2}H_0^{(1)}(\sqrt{e^2x^2/\pi})\right] & x\quad\text{timelike},\\[2ex] \dfrac{i}{2e^2}[\gamma_E+\ln\sqrt{-e^2x^2/4\pi}+K_0(\sqrt{-e^2x^2/\pi})] & x\quad\text{spacelike},\end{cases} \tag{39}$$

and is in fact a function of x^2 only. γ_E is here the Euler constant, and the functions $H_0^{(1)}$ and K_0 are the Hankel function of the first kind and Basset function respectively [29]. Since we have [1]

$$S(x)=\mathcal{S}_0(x)\exp[-ie^2\beta(x)], \tag{40}$$

we can write down the final formula for the four-fermion Green function:

$$G_{ab;cd}(x_1,x_2;x_3,x_4)=\frac{1}{2}[S_{ac}(x_1-x_3)S_{bd}(x_2-x_4)+[S(x_1-x_3)\gamma^5]_{ac}[S(x_2-x_4)\gamma^5]_{bd}]\exp\{ie^2[\beta(x_1-x_2)-\beta(x_1-x_4)$$
$$-\beta(x_2-x_3)+\beta(x_3-x_4)]\}+\frac{1}{2}\{S_{ac}(x_1-x_3)S_{bd}(x_2-x_4)-[S(x_1-x_3)\gamma^5]_{ac}[S(x_2-x_4)\gamma^5]_{bd}\}$$
$$\times\exp\{-ie^2[\beta(x_1-x_2)-\beta(x_1-x_4)-\beta(x_2-x_3)+\beta(x_3-x_4)]\}-\left\{\begin{matrix}x_3\leftrightarrow x_4\\ c\leftrightarrow d\end{matrix}\right\}. \quad (41)$$

We see that in coordinate space both four-point functions (four-fermion and two-boson–two-fermion) may perfectly be found and are given by compact formulas. Since β's are related to the full fermion propagator S, one can say that knowing S one knows ''everything.'' The calculation of higher functions may be led very much similarly to what was given in this section, and one will always obtain a product of electron propagators and exponentials of β function.

The exact expression for the four-fermion function we have obtained allows an analysis of its analytical properties. We concentrate below on the presence of the fermion-antifermion pole (t channel) corresponding to the Schwinger boson. Let us denote the first two terms on the right hand side of Eq. (41) by $G^1_{ab;cd}$ and $G^2_{ab;cd,}$ respectively. The remaining terms represented by the curly brackets can contribute to the eventual pole in the u channel only, and therefore we omit them in the present discussion.

While looking for a pole we first identify the ''in'' and ''out'' coordinates (in the t channel) of fermion and antifermion, $u\equiv x_1=x_3$, $v\equiv x_2=x_4$, and next consider the expression Fourier transformed in the variable $z\equiv v-u$. The identification has to be performed with care, for instance, in the following way.

(1) For the time coordinates we put

$$x_1^0=x_3^0\to u^0 \quad \text{and} \quad x_2^0=x_4^0\to v^0.$$

(2) For the spacial coordinates we assume

$$x_1^1\to u^1, \quad x_3^1\to u^1+\varepsilon, \quad x_2^1\to v^1, \quad x_4^1\to v^1+\eta.$$

(3) For the function depending on ε and η we take the fully symmetric limit

$$\lim_{\substack{\varepsilon\to0\\ \eta\to0}}^{sym} f(\varepsilon,\eta)\equiv\frac{1}{4}\lim_{\substack{\varepsilon\to0\\ \eta\to0}}[f(\varepsilon,\eta)+f(-\varepsilon,\eta)+f(\varepsilon,-\eta)$$
$$+f(-\varepsilon,-\eta)].$$

In that limit G^1 and G^2 become only z dependent. For instance, for G^1 we have

$$G^1(z)=\frac{1}{8\pi^2}(\gamma^0\otimes\gamma^0+\gamma^1\otimes\gamma^1)\lim_{\substack{\varepsilon\to0\\ \eta\to0}}\frac{1}{\varepsilon\eta}\exp\{ie^2[\beta(z)$$
$$-\beta(0,\varepsilon)-\beta(z^0,z^1+\eta)-\beta(z^0,z^1-\varepsilon)-\beta(0,\eta)$$
$$+\beta(z^0,z^1-\varepsilon+\eta)]\}, \quad (42)$$

where, when it was necessary, we wrote explicitly both coefficients of the two-vector argument of the β function

$$\beta(x)=\beta(-x)\equiv\beta(x^0,x^1).$$

The symmetric limit above may be performed in a straightforward way, since the β function is perfectly known, and we obtain

$$G^1(z)=-\frac{ie^2}{8\pi^2}(\gamma^0\otimes\gamma^0+\gamma^1\otimes\gamma^1)\frac{d^2}{dz^2}\beta(z). \quad (43)$$

The same limit for G^2 gives

$$G^2(z)=-\frac{ie^2}{8\pi^2}(\gamma^0\otimes\gamma^0-\gamma^1\otimes\gamma^1)\frac{d^2}{dz^2}\beta(z). \quad (44)$$

If we now apply explicitly the definition of β given by Eq. (39), and perform the Fourier transform over z, we find the following expression for the ''polar'' part of G:

$$G_{polar}(k)=-\frac{ie^2}{4\pi^2}\gamma^0\otimes\gamma^0\frac{(k^1)^2}{(k^2-e^2/\pi+i\epsilon)(k^2+i\epsilon)}$$
$$\to-\frac{i}{4\pi}\gamma^0\otimes\gamma^0\frac{(k^1)^2}{(k^2-e^2/\pi+i\epsilon)}, \quad (45)$$

from which a pole corresponding to the Schwinger boson may clearly be seen.

It should be noted that a similar analysis, although much more complicated, may be done without identyfing the ''in'' and ''out'' coordinates. One can, for example, introduce the new c.m. variables

$$u=\frac{1}{2}(x_1+x_3), \quad v=\frac{1}{2}(x_2+x_4),$$

and the relative ones

$$x=x_1-x_3, \quad y=x_2-x_4.$$

The Fourier transform of G performed over $z=v-u$ displays now much richer analytical structure (branch points at $k^2=n^2e^2/\pi,\ \ n=2,3,\ldots$) and the residue in the Schwinger pole depends on the relative coordinates x and y:

$$G_{polar}(x,y;k)=-4i\pi[S(x)\gamma^5]\otimes[S(y)\gamma^5]\times\frac{\sin[kx/2]\sin[ky/2]}{(k^2-e^2/\pi+i\epsilon)}, \tag{46}$$

where S is given by Eq. (40). For $x,y\to 0$ (in a symmetrical way) we reproduce the result given by Eq. (45). It may be noted that the form factor $F(x)\sim S(x)\gamma^5\sin kx/2$ is square normalizable in the sense $\int_{-\infty}^{\infty}dx^1|F(0,x^1)|^2$.

IV. SUMMARY

Below we would like to recapitulate the results we obtained in the present work. At first, in Sec. II, we considered Ward identities in momentun space satisfied by the four- and five-point Green functions. Thanks to the local chiral symmetry of the Lagrangian, apart from ordinary gauge invariance, we derived two identities. In the two-dimensional world these two identities suffice to entirely describe the considered Green function, and express it through lower ones. Each application of these identities allows us to reduce the number of external photons by 1. Following that approach we were able to reduce the two-boson–two-fermion function to the well-known electron propagator. In the case of the four-fermion function the situation turned out to be much more severe since we have no photon ‘‘legs’’ to reduce. An alternative approach was, therefore, introduced in Sec. II C. The starting point was here the Dyson-Schwinger equation which, on one hand, introduces the five-point function, but on the other permits one to reduce it to the function we are looking for. This leads to a self-consistent integral equation which contains, apart from the unknown function, only propagators which are perfectly known. We were, unfortunately, unable to solve this integral equation because of its complicated mathematical character, which is not unexpected since in the Schwinger model even the fermion propagator cannot be given an explicit form in momentum space. The self-consistent equation obtained in this section may, however, be a starting point for an analysis in momentum space constituting an alternative for taking the six-variable (two integrations may be separated out to give the Dirac delta function) Fourier transform.

In Sec. III we considered the same functions in coordinate space. We used the generating functional which had already been found in Schwinger's original work [1]. The Green functions are, of course, given as the appropriate derivatives of this functional over external currents. The problem which one only has to take care of is the tensor structure of the functions. Final compact expressions for all four-point functions were found and are shown to be expressible through the fermion propagator. All the methods of this, as well as of the preceding section, may easily be generalized to any higher Green functions.

For the most interesting case — the four-fermion function — we were able to show that Eq. (41) contains a pole, in the fermion-antifermion channel, corresponding to the Schwinger boson. It is interesting to note that the form factor in the residue of the pole turns out to be normalizable in the one-space direction if we set the relative time to zero. However, we do not treat this observation as any ‘‘proof’’ that the Schwinger boson is a ‘‘bound electron-positron state,’’ as is here and there suggested [2].

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APPENDIX: DEFINITIONS OF THE GREEN FUNCTIONS

In this appendix we give the definitions of various Green functions used in the formulas of Secs. II and III. If we introduce the generating functionals Z and W by the formula

$$Z(\eta,\bar{\eta},J)=\exp iW(\eta,\bar{\eta},J)=\int D\Psi D\bar{\Psi}DA\exp\left\{i\int d^2x[\mathcal{L}(x)+\bar{\eta}(x)\Psi(x)+\bar{\Psi}(x)\eta(x)+J^\mu(x)A_\mu(x)]\right\}, \tag{A1}$$

we can define the connected Green functions through derivatives of the functional over the external currents as follows:

$$\left.\frac{\delta^2 W}{\delta\bar{\eta}_a(x)\,\delta\eta_b(y)}\right|_{\text{currents}=0}=S_{ab}(x-y), \tag{A2}$$

$$\left.\frac{\delta^2 W}{\delta J^\mu(x)\,\delta J^\nu(y)}\right|_{\text{currents}=0}=-D_{\mu\nu}(x-y), \tag{A3}$$

$$\left.\frac{\delta^3 W}{\delta J^\mu(x)\,\delta\bar{\eta}_a(y)\,\delta\eta_b(z)}\right|_{\text{currents}=0}=-e\int d^2w_1d^2w_2d^2w_3D_{\mu\nu}(x-w_1)S_{ac}(y-w_2)\times\Gamma^\nu_{cd}(w_1;w_2,w_3)S_{db}(w_3-z). \tag{A4}$$

We also need the four- and five-point functions

$$\left.\frac{\delta^4 W}{\delta\overline{\eta}_a(x)\,\delta\overline{\eta}_b(y)\,\delta\eta_c(z)\,\delta\eta_d(u)}\right|_{\text{currents}=0} = -i\int d^2w_1 d^2w_2 d^2w_3 d^2w_4 S_{ae}(x-w_1)S_{bf}(y-w_2)\Gamma_{ef;gh}(w_1,w_2;w_3,w_4) \times S_{gc}(w_3-z)S_{hd}(w_4-u), \tag{A5}$$

$$\left.\frac{\delta^4 W}{\delta J^{\mu}(x)\,\delta J^{\nu}(y)\,\delta\overline{\eta}_a(z)\,\delta\eta_b(u)}\right|_{\text{currents}=0} = -i\int d^2w_1 d^2w_2 d^2w_3 d^2w_4 D_{\mu\alpha}(x-w_1)S_{ac}(z-w_3) \times \Gamma^{\alpha\beta}_{cd}(w_1,w_2;w_3,w_4)S_{db}(w_4-u)D_{\beta\nu}(w_2-y), \tag{A6}$$

$$\left.\frac{\delta^5 W}{\delta J^{\mu}(x)\,\delta\overline{\eta}_a(y)\,\delta\overline{\eta}_b(z)\,\delta\eta_c(u)\,\delta\eta_d(w)}\right|_{\text{currents}=0} = \int d^2w_1 d^2w_2 d^2w_3 d^2w_4 d^2w_5 D_{\mu\alpha}(x-w_1)S_{ae}(y-w_2)S_{bf}(z-w_3) \times \Gamma^{\alpha}_{ef;gh}(w_1;w_2,w_3;w_4,w_5)S_{gc}(w_4-u)S_{hd}(w_5-w). \tag{A7}$$

Thanks to the translational invariance of the theory, these functions depend in fact only on the differences of arguments. The corresponding definitions in momentum space, after having pulled apart the Dirac delta function of the whole two-momentum, are given on Fig. 1.

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